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Some remarks on parameterized inequalities involving conformable fractional operators

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Abstract: In this paper, we prove an identity for differentiable convex functions related to conformable fractional integrals. Moreover, some parameterized inequalities are established by using conformable fractional integrals. To be more precise, parameterized inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, previous and new results are presented by using special cases of the obtained theorems.

Key words: Hermite-Hadamard-type inequalities, Simpson-type inequalities, Bullen-type inequalities, Fractional conformable integrals, fractional calculus, convex function

1. Introduction

The inequalities, introduced by C. Hermite and J. Hadamard for convex functions are important topics in the literature. These inequalities state that if $\mathfrak{F}: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\sigma, \delta \in I$ with $\sigma < \delta$, then the following double inequality holds:

$$\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) \leq \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathfrak{F}(x)dx \leq \frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}. \tag{1.1}$$

Convex theory is an effective methodology in the case of considering the great issues that originate in a large number of areas of the pure and applied sciences. Moreover, convexity theory has a critical place in the advancement of the idea of inequality. There are several types of convexity in the literature.

Many mathematicians have studied midpoint-type and trapezoid-type inequalities for various types of convex functions, extensively. More precisely, an upper bound for which is the right-hand side of inequality (1.1) first was investigated by Dragomir and Agarwal in paper [8]. An upper bound for which is the left-hand side of inequality (1.1) first was considered by Kırmacı in paper [23]. Sarikaya et al. and Iqbal et al. proved several fractional trapezoid-type and midpoint-type inequalities involving convex functions in papers [27] and [17], respectively.

A great number of papers are focused on the results of Simpson-type inequalities by using convex functions. For instance, certain Simpson-type inequalities established by the case of s-convex differentiable

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functions in paper [3]. Moreover, the new variants of Simpson-type inequalities are considered by means of differentiable convex function in paper [26]. See Refs. [4] and [21] and the references therein for further information.

Bullen [6] established the inequality which is known as Bullen-type inequalities for the case of a convex function. In paper [10], the generalized Bullen-type inequalities are presented for the case of local fractional integrals on fractal sets. In [9], Du et al. used the generalized fractional integrals to obtain Bullen-type inequalities. Moreover, several generalizations of integral inequalities of Bullen-type inequalities were obtained for twice differentiable functions involving Riemann-Liouville fractional integrals in paper [7]. See [13, 14] for details and unexplained subjects.

Some midpoint-type inequalities are established by using the conformable fractional integrals in paper [12]. A new generalization of Hermite-Hadamard inequality [28] proved for fractional conformable integrals. In paper [24], conformable derivatives were improved by using the derivative's fundamental limit formulation. Several significant requirements that cannot be accomplished by the Riemann-Liouville and Caputo definitions are implemented by conformable derivatives. Abdelhakim [2] proves that the conformable approach in [24] cannot yield good results when compared to the Caputo definition for the case of specific functions. This flaw in the conformable definition is avoided by several extensions of the conformable approach [16, 30]. It can be referred to [1, 11, 18, 29] and the references cited therein for further information on recent developments to the above fractional integral inequalities.

The aim of this paper is to establish several parameterized inequalities with the help of conformable fractional integrals. The general structure of our paper contains four parts, including the introduction. The rest of the paper continues as follows: In Section 2, the fundamental definitions of convex functions, Riemann-Liouville integrals, and conformable integrals will be presented. In section 3, an equality will be given for the case of differentiable convex functions involving the conformable fractional integrals. With the help of this equality, we will prove several parameterized inequalities for the conformable fractional integrals. Moreover, we also give several corollaries and remarks. In section 4, summary and concluding remarks are given.

2. Preliminaries

Riemann-Liouville fractional integrals, conformable fractional integrals, and several types of fractional integrals have been investigated for these type of inequalities, which are mentioned above. The basic definitions of convex functions, Riemann-Liouville integrals, and conformable integrals, which are used throughout the paper, are given as follows:

Definition 2.1 (See [25]) *Let I denote an interval of real numbers. Then, a function $\mathfrak{F} : I \to \mathbb{R}$ is said to be convex, if the following inequality holds:*

$$\mathfrak{F}\left(\mu x+\left(1-\mu\right)y\right)\leq\mu\mathfrak{F}\left(x\right)+\left(1-\mu\right)\mathfrak{F}\left(y\right)$$

$\forall x, y \in I$ *and* $\forall \mu \in [0, 1]$.

Kilbas et al. [22] investigated fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 2.2 (See [22]) *Consider $\mathfrak{F} \in L_1[\sigma, \delta]$, $\sigma, \delta \in \mathbb{R}$ with $\sigma < \delta$. The Riemann-Liouville integrals*

$J_{\sigma+}^{\beta}\mathfrak{F}$ and $J_{\delta-}^{\beta}\mathfrak{F}$ of order $\beta>0$ are described by

$$J_{\sigma+}^{\beta}\mathfrak{F}(x)=\frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}(x-\mu)^{\beta-1}\mathfrak{F}(\mu)d\mu,\quad x>\sigma \tag{2.1}$$

and

$$J_{\delta-}^{\beta}\mathfrak{F}(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\delta}(\mu-x)^{\beta-1}\mathfrak{F}(\mu)d\mu,\quad x<\delta, \tag{2.2}$$

respectively. Here, Γ denotes the Gamma function defined by

$$\Gamma(\beta)=\int_{0}^{\infty}e^{-u}u^{\beta-1}du.$$

The fractional conformable integral operators were established by Jarad et al. in paper [19]. They also provided several characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 2.3 (See [19]) *Assume that $\beta>0$ and $\alpha\in(0,1]$. For $\mathfrak{F}\in L_1[\sigma,\delta]$, the generalized fractional Riemann-Liouville integrals ${}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}$ and ${}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}$ are defined by*

$${}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}(x)=\frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}\left(\frac{(x-\sigma)^{\alpha}-(\mu-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathfrak{F}(\mu)}{(\mu-\sigma)^{1-\alpha}}d\mu,\quad x>\sigma, \tag{2.3}$$

and

$${}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\delta}\left(\frac{(\delta-x)^{\alpha}-(\delta-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathfrak{F}(\mu)}{(\delta-\mu)^{1-\alpha}}d\mu,\quad x<\delta, \tag{2.4}$$

respectively.

If we choose $\alpha=1$ in equalities (2.3) and (2.4), then the fractional integrals in (2.3) and (2.4) become the Riemann-Liouville fractional integrals in (2.1) and (2.2), respectively.

3. Main results

Throughout the paper, we assume that $\alpha\in(0,1],\beta\in\mathbb{R}^{+}$, and $\lambda\in[0,1]$.

Lemma 3.1 *Suppose $\mathfrak{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable function on (σ,δ). If $\mathfrak{F}'$ belongs to $L[\sigma,\delta]$, then the following equality holds:*

$$\begin{aligned}&\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\\&-\left(\lambda\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+(1-\lambda)\left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right)\right)\\&=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_{0}^{1}\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right.\\&\left.-\int_{0}^{1}\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right],\end{aligned} \tag{3.1}$$

where $\Gamma(\beta)$ *is Euler Gamma function.*

Proof Let us consider

$$\begin{aligned}
&\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right.\\
&\left.-\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right]\\
&=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[I_1-I_2\right]. \qquad (3.2)
\end{aligned}$$

With the help of the integrating by parts and changing variables with $x=\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta$, we get

$$\begin{aligned}
I_1&=\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\\
&=\frac{2}{\delta-\sigma}\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\Bigg|_0^1\\
&\quad+\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathfrak{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\\
&=-\frac{2(1-\lambda)}{(\delta-\sigma)\alpha^{\beta}}\mathfrak{F}(\delta)-\frac{2\lambda}{(\delta-\sigma)\alpha^{\beta}}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\\
&\quad+\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_{\frac{\sigma+\delta}{2}}^{\delta}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-(\delta-x)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathfrak{F}(x)}{(\delta-x)^{1-\alpha}}dx\\
&=-\frac{2(1-\lambda)}{(\delta-\sigma)\alpha^{\beta}}\mathfrak{F}(\delta)-\frac{2\lambda}{(\delta-\sigma)\alpha^{\beta}}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\\
&\quad+\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\Gamma(\beta+1)\left[{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]. \qquad (3.3)
\end{aligned}$$

Similar to foregoing process, changing variables with $x=\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta$ and integrating on $[0,1]$, we obtain

$$I_2 = \int_0^1 \left(\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^\beta \right) \mathfrak{F}' \left(\frac{1+\mu}{2}\sigma + \frac{1-\mu}{2}\delta \right) d\mu$$

$$= -\frac{2}{\delta-\sigma} \left(\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^\beta \right) \mathfrak{F} \left(\frac{1+\mu}{2}\sigma + \frac{1-\mu}{2}\delta \right) \Big|_0^1$$

$$- \frac{2\beta}{\delta-\sigma} \int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^{\beta-1} (1-\mu)^{\alpha-1} \mathfrak{F} \left(\frac{1+\mu}{2}\sigma + \frac{1-\mu}{2}\delta \right) d\mu$$

$$= \frac{2(1-\lambda)}{(\delta-\sigma)\alpha^\beta} \mathfrak{F}(\sigma) + \frac{2\lambda}{(\delta-\sigma)\alpha^\beta} \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right)$$

$$- \left(\frac{2}{\delta-\sigma} \right)^{1+\alpha\beta} \beta \int_\sigma^{\frac{\sigma+\delta}{2}} \left(\frac{\left(\frac{\delta-\sigma}{2}\right)^\alpha - (x-\sigma)^\alpha}{\alpha} \right)^{\beta-1} \frac{\mathfrak{F}(x)}{(x-\sigma)^{1-\alpha}} dx$$

$$= \frac{2(1-\lambda)}{(\delta-\sigma)\alpha^\beta} \mathfrak{F}(\sigma) + \frac{2\lambda}{(\delta-\sigma)\alpha^\beta} \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right)$$

$$- \left(\frac{2}{\delta-\sigma} \right)^{1+\alpha\beta} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma} \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right) \right]. \tag{3.4}$$

Substituting equalities (3.3) and (3.4) in the equality (3.2), we can write

$$\frac{\alpha^\beta(\delta-\sigma)}{4}[I_1 - I_2] = \frac{\alpha^\beta 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma} \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta} \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right) \right]$$

$$- \left(\lambda \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right) + (1-\lambda) \left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2} \right) \right).$$

Hence, the proof of Lemma 3.1 is completed. □

Remark 3.2 *Let us consider* $\lambda = 0$ *in (3.1). Then, the equality (3.1) reduces to*

$$\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2} - \frac{\alpha^\beta 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma} \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta} \mathfrak{F}\left(\frac{\sigma+\delta}{2} \right) \right]$$

$$= \frac{\alpha^\beta(\delta-\sigma)}{4} \left[\int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^\beta \mathfrak{F}' \left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta \right) d\mu \right.$$

$$\left. - \int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^\beta \mathfrak{F}' \left(\frac{1+\mu}{2}\sigma + \frac{1-\mu}{2}\delta \right) d\mu \right],$$

which is given in paper [15, Lemma 2].

Remark 3.3 *If we assign $\lambda = 1$ in (3.1), then (3.1) equals to*

$$\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)$$

$$=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right.$$

$$\left.-\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right],$$

which is established by Hyder et al. in paper [15, Lemma 1].

Remark 3.4 *Assume that λ equals to $\frac{2}{3}$ in (3.1). Then, (3.1) reduces to*

$$\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]$$

$$-\frac{1}{6}\left(\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)+4\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right)$$

$$=\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\int_0^1\left(\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right.$$

$$\left.+\int_0^1\left(\frac{1}{2}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right)\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right],$$

which is given in paper [5].

Remark 3.5 *Suppose λ is equal to $\frac{1}{2}$. Then, the equality (3.1) becomes to*

$$\frac{1}{2}\left(\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right)$$

$$-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]$$

$$=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right)\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right.$$

$$\left.-\int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right)\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right].$$

This coincides with [20, Lemma 1].

Theorem 3.6 *Suppose that all assumptions of the Lemma 3.1 are valid. Suppose also that* $|\mathfrak{F}'|$ *is a convex function on* $[\sigma,\delta]$. *Then, it follows*

$$\begin{aligned}&\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right.\\&\left.-\left(\lambda\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+(1-\lambda)\left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right)\right)\right|\\&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1(\alpha,\beta,\lambda)\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right).\end{aligned}\tag{3.5}$$

Here,

$$\begin{aligned}A_1(\alpha,\beta,\lambda)&=\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu\\&=\frac{1}{\alpha^{\beta}}\left[\lambda(2C-1)+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\lambda^{\frac{1}{\beta}}\right)\right)\right]\end{aligned}$$

with $C=1-\left(1-\lambda^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$, *the functions* $\mathfrak{B}(\cdot,\cdot)$ *and* $\mathscr{B}(\cdot,\cdot,\cdot)$ *are the Beta function and the incomplete Beta function defined as*

$$\begin{cases}\mathfrak{B}(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}du,\\\mathscr{B}(x,y,r)=\int_0^r u^{x-1}(1-u)^{y-1}du\end{cases}$$

for $x,y>0$ *and* $r\in[0,1]$.

Proof By Lemma 3.1 and convexity of $|\mathfrak{F}'|$, we obtain

$$\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right.$$
$$\left.-\left(\lambda\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+(1-\lambda)\left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right)\right)\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\left|\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right|\right.$$
$$\left.+\left|\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right|\right]$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|d\mu\right.$$
$$\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|d\mu\right]$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\frac{1-\mu}{2}|\mathfrak{F}'(\sigma)|+\frac{1+\mu}{2}|\mathfrak{F}'(\delta)|\right]d\mu\right.$$
$$\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\frac{1+\mu}{2}|\mathfrak{F}'(\sigma)|+\frac{1-\mu}{2}|\mathfrak{F}'(\delta)|\right]d\mu\right]$$
$$=\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1(\alpha,\beta,\lambda)\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right).$$

This ends the proof of Theorem 3.6. □

Remark 3.7 *If we select $\lambda=0$ in (3.5), then (3.5) reduces to the following trapezoid-type inequality*

$$\left|\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|$$
$$\leq\frac{\delta-\sigma}{4\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right),$$

which is given in [15, Theorem 5].

Remark 3.8 *If we assign $\lambda=1$ in (3.5), then we obtain the following midpoint-type inequality*

$$\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right|$$
$$\leq\frac{\delta-\sigma}{4}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right]\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right),$$

which is established by Hyder et al. in paper [15, Theorem 2].

Remark 3.9 *Consider $\lambda = \frac{2}{3}$ in the inequality (3.5). Then, we get the following Simpson-type inequality*

$$\left| \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma} \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta} \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) \right] - \frac{1}{6}\left[\mathfrak{F}(\sigma) + 4\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) + \mathfrak{F}(\delta) \right] \right|$$
$$\leq \frac{\delta-\sigma}{2}\left[\frac{2D-1}{3} + \frac{1}{\alpha}\left(\frac{1}{2}\mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) - \mathscr{B}\left(\beta+1, \frac{1}{\alpha}, \left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right) \right) \right] \left(|\mathfrak{F}'(\sigma)| + |\mathfrak{F}'(\delta)| \right),$$

which is presented in [5]. Here, $D = 1 - \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$.

Remark 3.10 *Let us note $\lambda = \frac{1}{2}$ in (3.1). Then, we have the following Bullen-type inequality*

$$\left| \frac{1}{2}\left[\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) + \frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2} \right] - \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma} \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta} \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) \right] \right|$$
$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left\{ \frac{1}{\alpha^{\beta}}\left[\frac{1}{2} - \left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} \right] + \frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) - 2\mathscr{B}\left(\beta+1, \frac{1}{\alpha}, \left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right) \right] \right\} \left(|\mathfrak{F}'(\sigma)| + |\mathfrak{F}'(\delta)| \right).$$

This coincides with [20, Theorem 1].

Theorem 3.11 *Let us note that all assumptions of the Lemma 3.1 are valid. Let us also note that $|\mathfrak{F}'|^q$ is a convex function on $[\sigma, \delta]$ where $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Then we obtain*

$$\left| \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma} \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta} \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) \right] - \left(\lambda\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) + (1-\lambda)\left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right) \right) \right|$$
$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda) \left\{ \left[\frac{|\mathfrak{F}'(\sigma)|^q + 3|\mathfrak{F}'(\delta)|^q}{4} \right]^{\frac{1}{q}} + \left[\frac{3|\mathfrak{F}'(\sigma)|^q + |\mathfrak{F}'(\delta)|^q}{4} \right]^{\frac{1}{q}} \right\}$$
$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} \left[4A_2(\alpha,\beta,p,\lambda) \right]^{\frac{1}{p}} \left(|\mathfrak{F}'(\sigma)| + |\mathfrak{F}'(\delta)| \right). \tag{3.6}$$

Here,

$$A_2(\alpha,\beta,p,\lambda) = \int_0^1 \left| \frac{\lambda}{\alpha^{\beta}} - \left(\frac{1-(1-\mu)^{\alpha}}{\alpha} \right)^{\beta} \right|^p d\mu.$$

Proof From Lemma 3.1, we have

$$\begin{aligned}
&\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right.\\
&\qquad\left.-\left(\lambda\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+(1-\lambda)\left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right)\right)\right|\\
&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|d\mu\right.\\
&\qquad\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|d\mu\right]. \qquad (3.7)
\end{aligned}$$

Now, we consider the integrals on the right side of (3.7). Using the convexity of $|\mathfrak{F}'|^q$ and well-known Hölder inequality, we get

$$\begin{aligned}
&\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|d\mu\\
&\leq\left(\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&\leq A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left(\int_0^1\left(\frac{1-\mu}{2}|\mathfrak{F}'(\sigma)|^q+\frac{1+\mu}{2}|\mathfrak{F}'(\delta)|^q\right)d\mu\right)^{\frac{1}{q}}\\
&=A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left[\frac{|\mathfrak{F}'(\sigma)|^q+3|\mathfrak{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}} \qquad (3.8)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|d\mu\\
&\leq\left(\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&\leq A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left(\int_0^1\left(\frac{1+\mu}{2}|\mathfrak{F}'(\sigma)|^q+\frac{1-\mu}{2}|\mathfrak{F}'(\delta)|^q\right)d\mu\right)^{\frac{1}{q}}\\
&=A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left[\frac{3|\mathfrak{F}'(\sigma)|^q+|\mathfrak{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}. \qquad (3.9)
\end{aligned}$$

If we consider (3.8) and (3.9) in (3.7), then we have

$$\begin{aligned}&\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right.\\&\quad\left.-\left(\lambda\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+(1-\lambda)\left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right)\right)\right|\\&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left\{\left[\frac{|\mathfrak{F}'(\sigma)|^{q}+3|\mathfrak{F}'(\delta)|^{q}}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathfrak{F}'(\sigma)|^{q}+|\mathfrak{F}'(\delta)|^{q}}{4}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

The remaining part of the Theorem 3.11 can be easily seen by setting $\rho_1 = |\mathfrak{F}'(\sigma)|^q$, $\rho_2 = |3\mathfrak{F}'(\sigma)|^q$, $\tau_1 = 3|\mathfrak{F}'(\delta)|^q$, $\tau_2 = |\mathfrak{F}'(\delta)|^q$ and applying the well-known inequality

$$\sum_{k=1}^{n}(\rho_k+\tau_k)^s\leq\sum_{k=1}^{n}\rho_k^s+\sum_{k=1}^{n}\tau_k^s$$

with $0 \leq s < 1$. This finishes the proof of Theorem 3.11. □

Corollary 3.12 *If it is chosen $\lambda = 1$ in (3.6), then we have the following midpoint-type inequality*

$$\begin{aligned}&\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[4A_2(\alpha,\beta,p,1)\right]^{\frac{1}{p}}\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right),\end{aligned}$$

where

$$A_2(\alpha,\beta,p,1)=\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)^{p}d\mu.$$

Remark 3.13 *Let us note $\lambda = 0$ in (3.6). Then we get the following trapezoid-type inequalities*

$$\begin{aligned}&\left|\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\&\leq\frac{\delta-\sigma}{4}\left(\frac{1}{\alpha}\mathfrak{B}\left(\beta p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left\{\left[\frac{|\mathfrak{F}'(\sigma)|^{q}+3|\mathfrak{F}'(\delta)|^{q}}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathfrak{F}'(\sigma)|^{q}+|\mathfrak{F}'(\delta)|^{q}}{4}\right]^{\frac{1}{q}}\right\}\\&\leq\frac{\delta-\sigma}{4}\left(\frac{4}{\alpha}\mathfrak{B}\left(\beta p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right),\end{aligned}$$

which is established in [15, Theorem 6].

Remark 3.14 *Consider that λ equals to $\frac{2}{3}$ in the inequalities (3.6). Then, we have the following Simpson-type* *inequalities*

$$\begin{aligned}
&\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right.\\
&\qquad\left.-\frac{1}{6}\left[\mathfrak{F}(\sigma)+4\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+\mathfrak{F}(\delta)\right]\right|\\
&\leq\frac{\delta-\sigma}{2}\left(\int_0^1\left|\frac{(1-(1-\mu)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|^p d\mu\right)^{\frac{1}{p}}\\
&\qquad\times\left\{\left[\frac{|\mathfrak{F}'(\sigma)|^q+3|\mathfrak{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathfrak{F}'(\sigma)|^q+|\mathfrak{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}\\
&\leq\frac{\delta-\sigma}{2^{\frac{2}{q}-1}}\left(\int_0^1\left|\frac{(1-(1-\mu)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|^p d\mu\right)^{\frac{1}{p}}\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right),
\end{aligned}$$

which is given by Budak et al. in paper [5].

Remark 3.15 *Assume $\lambda=\frac{1}{2}$ in the inequalities (3.6). Then, we get the following Bullen-type double inequality*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right]\right.\\
&\qquad\left.-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_2^{\frac{1}{p}}\left(\alpha,\beta,p,\frac{1}{2}\right)\left\{\left[\frac{|\mathfrak{F}'(\sigma)|^q+3|\mathfrak{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathfrak{F}'(\sigma)|^q+|\mathfrak{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}\\
&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left(4\int_0^1\left|\frac{1}{2\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left(|\mathfrak{F}'(\sigma)|+|\mathfrak{F}'(\delta)|\right),
\end{aligned}$$

which is presented in [20, Theorem 2].

Theorem 3.16 *Let us consider that all assumptions of the Lemma 3.1 hold. If $|\mathfrak{F}'|^q$ is convex on $[\sigma,\delta]$ where $q\geq 1$, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right.\\
&\qquad\left.-\left(\lambda\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+(1-\lambda)\left(\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right)\right)\right|\\
&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left\{\left(A_3(\alpha,\beta,\lambda)|\mathfrak{F}'(\sigma)|^q+A_4(\alpha,\beta,\lambda)|\mathfrak{F}'(\delta)|^q\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(A_4(\alpha,\beta,\lambda)|\mathfrak{F}'(\sigma)|^q+A_3(\alpha,\beta,\lambda)|\mathfrak{F}'(\delta)|^q\right)^{\frac{1}{q}}\right\}. \qquad (3.10)
\end{aligned}$$

Here, A_1 is defined in Theorem 3.6 and

$$A_3(\alpha,\beta,\lambda) = \int_0^1 \frac{1-\mu}{2}\left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| d\mu$$

$$= \frac{1}{2}\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| d\mu - \int_0^1 \frac{\mu}{2}\left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| d\mu$$

$$= \frac{1}{2\alpha^\beta}\left\{\lambda\left(2C - C^2 - \frac{1}{2}\right) + \frac{1}{\alpha}\left[-2\mathscr{B}\left(\beta+1, \frac{2}{\alpha}, \lambda^{\frac{1}{\beta}}\right) + \mathfrak{B}\left(\beta+1, \frac{2}{\alpha}\right)\right]\right\},$$

$$A_4(\alpha,\beta,\lambda) = \int_0^1 \frac{1+\mu}{2}\left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| d\mu$$

$$= \frac{1}{2}\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| d\mu + \int_0^1 \frac{\mu}{2}\left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| d\mu$$

$$= \frac{1}{2\alpha^\beta}\left\{\lambda\left(2C + C^2 - \frac{3}{2}\right) + \frac{1}{\alpha}\left[-4\mathscr{B}\left(\beta+1, \frac{1}{\alpha}, \lambda^{\frac{1}{\beta}}\right) + 2\mathscr{B}\left(\beta+1, \frac{2}{\alpha}, \lambda^{\frac{1}{\beta}}\right)\right.\right.$$
$$\left.\left. +2\mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) - \mathfrak{B}\left(\beta+1, \frac{2}{\alpha}\right)\right]\right\}.$$

Proof If we consider the convexity of $|\mathfrak{F}'|^q$ and power mean inequality, then we get

$$\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| \left|\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta\right)\right| d\mu$$

$$\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| d\mu\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| \left|\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}$$

$$= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| \left|\mathfrak{F}'\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}$$

$$\leq A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| \left[\frac{1-\mu}{2}|\mathfrak{F}'(\sigma)|^q + \frac{1+\mu}{2}|\mathfrak{F}'(\delta)|^q\right] d\mu\right)^{\frac{1}{q}}$$

$$= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(A_3(\alpha,\beta,\lambda)|\mathfrak{F}'(\sigma)|^q + A_4(\alpha,\beta,\lambda)|\mathfrak{F}'(\delta)|^q\right)^{\frac{1}{q}} \tag{3.11}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|d\mu \\
&\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|d\mu\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathfrak{F}'\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left[\frac{1+\mu}{2}\left|\mathfrak{F}'(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathfrak{F}'(\delta)\right|^q\right]d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(A_4(\alpha,\beta,\lambda)\left|\mathfrak{F}'(\sigma)\right|^q+A_3(\alpha,\beta,\lambda)\left|\mathfrak{F}'(\delta)\right|^q\right)^{\frac{1}{q}}. \qquad (3.12)
\end{aligned}$$

If we consider (3.11) and (3.12) in (3.7), then the proof of Theorem 3.16 is completed. □

Remark 3.17 *If we consider (3.10) as $\lambda = 0$, then we obtain the following trapezoid-type inequality*

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}-\frac{\alpha^\beta 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\leq \frac{\delta-\sigma}{4\alpha}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right]^{1-\frac{1}{q}} \\
&\quad\times\left\{\left[\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\left|\mathfrak{F}'(\sigma)\right|^q+\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right)\left|\mathfrak{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right)\left|\mathfrak{F}'(\sigma)\right|^q+\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\left|\mathfrak{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

which is presented in [15, Theorem 7].

Remark 3.18 *If we choose* $\lambda = 1$ *in (3.10), then we have the following midpoint-type inequality*

$$\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right|$$

$$\leq\frac{\delta-\sigma}{4}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right]^{1-\frac{1}{q}}\left\{\left[\left(\frac{1}{4}-\frac{1}{2\alpha}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right)|\mathfrak{F}'(\sigma)|^{q}\right.\right.$$

$$\left.+\left(\frac{3}{4}-\frac{1}{2\alpha}\left(2\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right)\right)|\mathfrak{F}'(\delta)|^{q}\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{3}{4}-\frac{1}{2\alpha}\left(2\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right)\right)|\mathfrak{F}'(\sigma)|^{q}\right.$$

$$\left.\left.+\left(\frac{1}{4}-\frac{1}{2\alpha}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right)|\mathfrak{F}'(\delta)|^{q}\right]^{\frac{1}{q}}\right\},$$

which is established by Hyder et al. in paper [15, Theorem 4].

Remark 3.19 *Suppose* $\lambda = \frac{2}{3}$ *in the inequality (3.10). Then, the following Simpson-type inequality holds:*

$$\left|\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right.$$

$$\left.-\frac{1}{6}\left[\mathfrak{F}(\sigma)+4\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+\mathfrak{F}(\delta)\right]\right|$$

$$\leq\frac{\delta-\sigma}{2^{1+\frac{1}{q}}}\left[\frac{2D-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right]^{1-\frac{1}{q}}$$

$$\times\left\{\left(A_3\left(\alpha,\beta,\frac{2}{3}\right)|\mathfrak{F}'(\sigma)|^{q}+A_4\left(\alpha,\beta,\frac{2}{3}\right)|\mathfrak{F}'(\delta)|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(A_4\left(\alpha,\beta,\frac{2}{3}\right)|\mathfrak{F}'(\sigma)|^{q}+A_3\left(\alpha,\beta,\frac{2}{3}\right)|\mathfrak{F}'(\delta)|^{q}\right)^{\frac{1}{q}}\right\},$$

where

$$A_3\left(\alpha,\beta,\frac{2}{3}\right)=\frac{4D-2D^2-1}{6}+\frac{1}{\alpha}\left(\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)$$

and

$$A_4\left(\alpha,\beta,\frac{2}{3}\right)=\frac{4D+2D^2-3}{6}+\frac{1}{\alpha}\left(\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right.$$

$$\left.-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)+\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right),$$

which is given in [5]. Here, $D=1-\left(1-\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$.

Remark 3.20 *For $\lambda=\frac{1}{2}$, the inequality (3.10) reduces to the following Bullen-type inequality*

$$\begin{aligned}&\Bigg|\frac{1}{2}\left[\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}\right]\\&-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+{}_{-}^{\beta}\mathcal{J}_{\delta}^{\alpha}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right]\Bigg|\\&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_{1}^{1-\frac{1}{q}}\left(\alpha,\beta,\frac{1}{2}\right)\left\{\left(A_{3}\left(\alpha,\beta,\frac{1}{2}\right)|\mathfrak{F}'(\sigma)|^{q}+A_{4}\left(\alpha,\beta,\frac{1}{2}\right)|\mathfrak{F}'(\delta)|^{q}\right)^{\frac{1}{q}}\right.\\&\left.+\left(A_{4}\left(\alpha,\beta,\frac{1}{2}\right)|\mathfrak{F}'(\sigma)|^{q}+A_{3}\left(\alpha,\beta,\frac{1}{2}\right)|\mathfrak{F}'(\delta)|^{q}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where

$$\begin{aligned}A_{1}\left(\alpha,\beta,\frac{1}{2}\right)&=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]\\&+\frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],\end{aligned}$$

$$\begin{aligned}A_{3}\left(\alpha,\beta,\frac{1}{2}\right)&=\frac{1}{2\alpha^{\beta}}\left[\frac{3}{4}-\frac{1}{2}\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)^{2}-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right]\\&+\frac{1}{\alpha^{\beta+1}}\left[\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)\right],\end{aligned}$$

and

$$\begin{aligned}A_{4}\left(\alpha,\beta,\frac{1}{2}\right)&=\frac{1}{2\alpha^{\beta}}\left[\frac{1}{4}+\frac{1}{2}\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)^{2}-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right]\\&+\frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right.\\&\left.+\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)\right].\end{aligned}$$

This coincides with [20, Theorem 3].

4. Summary & concluding remarks

In this paper, we established an identity for the case of convex differentiable functions including conformable fractional integrals. By using this identity, we proved parameterized inequalities for the case of conformable fractional integrals. Moreover, we offered previous and new results by using special cases of the obtained theorems.

In future studies, the ideas and strategies for our results related to parameterized inequalities by conformable fractional integrals may open new ways for mathematicians in this field. In addition to this, one can apply these resulting inequalities to different types of fractional integrals. Furthermore, one can obtain likewise parameterized inequalities by conformable fractional integrals for convex functions by using quantum calculus.

Availability of data and material

Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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