Turkish Journal of Mathematics

Volume 47 | Number 2 Article 14

1-1-2023

On conditions of regular solvability for two classes of third-order operator-differential equations in a fourth-order Sobolev-type space

ARAZ R. ALIEV

NAZILA L. MURADOVA

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 608 – 619
© TÜBİTAK
doi:10.55730/1300-0098.3382

On conditions of regular solvability for two classes of third-order operator-differential equations in a fourth-order Sobolev-type space

Araz R. ALIEV[1,2,*], **Nazila L. MURADOVA**[3]

[1]Department of General and Applied Mathematics, Faculty of Information Technology and Control, Azerbaijan State Oil and Industry University, Baku, Azerbaijan

[2]Department of Functional Analysis, Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan

[3]Department of Mathematics and Teaching Methods, Nakhchivan Teachers' Institute, Nakhchivan, Azerbaijan

Received: 31.10.2022 • **Accepted/Published Online:** 10.01.2023 • **Final Version:** 09.03.2023

Abstract: In this paper, we study two classes of operator-differential equations of the third order with a multiple characteristic, considered on the whole axis. We introduce the concept of a smooth regular solution of order 1 and obtain sufficient conditions for the "smoothly" regular solvability of these equations.

Key words: Operator-differential equation, Sobolev-type space, smooth regular solution, operator pencils, eigenvalue

1. Introduction

When modeling some problems in mechanics and engineering, particularly, in filtration problems [6], in stability problems for plates made of a plastic material [16], in problems of the dynamics of arches and rings [15], etc., partial differential equations with real and real multiple characteristics are used. These equations can be reduced to operator-differential equations with a multiple characteristic.

In [1], [3], [5], [8], [10], the main attention is paid to various issues of well-posed and unique solvability of fourth-order operator-differential equations with a multiple characteristic in Sobolev-type spaces. Despite the considerable number of journal publications (see, for example, [4], [7], [11], [13], [14]) devoted to the study of various aspects of the theory of operator-differential equations of an odd order, there are comparatively few works in which operator-differential equations of the third order with a multiple characteristic are studied in a broad aspect (see, for example, [2]).

Consider a third-order operator-differential equation of the form

$$\left(-\frac{d}{dt}+A\right)^{k}\left(\frac{d}{dt}+A\right)^{3-k}u(t)+\sum_{j=1}^{2}A_j u^{(3-j)}(t)=f(t),\ t\in\mathbb{R}=(-\infty,+\infty), \tag{1.1}$$

where A is a self-adjoint positive-definite operator in a separable Hilbert space H, A_j, $j=1,2$, are linear unbounded operators in H, $f(t)\in W_2^1(\mathbb{R};H)$, $u(t)\in W_2^4(\mathbb{R};H)$, $k=1$ or $k=2$. Here by $W_2^m(\mathbb{R};H)$ for

*Correspondence: alievaraz@yahoo.com, alievaraz@asoiu.edu.az

2010 *AMS Mathematics Subject Classification:* 34G10, 35G05, 47A68, 47E05, 47N20 integers $m \geq 1$we understand a Hilbert space (see [12]):

$$W_2^m(\mathbb{R};H) = \left\{ u(t) : \frac{d^m u(t)}{dt^m} \in L_2(\mathbb{R};H),\ A^m u(t) \in L_2(\mathbb{R};H) \right\}$$

with the norm

$$\|u\|_{W_2^m(\mathbb{R};H)} = \left(\left\| \frac{d^m u}{dt^m} \right\|^2_{L_2(\mathbb{R};H)} + \|A^m u\|^2_{L_2(\mathbb{R};H)} \right)^{1/2},$$

where $L_2(\mathbb{R};H)$ denotes the Hilbert space of vector-functions $f(t)$ defined in $\mathbb{R}$ with the values in H (see [9]), and for which

$$\|f\|_{L_2(\mathbb{R};H)} = \left(\int_{-\infty}^{+\infty} \|f(t)\|_H^2 \, dt \right)^{1/2} < +\infty.$$

Derivatives are understood in the sense of the theory of distributions (see [9]).

Writing an equation of the form (1.1) allows us to consider in one equation two classes of operator-differential equations of the third order with multiple characteristics. Note that equations of the form (1.1) cover equations with real and real multiple characteristics and can be applied, for example, in modeling of filtration problems [6] and problems of the dynamics of arches and rings [15].

Definition 1.1 *If a vector-function $u(t) \in W_2^4(\mathbb{R};H)$ satisfies Equation (1.1) for all $t \in \mathbb{R}$, then we will call it a smooth regular solution of order 1 to Equation (1.1).*

Definition 1.2 *If for any $f(t) \in W_2^1(\mathbb{R};H)$ there exists a smooth regular solution of order 1 to Equation (1.1) satisfying the inequality*

$$\|u\|_{W_2^4(\mathbb{R};H)} \leq const\ \|f\|_{W_2^1(\mathbb{R};H)},$$

then Equation (1.1) will be called "smoothly" regularly solvable.

In the present paper, coefficient conditions have been found that ensure the "smoothly" regular solvability of Equation (1.1). These conditions are sufficient. Such issues are studied in [5] for a class of fourth-order operator-differential equations with a multiple characteristic.

2. Boundedness of operators

Denote by H_θ the scale of Hilbert spaces generated by the operator A, i.e.

$$H_\theta = Dom\left(A^\theta\right),\ \theta \geq 0,\ (x,y)_\theta = (A^\theta x, A^\theta y),\ x,y \in Dom(A^\theta).$$

Throughout the entire work, $L(X,Y)$ is traditionally understood as the set of linear bounded operators acting from a Hilbert space X to another Hilbert space Y.

Denote, respectively, by $P_{0,k}$, $P_{1,k}$, and $P^{(k)}$ the operators acting from the space $W_2^4(\mathbb{R};H)$ into the space $W_2^1(\mathbb{R};H)$ as follows:

$$P_{0,k}u(t) \equiv \left(-\frac{d}{dt} + A\right)^k \left(\frac{d}{dt} + A\right)^{3-k} u(t),\ \ u(t) \in W_2^4(\mathbb{R};H),$$

$$P_{1,k}u(t) \equiv \sum_{j=1}^{2} A_j u^{(3-j)}(t), \quad u(t) \in W_2^4(\mathbb{R};H),$$

$$P^{(k)}u(t) \equiv P_{0,k}u(t) + P_{1,k}u(t), \;\; u(t) \in W_2^4(\mathbb{R};H).$$

The following two lemmas hold.

Lemma 2.1 *Let A be a self-adjoint positive-definite operator in H. Then the operator $P_{0,k}$ acts boundedly from the space $W_2^4(\mathbb{R};H)$ into the space $W_2^1(\mathbb{R};H)$.*

Proof For any $u(t) \in W_2^4(\mathbb{R};H)$, in case $k=1$, we have:

$$\|P_{0,1}u\|^2_{W_2^1(\mathbb{R};H)} = \left\| -\frac{d^3u}{dt^3} - A\frac{d^2u}{dt^2} + A^2\frac{du}{dt} + A^3u \right\|^2_{W_2^1(\mathbb{R};H)} =$$

$$\left\| -\frac{d^4u}{dt^4} - A\frac{d^3u}{dt^3} + A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} \right\|^2_{L_2(\mathbb{R};H)} +$$

$$\left\| -A\frac{d^3u}{dt^3} - A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} + A^4u \right\|^2_{L_2(\mathbb{R};H)} \leq$$

$$\left(\left\|\frac{d^4u}{dt^4}\right\|_{L_2(\mathbb{R};H)} + \left\|A\frac{d^3u}{dt^3}\right\|_{L_2(\mathbb{R};H)} + \left\|A^2\frac{d^2u}{dt^2}\right\|_{L_2(\mathbb{R};H)} + \left\|A^3\frac{du}{dt}\right\|_{L_2(\mathbb{R};H)} \right)^2 +$$

$$\left(\left\|A\frac{d^3u}{dt^3}\right\|_{L_2(\mathbb{R};H)} + \left\|A^2\frac{d^2u}{dt^2}\right\|_{L_2(\mathbb{R};H)} + \left\|A^3\frac{du}{dt}\right\|_{L_2(\mathbb{R};H)} + \left\|A^4u\right\|_{L_2(\mathbb{R};H)} \right)^2. \tag{2.1}$$

Using the theorem on intermediate derivatives [12]

$$\left\| A^j \frac{d^{4-j}u}{dt^{4-j}} \right\|_{L_2(\mathbb{R};H)} \leq c_j \left\|u\right\|_{W_2^4(\mathbb{R};H)}, \;\; j = 0,1,2,3,4,$$

from inequality (2.1) we obtain

$$\|P_{0,1}u\|_{W_2^1(\mathbb{R};H)} \leq \text{const} \, \|u\|_{W_2^4(\mathbb{R};H)}. \tag{2.2}$$

Carrying out the same reasoning in the case of $k=2$, we have:

$$\|P_{0,2}u\|^2_{W_2^1(\mathbb{R};H)} = \left\| \frac{d^3u}{dt^3} - A\frac{d^2u}{dt^2} - A^2\frac{du}{dt} + A^3u \right\|^2_{W_2^1(\mathbb{R};H)} =$$

$$\left\| \frac{d^4u}{dt^4} - A\frac{d^3u}{dt^3} - A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} \right\|^2_{L_2(\mathbb{R};H)} +$$

$$\left\| A\frac{d^3u}{dt^3} - A^2\frac{d^2u}{dt^2} - A^3\frac{du}{dt} + A^4u \right\|^2_{L_2(\mathbb{R};H)} \leq$$

$$\left(\left\|\frac{d^4u}{dt^4}\right\|_{L_2(\mathbb{R};H)}+\left\|A\frac{d^3u}{dt^3}\right\|_{L_2(\mathbb{R};H)}+\left\|A^2\frac{d^2u}{dt^2}\right\|_{L_2(\mathbb{R};H)}+\left\|A^3\frac{du}{dt}\right\|_{L_2(\mathbb{R};H)}\right)^2+$$

$$\left(\left\|A\frac{d^3u}{dt^3}\right\|_{L_2(\mathbb{R};H)}+\left\|A^2\frac{d^2u}{dt^2}\right\|_{L_2(\mathbb{R};H)}+\left\|A^3\frac{du}{dt}\right\|_{L_2(\mathbb{R};H)}+\left\|A^4u\right\|_{L_2(\mathbb{R};H)}\right)^2\leq$$

$$\leq \text{const}\,\|u\|^2_{W_2^4(\mathbb{R};H)}\,. \tag{2.3}$$

□

Lemma 2.2 *Let A be a self-adjoint positive-definite operator in H, and the operators $A_j\in L(H_j,H)\cap L(H_{j+1},H_1)$, $j=1,2$. Then the operator $P_{1,k}$ acts boundedly from the space $W_2^4(\mathbb{R};H)$ into the space $W_2^1(\mathbb{R};H)$.*

Proof For any $u(t)\in W_2^4(\mathbb{R};H)$, we have:

$$\|P_{1,k}u\|^2_{W_2^1(\mathbb{R};H)}=\left\|\sum_{j=1}^{2}A_ju^{(4-j)}\right\|^2_{L_2(\mathbb{R};H)}+\left\|\sum_{j=1}^{2}AA_ju^{(3-j)}\right\|^2_{L_2(\mathbb{R};H)}\leq$$

$$\left(\sum_{j=1}^{2}\left\|A_jA^{-j}A^ju^{(4-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2+\left(\sum_{j=1}^{2}\left\|AA_jA^{-(j+1)}A^{j+1}u^{(3-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2\leq$$

$$\left(\sum_{j=1}^{2}\left\|A_j\quad A^{-j}\right\|\left\|A^j\quad u^{(4-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2+\left(\sum_{j=1}^{2}\left\|AA_j\quad A^{-(j+1)}\right\|\left\|A^{j+1}\quad u^{(3-j)}\right\|_{L_2(\mathbb{R};H)}\right)^2.$$

Since $A_j\in L(H_j,H)\cap L(H_{j+1},H_1)$, $j=1,2$, then the operators A_jA^{-j} and $AA_jA^{-(j+1)}$, $j=1,2$, are bounded in H. Taking into account again the theorem on intermediate derivatives [12], we obtain

$$\|P_{1,k}u\|_{W_2^1(\mathbb{R};H)}\leq \text{const}\,\|u\|_{W_2^4(\mathbb{R};H)}\,. \tag{2.4}$$

□

Lemmas 2.1 and 2.2 imply the validity of the following theorem.

Theorem 2.3 *Let the conditions of Lemma 2.2 be satisfied. Then the operator $P^{(k)}$ acts boundedly from the space $W_2^4(\mathbb{R};H)$ into the space $W_2^1(\mathbb{R};H)$.*

Proof For any $u(t)\in W_2^4(\mathbb{R};H)$, from the inequalities (2.2), (2.3), and (2.4), it follows that

$$\left\|P^{(k)}u\right\|_{W_2^1(\mathbb{R};H)}\leq\|P_{0,k}u\|_{W_2^1(\mathbb{R};H)}+\|P_{1,k}u\|_{W_2^1(\mathbb{R};H)}\leq \text{const}\,\|u\|_{W_2^4(\mathbb{R};H)}\,.$$

□

3. Solvability of the equation $P_{0,k}u(t) = f(t)$

Now we study the solvability of the main part of Equation (1.1).

The following theorem holds.

Theorem 3.1 *The equation $P_{0,k}u(t) = f(t)$ has a unique smooth solution regular of order 1, $u(t)$, for any $f(t) \in W_2^1(\mathbb{R}; H)$, and the following inequality holds*

$$\|u\|_{W_2^4(\mathbb{R};H)} \leq const \|f\|_{W_2^1(\mathbb{R};H)}.$$

Proof Let $f(t) \in W_2^1(\mathbb{R}; H)$, i.e. the following norm is finite

$$\left\|\frac{df}{dt}\right\|^2_{L_2(\mathbb{R};H)} + \|Af\|^2_{L_2(\mathbb{R};H)} = \|f\|^2_{W_2^1(\mathbb{R};H)}.$$

Then it follows from Parseval's equality that

$$\left\|i\lambda\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} + \left\|A\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} < +\infty,$$

where $\widehat{f}(\lambda)$ is the Fourier transform of the function $f(t)$. Denoting by $\widehat{u}(\lambda)$ the Fourier transform of the function $u(t)$, from the equation $P_{0,k}u(t) = f(t)$ we have

$$P_{0,k}(i\lambda)\,\widehat{u}(\lambda) = \widehat{f}(\lambda)$$

or

$$\widehat{u}(\lambda) = P_{0,k}^{-1}(i\lambda)\,\widehat{f}(\lambda), \lambda \in \mathbb{R}. \tag{3.1}$$

From here, we determine

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P_{0,k}^{-1}(i\lambda)\,\widehat{f}(\lambda) e^{i\lambda t} d\lambda.$$

Let us show that $u(t)$ is a smooth regular solution of order 1 of the equation $P_{0,k}u(t) = f(t)$.

Indeed, from Parseval's equality, taking into account equality (3.1), we obtain:

$$\|u\|^2_{W_2^4(\mathbb{R};H)} = \left\|\frac{d^4u}{dt^4}\right\|^2_{L_2(\mathbb{R};H)} + \left\|A^4u\right\|^2_{L_2(\mathbb{R};H)} =$$

$$\left\|\lambda^4\widehat{u}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} + \left\|A^4\widehat{u}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} =$$

$$\left\|\lambda^4 P_{0,k}^{-1}(i\lambda)\,\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} + \left\|A^4 P_{0,k}^{-1}(i\lambda)\,\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} \leq$$

$$\sup_{\lambda\in\mathbb{R}} \left\|-i\lambda^3 P_{0,k}^{-1}(i\lambda)\right\|^2_{H\to H} \left\|i\lambda\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)} +$$

$$\sup_{\lambda\in\mathbb{R}} \left\|A^3 P_{0,k}^{-1}(i\lambda)\right\|^2_{H\to H} \cdot \left\|A\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)}. \tag{3.2}$$

On the other hand, from the spectral expansion of the operator A, we have:

$$\sup_{\lambda\in\mathbb{R}}\left\|-i\lambda^3 P_{0,k}^{-1}(i\lambda)\right\|_{H\to H}=\sup_{\lambda\in\mathbb{R}}\sup_{\sigma\in\sigma(A)}\left|-i\lambda^3(i\lambda+\sigma)^{-3+k}(-i\lambda+\sigma)^{-k}\right|=$$

$$\sup_{\lambda\in\mathbb{R}}\sup_{\sigma\in\sigma(A)}\left|-i\lambda^3(i\lambda+\sigma)^{-3+2k}\left(\lambda^2+\sigma^2\right)^{-k}\right|=\sup_{\lambda\in\mathbb{R}}\sup_{\sigma\in\sigma(A)}\frac{\lambda^3}{\left(\lambda^2+\sigma^2\right)^{3/2}}\le 1, \tag{3.3}$$

$$\sup_{\lambda\in\mathbb{R}}\left\|A^3 P_{0,k}^{-1}(i\lambda)\right\|_{H\to H}=\sup_{\lambda\in\mathbb{R}}\sup_{\sigma\in\sigma(A)}\left|\sigma^3(i\lambda+\sigma)^{-3+k}(-i\lambda+\sigma)^{-k}\right|=$$

$$\sup_{\lambda\in\mathbb{R}}\sup_{\sigma\in\sigma(A)}\frac{\sigma^3}{\left(\lambda^2+\sigma^2\right)^{3/2}}\le 1, \tag{3.4}$$

where $\sigma(A)$ denotes the spectrum of the operator A. Taking (3.3) and (3.4) into account in the inequality (3.2), we obtain:

$$\|u\|^2_{W_2^4(\mathbb{R};H)}\le\left\|i\lambda\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)}+\left\|A\widehat{f}(\lambda)\right\|^2_{L_2(\mathbb{R};H)}=\|f\|^2_{W_2^1(\mathbb{R};H)}.$$

Obviously, $u(t)$ satisfies the equation $P_{0,k}u(t)=f(t)$. Therefore, $u(t)$ is a smooth regular solution of order 1 of the equation $P_{0,k}u(t)=f(t)$. □

Corollary 3.2 *The norms $\|P_{0,k}u\|_{W_2^1(\mathbb{R};H)}$ and $\|u\|_{W_2^4(\mathbb{R};H)}$ are equivalent in the space $W_2^4(\mathbb{R};H)$.*

4. Estimation of the norms of intermediate derivatives operators

By the theorem on intermediate derivatives [12], according to Corollary 3.2, the following numbers are finite:

$$n_{j,k}=\sup_{0\ne u\in W_2^4(\mathbb{R};H)}\left\|A^{3-j}u^{(j)}\right\|_{W_2^1(\mathbb{R};H)}\cdot\|P_{0,k}u\|^{-1}_{W_2^1(\mathbb{R};H)},\ j=1,2.$$

There arises a problem of calculating $n_{j,k}$, $j=1,2$. Before turning to this problem, we prove the following lemma.

Lemma 4.1 *Let $\beta\in\left[0,\frac{27}{4}\right)$. Then the operator pencils*

$$\widetilde{P}_j(\lambda;\beta;A)=\left(-\lambda^2E+A^2\right)P_j(\lambda;\beta;A),\ \ j=1,2, \tag{4.1}$$

where

$$P_j(\lambda;\beta;A)=\left(-\lambda^2E+A^2\right)^3-\beta\,(i\lambda)^{2j}\,A^{6-2j},\ \ j=1,2,$$

(E is a unit operator) that depends on the parameter β, are invertible on the imaginary axis and there exist points $\xi_{0,j}\in\mathbb{R}$, $j=1,2$, such that the characteristic polynomials

$$\widetilde{P}_j(i\xi;\beta;\sigma)=\left(\xi^2+\sigma^2\right)\left(\left(\xi^2+\sigma^2\right)^3-\beta\xi^{2j}\sigma^{6-2j}\right),\ \ j=1,2,\ \ \sigma\in\sigma(A),$$

satisfy the following properties:

$\widetilde{P}_j\left(i\xi_{0,j};\beta;\sigma\right) > 0 \ \ at \ \beta\in\left[0,\frac{27}{4}\right), \ j=1,2, \ \sigma\in\sigma(A);$

$\widetilde{P}_j\left(i\xi_{0,j};\beta;\sigma\right) = 0 \ \ at \ \beta=\frac{27}{4}, \ j=1,2, \ \sigma\in\sigma(A);$

$\widetilde{P}_j\left(i\xi_{0,j};\beta;\sigma\right) < 0 \ \ at \ \beta>\frac{27}{4}, \ j=1,2, \ \sigma\in\sigma(A).$

Proof Since A is a self-adjoint positive-definite operator, i.e. $A=A^*\geq cE, \ c>0,$ then for $\sigma\in\sigma(A)$ $(\sigma\geq\sigma_0\geq c>0)$ the characteristic polynomials of the operator pencils $\widetilde{P}_j(\lambda;\beta;A), \ j=1,2,$ have the form

$$\widetilde{P}_j(\lambda;\beta;\sigma)=\left(-\lambda^2+\sigma^2\right)\left(\left(-\lambda^2+\sigma^2\right)^3-\beta\left(i\lambda\right)^{2j}\sigma^{6-2j}\right), \ \ j=1,2.$$

For $\lambda=i\xi, \ \xi\in\mathbb{R}$, we have

$$\widetilde{P}_j(i\xi;\beta;\sigma)=\left(\xi^2+\sigma^2\right)\left(\left(\xi^2+\sigma^2\right)^3-\beta\xi^{2j}\sigma^{6-2j}\right)=$$

$$\left(\xi^2+\sigma^2\right)^4\left(1-\beta\frac{\xi^{2j}\sigma^{6-2j}}{\left(\xi^2+\sigma^2\right)^3}\right)\geq\left(\xi^2+\sigma^2\right)\left[1-\beta\frac{\frac{\xi^{2j}}{\sigma^{2j}}}{\left(1+\frac{\xi^2}{\sigma^2}\right)^3}\right]\geq$$

$$\left(\xi^2+\sigma^2\right)\left[1-\beta\sup_{r\geq 0}\frac{r^j}{(r+1)^3}\right]\geq\sigma_0^2\left(1-\beta\frac{4}{27}\right), \ \ j=1,2.$$

Therefore, for $\beta\in\left[0,\frac{27}{4}\right)$, we have $\widetilde{P}_j(i\xi;\beta;\sigma)>0, \ j=1,2$. Then it follows from the spectral expansion of the operator A that $\widetilde{P}_j(i\xi;\beta;A)>0, \ j=1,2$, for any $\xi\in\mathbb{R}$, i.e. the operator pencils $\widetilde{P}_j(\lambda;\beta;A), \ j=1,2$, are invertible on the imaginary axis. Obviously, the minima of the characteristic functions $\widetilde{P}_j(i\xi;\beta;\sigma), \ j=1,2$, are positive for all $\beta\in\left[0,\frac{27}{4}\right)$. Indeed, as shown above,

$$\widetilde{P}_j(i\xi;\beta;\sigma)\geq\left(\xi^2+\sigma^2\right)\left(1-\beta\sup_{r\geq 0}\frac{r^j}{(r+1)^3}\right), \ \ j=1,2.$$

But the functions $\varphi_j(r)=\frac{r^j}{(r+1)^3}, \ j=1,2$, attain their maximum values for some $r_{0,j}=\frac{\xi_{0,j}^2}{\sigma^2}, \ j=1,2.$ Therefore, for these values $\xi_{0,j}, \ j=1,2,$

$$\widetilde{P}_j\left(i\xi_0;\beta;\sigma\right)\geq\left(\xi_{0,j}^2+\sigma^2\right)\left(1-\beta\frac{4}{27}\right), \ \ j=1,2.$$

□

Theorem 4.2 *The following equalities hold* $n_{j,k}=\frac{2}{3^{3/2}}, \ j=1,2.$

Proof Let the functions $u(t)\in W_2^4(\mathbb{R};H)$ have compact supports and be infinitely differentiable. Then, in case $k=1$, by Parseval's equality, we have:

$$\left\|P_{0,1}u\right\|^2_{W_2^1(\mathbb{R};H)}-\beta\left\|A^{3-j}u^{(j)}\right\|^2_{W_2^1(\mathbb{R};H)}=$$

$$\left\| -\frac{d^3u}{dt^3} - A\frac{d^2u}{dt^2} + A^2\frac{du}{dt} + A^3u \right\|^2_{W_2^1(\mathbb{R};H)} -$$

$$\beta \left\| A^{3-j}u^{(j)} \right\|^2_{W_2^1(\mathbb{R};H)} = \left\| -\frac{d^4u}{dt^4} - A\frac{d^3u}{dt^3} + A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} \right\|^2_{L_2(\mathbb{R};H)} +$$

$$\left\| -A\frac{d^3u}{dt^3} - A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} + A^4u \right\|^2_{L_2(\mathbb{R};H)} - \beta\left(\left\| A^{3-j}u^{(j+1)} \right\|^2_{L_2(\mathbb{R};H)} + \left\| A^{4-j}u^{(j)} \right\|^2_{L_2(\mathbb{R};H)} \right) =$$

$$\left\| \left(-(i\xi)^4 E - (i\xi)^3 A + (i\xi)^2 A^2 + (i\xi) A^3 \right) \widehat{u}(\xi) \right\|^2_{L_2(\mathbb{R};H)} +$$

$$\left\| \left(-(i\xi)^3 A - (i\xi)^2 A^2 + (i\xi) A^3 + A^4 \right) \widehat{u}(\xi) \right\|^2_{L_2(\mathbb{R};H)} -$$

$$\beta\left(\left\| (i\xi)^{j+1} A^{3-j}\widehat{u}(\xi) \right\|^2_{L_2(\mathbb{R};H)} + \left\| (i\xi)^{j} A^{4-j}\widehat{u}(\xi) \right\|^2_{L_2(\mathbb{R};H)} \right) =$$

$$\int_{-\infty}^{+\infty} \left(\left(-(i\xi)^4 E - (i\xi)^3 A + (i\xi)^2 A^2 + (i\xi) A^3 \right) \widehat{u}(\xi), \left(-(i\xi)^4 E - (i\xi)^3 A + (i\xi)^2 A^2 + (i\xi) A^3 \right) \widehat{u}(\xi) \right)_H d\xi +$$

$$\int_{-\infty}^{+\infty} \left(\left(-(i\xi)^3 A - (i\xi)^2 A^2 + (i\xi) A^3 + A^4 \right) \widehat{u}(\xi), \left(-(i\xi)^3 A - (i\xi)^2 A^2 + (i\xi) A^3 + A^4 \right) \widehat{u}(\xi) \right)_H d\xi -$$

$$\beta\int_{-\infty}^{+\infty} \left((i\xi)^{j+1} A^{3-j}\widehat{u}(\xi), (i\xi)^{j+1} A^{3-j}\widehat{u}(\xi) \right)_H d\xi - \beta\int_{-\infty}^{+\infty} \left((i\xi)^{j} A^{4-j}\widehat{u}(\xi), (i\xi)^{j} A^{4-j}\widehat{u}(\xi) \right)_H d\xi =$$

$$\int_{-\infty}^{+\infty} \left((\xi^2E + A^2)\left((\xi^2E + A^2)^3 - \beta\xi^{2j}A^{6-2j} \right)\widehat{u}(\xi), \widehat{u}(\xi) \right)_H d\xi, \ \ j = 1, 2.$$

Thus, for any $\beta \geq 0$ and sufficiently smooth $u(t)$, we have the equalities

$$\|P_{0,1}u\|^2_{W_2^1(\mathbb{R};H)} - \beta \left\| A^{3-j}u^{(j)} \right\|^2_{W_2^1(\mathbb{R};H)} = \int_{-\infty}^{+\infty} \left(\widetilde{P}_j(i\xi;\beta;A)\widehat{u}(\xi), \widehat{u}(\xi) \right)_H d\xi, \ \ j = 1, 2, \tag{4.2}$$

where $\widetilde{P}_j\,(i\xi;\beta;A)$ are defined in (4.1). By Lemma 4.1, for $\beta \in \left[0, \frac{27}{4}\right)$, the operator pencils $\widetilde{P}_j\,(i\xi;\beta;A) > 0$, $j = 1, 2$. Therefore, for $\beta \in \left[0, \frac{27}{4}\right)$, it follows from (4.2) that

$$\|P_{0,1}u\|^2_{W_2^1(\mathbb{R};H)} - \beta \left\| A^{3-j}u^{(j)} \right\|^2_{W_2^1(\mathbb{R};H)} > 0, \ \ j = 1, 2.$$

Proceeding here to the limit as $\beta \to \frac{27}{4}$, $j = 1, 2$, we have

$$\left\| A^{3-j}u^{(j)} \right\|_{W_2^1(\mathbb{R};H)} \leq \frac{2}{3^{3/2}} \|P_{0,1}u\|_{W_2^1(\mathbb{R};H)}, \ \ j = 1, 2,$$

i.e. $n_{j,1} \leq \frac{2}{3^{3/2}}$, $j = 1, 2$. To prove the equalities $n_{j,1} = \frac{2}{3^{3/2}}$, $j = 1, 2$, we define the functional

$$E(u) = \|P_{0,1}u\|^2_{W_2^1(\mathbb{R};H)} - \beta \left\| A^{3-j}u^{(j)} \right\|^2_{W_2^1(\mathbb{R};H)}$$

in the space $W_2^4(\mathbb{R};H)$ and $\forall\varepsilon > 0$, we look for the vector-function $u_\varepsilon(t) = g_\varepsilon(t)\psi_\varepsilon$, for which $E(u_\varepsilon) < 0$, where $\psi_\varepsilon \in Dom\left(A^8\right)$, $g_\varepsilon(t)$ is a scalar function. For this purpose, we write the inequality $E(u_\varepsilon) < 0$ as

$$E(u_\varepsilon) = \int_{-\infty}^{+\infty} \left(\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;A\right)\psi_\varepsilon,\psi_\varepsilon\right)\left|\widehat{g}_\varepsilon(\xi)\right|^2 d\xi < 0.$$

If A has at least one eigenvalue σ, then we choose the corresponding eigenvector ψ_ε: $A\psi_\varepsilon = \sigma\psi_\varepsilon$, $\|\psi_\varepsilon\| = 1$. Then it is obvious that

$$\left(\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;A\right)\psi_\varepsilon,\psi_\varepsilon\right)_H = \widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;\sigma\right),$$

but by the lemma $\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;\sigma\right) < 0$, $\forall\varepsilon > 0$ at $\xi = \xi_{0,j}$. If the operator A does not have an eigenvalue, then for any $\sigma \in \sigma(A)$ and for any $\delta > 0$, we can find a vector ψ_δ ($\|\psi_\delta\| = 1$) such that for any $s > 0$

$$A^s\psi_\delta = \sigma^s\psi_\delta + o(1,\delta) \text{ as } \delta \to 0,\ s > 0.$$

Then

$$\left(\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;A\right)\psi_\delta,\psi_\delta\right)_H = \widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;\sigma\right) + o(1,\delta) \text{ as } \delta \to 0.$$

For sufficiently small $\delta > 0$

$$\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;\sigma\right) + o(1,\delta) < 0.$$

Thus, for some $\xi = \xi_{0,j}$ and $\psi_\varepsilon \in Dom\left(A^8\right)$, $\varepsilon > 0$

$$\left(\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;A\right)\psi_\varepsilon,\psi_\varepsilon\right) < 0. \tag{4.3}$$

Since $\left(\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;A\right)\psi_\varepsilon,\psi_\varepsilon\right)$ is a continuous function of the argument ξ, the inequality (4.3) holds for some $\xi \in (\eta_1,\eta_2)$. Now we construct the function $g_\varepsilon(t)$ as follows. Let $\widehat{g}_\varepsilon(t)$ be an infinitely differentiable function with the support in the interval (η_1,η_2). Denote by

$$g_\varepsilon(t) = \frac{1}{\sqrt{2\pi}}\int_{\eta_1}^{\eta_2}\widehat{g}_\varepsilon(\xi)e^{i\xi t}d\xi.$$

Obviously, $g_\varepsilon(t) \in W_2^4(\mathbb{R})$. Then

$$E(u_\varepsilon) = E(g_\varepsilon(t)\psi_\varepsilon) = \int_{\eta_1}^{\eta_2}\left(\widetilde{P}_j\left(i\xi;\frac{27}{4}+\varepsilon;A\right)\psi_\varepsilon,\psi_\varepsilon\right)\left|\widehat{g}_\varepsilon(\xi)\right|^2 d\xi < 0,$$

it is hereby proved that $n_{j,1} = \frac{2}{3^{3/2}}$, $j = 1,2$.

The proof in the case of $k = 2$ is carried out in a similar way. $\square$

5. Solvability conditions for Equation (1.1)

We now formulate conditions on the "smoothly" regular solvability of Equation (1.1).

Theorem 5.1 *Let A be a self-adjoint positive-definite operator in H and the operators*

$$A_j \in L\left(H_j, H\right) \cap L\left(H_{j+1}, H_1\right), \quad j = 1, 2,$$

moreover, the following inequality holds

$$\sum_{j=1}^{2} \max\left\{\left\|A_{3-j}A^{-(3-j)}\right\|_{H\to H}, \left\|AA_{3-j}A^{-(4-j)}\right\|_{H\to H}\right\} < \frac{3^{3/2}}{2}.$$

Then Equation (1.1) is "smoothly" regularly solvable.

Proof Using the notation introduced earlier, we represent Equation (1.1) as an operator equation

$$P_{0,k}u(t) + P_{1,k}u(t) = f(t), \tag{5.1}$$

where $f(t) \in W_2^1(\mathbb{R}; H)$, $u(t) \in W_2^4(\mathbb{R}; H)$. By Theorem 2.3, the operator $P^{(k)} = P_{0,k} + P_{1,k}$ is a bounded operator from the space $W_2^4(\mathbb{R}; H)$ to the space $W_2^1(\mathbb{R}; H)$, and, by Theorem 3.1, the operator $P_{0,k}$ maps $W_2^4(\mathbb{R}; H)$ isomorphically onto $W_2^1(\mathbb{R}; H)$. Then there is a bounded inverse $P_{0,k}^{-1}$ acting from $W_2^1(\mathbb{R}; H)$ to $W_2^4(\mathbb{R}; H)$. If in Equation (5.1) we make the substitution $u(t) = P_{0,k}^{-1}\upsilon(t)$, where $\upsilon(t) \in W_2^1(\mathbb{R}; H)$, then we obtain

$$\left(E + P_{1,k}P_{0,k}^{-1}\right)\upsilon(t) = f(t).$$

Let us show that, under the conditions of the theorem, the norm of the operator $P_{1,k}P_{0,k}^{-1}$ is less than one. We have

$$\left\|P_{1,k}P_{0,k}^{-1}\upsilon\right\|_{W_2^1(\mathbb{R};H)} = \left\|P_{1,k}u\right\|_{W_2^1(\mathbb{R};H)} \le \sum_{j=1}^{2}\left\|A_j u^{(3-j)}\right\|_{W_2^1(\mathbb{R};H)} =$$

$$\sum_{j=1}^{2}\left(\left\|A_{3-j}u^{(j+1)}\right\|_{L_2(\mathbb{R};H)}^2 + \left\|AA_{3-j}u^{(j)}\right\|_{L_2(\mathbb{R};H)}^2\right)^{\frac{1}{2}} \le$$

$$\sum_{j=1}^{2}\left(\left\|A_{3-j}A^{-(3-j)}\right\|_{H\to H}^2\left\|A^{3-j}u^{(j+1)}\right\|_{L_2(\mathbb{R};H)}^2 + \left\|AA_{3-j}A^{-(4-j)}\right\|_{H\to H}^2\left\|A^{4-j}u^{(j)}\right\|_{L_2(\mathbb{R};H)}^2\right)^{\frac{1}{2}} \le$$

$$\sum_{j=1}^{2}\max\left\{\left\|A_{3-j}A^{-(3-j)}\right\|_{H\to H}, \left\|AA_{3-j}A^{-(4-j)}\right\|_{H\to H}\right\} \times$$

$$\left(\left\|A^{3-j}u^{(j+1)}\right\|_{L_2(\mathbb{R};H)}^2 + \left\|A^{4-j}u^{(j)}\right\|_{L_2(\mathbb{R};H)}^2\right)^{\frac{1}{2}}.$$

Since

$$\left\|A^{3-j}u^{(j)}\right\|_{W_2^1(\mathbb{R};H)} = \left(\left\|A^{3-j}u^{(j+1)}\right\|_{L_2(\mathbb{R};H)}^2 + \left\|A^{4-j}u^{(j)}\right\|_{L_2(\mathbb{R};H)}^2\right)^{\frac{1}{2}},$$

then

$$\left\|P_{1,k}P_{0,k}^{-1}v\right\|_{W_2^1(\mathbb{R};H)} \leq$$

$$\sum_{j=1}^{2}\max\left\{\left\|A_{3-j}A^{-(3-j)}\right\|_{H\to H}, \left\|AA_{3-j}A^{-(4-j)}\right\|_{H\to H}\right\} \times \left\|A^{3-j}u^{(j)}\right\|_{W_2^1(\mathbb{R};H)} \leq$$

$$\frac{2}{3^{3/2}}\sum_{j=1}^{3}\max\left\{\left\|A_{3-j}A^{-(3-j)}\right\|_{H\to H}, \left\|AA_{3-j}A^{-(4-j)}\right\|_{H\to H}\right\} \times \left\|P_{0,k}u\right\|_{W_2^1(\mathbb{R};H)} =$$

$$\frac{2}{3^{3/2}}\sum_{j=1}^{2}\max\left\{\left\|A_{3-j}A^{-(3-j)}\right\|_{H\to H} \quad \left\|AA_{3-j}A^{-(4-j)}\right\|_{H\to H}\right\} \times \left\|v\right\|_{W_2^1(\mathbb{R};H)}.$$

Thus, $\left\|P_{1,k}P_{0,k}^{-1}\right\|_{W_2^1(\mathbb{R};H)\to W_2^1(\mathbb{R};H)} < 1$. Therefore, the operator $E+P_{1,k}P_{0,k}^{-1}$ has an inverse in the space $W_2^1(\mathbb{R};H)$ and, hence, $u(t)$ can be defined by the formula

$$u(t) = P_{0,k}^{-1}\left(E+P_{1,k}P_{0,k}^{-1}\right)^{-1}f(t),$$

and

$$\|u\|_{W_2^4(\mathbb{R};H)} \leq \left\|P_{0,k}^{-1}\right\|_{W_2^1(\mathbb{R};H)\to W_2^4(\mathbb{R};H)}\left\|\left(E+P_{1,k}P_{0,k}^{-1}\right)^{-1}\right\|_{W_2^1(\mathbb{R};H)\to W_2^1(\mathbb{R};H)} \times$$

$$\|f\|_{W_2^1(\mathbb{R};H)} \leq const\,\|f\|_{W_2^1(\mathbb{R};H)}.$$

□

Corollary 5.2. *The operator $P^{(k)}$ is an isomorphism between the spaces $W_2^4(\mathbb{R};H)$ and $W_2^1(\mathbb{R};H)$.*

References

[1] Aliev AR. On the solvability of a fourth-order operator-differential equation with multiple characteristic. Ukrainian Mathematical Journal 2014; 66 (5): 781-791. https://doi.org/10.1007/s11253-014-0972-1

[2] Aliev AR, Elbably AL. On a class of operator-differential equations of the third order with multiple characteristics on the whole axis in the weighted space. Mathematica Slovaca 2015; 65 (3): 667-682. https://doi.org/10.1515/ms-2015-0046

[3] Aliev AR, Mohamed AS. On the well-posedness of a boundary value problem for a class of fourth-order operator-differential equations. Differential Equations 2012; 48 (4): 596-598. https://doi.org/10.1134/S0012266112040131

[4] Aliev AR, Muradova NL. Third-order operator-differential equations with discontinuous coefficients and operators in the boundary conditions. Electronic Journal of Differential Equations 2013; 2013 (219): 1-13.

[5] Aliev AR, Muradova NL. Conditions for the existence of smooth solutions for a class of fourth order operator-differential equations. Baku Mathematical Journal 2022; 1 (1): 3-14. https://doi.org/10.32010/j.bmj.2022.01

[6] Barenblatt GI, Zheltov IuP, Kochina IN. Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata]. Journal of Applied Mathematics and Mechanics 1960; 24 (5): 1286-1303. https://doi.org/10.1016/0021-8928(60)90107-6

[7] Favini A, Yakubov Y. Higher order ordinary differential-operator equations on the whole axis in UMD Banach spaces. Differential and Integral Equations 2008; 21 (5-6): 497-512.

[8] Gasymov AA. On solvability of a class of complicated characteristic operator-differential equations of fourth order. Transactions of National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences 2008; 28 (1): 49-54.

[9] Hille E, Phillips RS. Functional Analysis and Semigroups. Moscow, Russia: Izdatel'stvo Innostrannoj Literatury, 1962 (in Russian).

[10] Humbataliyev RZ. On the conditions of existence of smooth solutions for a class of operator-differential equations on the whole axis. Transactions of National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences 2003; 23 (1): 59-66.

[11] Kalantarov V, Tiryaki A. On the stability results for third order differential-operator equations. Turkish Journal of Mathematics 1997; 21 (2): 179-186.

[12] Lions JL, Magenes E. Non-Homogeneous Boundary Value Problems and Applications. Paris, France: Dunod, 1968; Moscow, Russia: Mir, 1971; Berlin, Germany: Springer, 1972.

[13] Mirzoev SS. Multiple completeness of root vectors of polynomial operator pencils corresponding to boundary-value problems on the semiaxis. Functional Analysis and Its Applications 1983; 17 (2): 151-153. https://doi.org/10.1007/BF01083147

[14] Mirzoyev SS, Aliyev AR. Initial boundary value problems for a class of third order operator-differential equations with variable coefficients. Transactions of National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences 2006; 26 (4): 153-164.

[15] Pilipchuk VN. On essentially nonlinear dynamics of arches and rings. Journal of Applied Mathematics and Mechanics 1982; 46 (3): 360-364. https://doi.org/10.1016/0021-8928(82)90112-5

[16] Teters GA. Complex Loading and Stability of Shells Made of Polymeric Materials. Riga, Latvia: Zinatne Press, 1969 (in Russian).