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On max-min solutions of fuzzy games with nonlinear memberships functions

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Abstract: In this paper, we deal with two-person zero-sum games with fuzzy goals. We investigated the cases where the membership functions of the players are nonlinear. We examined how the solutions should be if the membership functions of players were exponential functions. In case players' membership functions are exponential, we developed a new method for the maximin solution according to a degree of attainment of the fuzzy goals. An application was made to show the effectiveness of the method.

Key words: Fuzzy goal, zero-sum games, maximin solution

1. Introduction

Game theory has been used as a solution of decision making problems [9, 10, 15]. With the development of the fuzzy theory [8, 18, 20], uncertain events were indicated by fuzzy sets.

Butnariu was the first to study game theory in a fuzzy environment [3]. Using fuzzy games, Buckley studied behavior of decision makers [2].

Campos examined maximin problems [4]. Later extended by Nishizaki for the multiobjective situation [7, 13, 14, 17]. In the literature, there are many models of the two-person zero sum fuzzy games with fuzzy payoffs [1, 5, 6, 11, 12].

This paper is related to games with fuzzy goals. We investigated the cases where the membership functions of the players are not linear. We examined how the solutions should be if the membership functions of players were exponential functions. In case players' membership functions are exponential, we developed a new method for computing the maximin solution of games with fuzzy goals.

2. Games with exponential membership function

Let our payment matrix be A:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \tag{1}$$

where we assume that pure strategies correspond to the rows and the columns of the matrix A for Player

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1 and Player 2, respectively. When Player 1 chooses a pure strategy $i \in I = \{1, ..., m\}$ and Player 2 chooses a pure strategy $j \in J = \{1, ..., n\}$, Player 1 receives the payoff a_{ij} from Player 2.

$X = \{x \in R^m \mid x_1 + x_2 + ... + x_m = 1,\ x_i \geq 0, i = 1, ..., m\ \}$ is a mixed strategy of Player 1,

$Y = \{y \in R^n \mid y_1 + y_2 + ... + y_n = 1,\ y_j \geq 0,\ j = 1, ..., n\}$ is a mixed strategy of Player 2.

Suppose that a player has a fuzzy goal, which expresses the player's degree of satisfaction for a payoff.

Definition 2.1 *: Let a domain of the payoff for Player 1 be $D \in R$. Then the fuzzy goal $\mu_{\widetilde{G}}$ according to the payoff for Player 1 is a fuzzy set on the set D characterized by a linear or an exponential membership function.*

For a linear membership function:

$$\mu_{\widetilde{G}}: \quad D \to [0,1]$$

$$p \to \mu_{\widetilde{G}}(p) = \left\{ \begin{array}{ccc} 0 & if & p \leq \underline{a} \\ 1 - \frac{\overline{a}-p}{\overline{a}-\underline{a}} & if & \underline{a} \leq p \leq \overline{a} \\ 1 & if & \overline{a} \leq p \end{array} \right\} \tag{2}$$

This membership function is given in Figure 1 and for an exponential membership function[19]:

$$\mu_{\widetilde{G}}: \quad D \to [0,1]$$

$$p \to \mu_{\widetilde{G}}(p) = \left\{ \begin{array}{ccc} 1 & if & p \leq \underline{a} \\ \frac{e^{-s\left(\frac{p-a}{\overline{a}-\underline{a}}\right)} - e^{-s}}{1-e^{-s}} & if & \underline{a} \leq p \leq \overline{a} \\ 0 & if & \overline{a} \leq p \end{array} \right\} \tag{3}$$

This membership function is given in Figure 2. We assume the following: Player 1 specifies the finite value $\underline{a}$ of the payoff for which the degree of satisfaction is 0, the finite value $\overline{a}$ of the payoff for which the degree of satisfaction is 1.

$\mu_{\widetilde{G}}(p) = 0$ for the value p smaller than $\underline{a}$, $\mu_{\widetilde{G}}(p) = 1$ for the value p larger than $\overline{a}$.

A membership function value for a fuzzy goal can be interpreted as the degree of attainment of the fuzzy goal for the payoff. When a player has two different payoffs, player chooses the payoff with the larger membership function.

Definition 2.2 *: Player 1's maximin value is:*

$$\max_{x \in X} \min_{y \in Y} \mu(x, y) \tag{4}$$

such a strategy x is called the maximin solution according to a degree of attainment of the fuzzy goal. Similarly, Player 2's minimax value is:

$$\min_{y \in Y} \max_{x \in X} \overline{\mu}(x, y) \tag{5}$$

such a strategy y is called the minimax solution, $\overline{\mu}$ is a membership function of Player 2.

3. Computational method

This section is devoted to developing the method for computing the maximin solution. Let (x, y) be any strategy pair and xAy be an expected payoff. Then $\mu(xAy)$ is membership function of the fuzzy goal.

If the membership function is a linear function:

$$\mu(xAy) = \left\{ \begin{array}{ccc} 0 & if & xAy \leq \underline{a} \\ 1 - \frac{\overline{a} - xAy}{\overline{a} - \underline{a}} & if & \underline{a} \leq xAy \leq \overline{a} \\ 1 & if & \overline{a} \leq xAy \end{array} \right\} \tag{6}$$

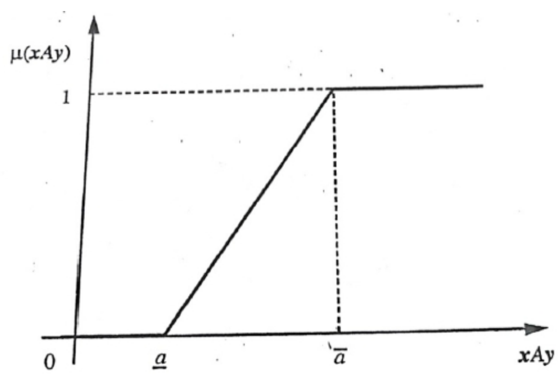

Figure 1. Linear membership function.

If the membership function is an exponential function:

$$\mu(xAy) = \left\{ \begin{array}{ccc} 1 & if & xAy \leq \underline{a} \\ \frac{e^{-s\left(\frac{xAy - \underline{a}}{\overline{a} - \underline{a}}\right)} - e^{-s}}{1 - e^{-s}} & if & \underline{a} \leq xAy \leq \overline{a} \\ 0 & if & \overline{a} \leq xAy \end{array} \right\} \tag{7}$$

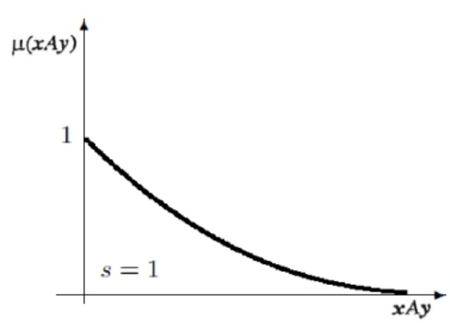

Figure 2. Exponential membership function

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$\underline{a}$ is the payoff to Player 1 and $\overline{a}$ is the payoff giving the best to Player 1. The parameter according to the worst degree of satisfaction of Player 1 is:

$$\underline{a} = \min_{x\in X}\min_{y\in Y} xAy = \min_{i\in I}\min_{j\in J} a_{ij} \tag{8}$$

the parameter according to the best degree of satisfaction of Player 1 is:

$$\overline{a} = \max_{x\in X}\max_{y\in Y} xAy = \max_{i\in I}\max_{j\in J} a_{ij} \tag{9}$$

Theorem 3.1 *Let membership function be an exponential function, Player 1's maximin solution is equal to an optimal solution of (10) mathematical programming problem:*

$$\begin{array}{l} \max imize \quad \lambda \\ subject\ to \\ \qquad \widehat{a}_{1j}x_1 + ... + \widehat{a}_{mj}x_m + c \geq \lambda, \quad j = 1, ..., n \\ \qquad x_1 + ... + x_m = 1 \\ \qquad x_i \geq 0,\ i = 1, ..., m \end{array} \tag{10}$$

where

$$\widehat{a}_{ij} = \frac{e^{-\left(\frac{xAy-a}{\overline{a}-\underline{a}}\right)}}{1-e^{-1}} \quad and \quad c = -\frac{e^{-1}}{1-e^{-1}}$$

Proof Problem (4) can be transformed into:

$$\begin{aligned} \max_{x\in X}\min_{y\in Y} \mu(x,y) &= \max_{x\in X}\min_{y\in Y}\left(\frac{e^{-\left(\frac{xAy-a}{\overline{a}-\underline{a}}\right)}-e^{-1}}{1-e^{-1}}\right) \\ &= \max_{x\in X}\min_{y\in Y}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\widehat{a}_{ij}x_iy_j + c\right) \\ &= \max_{x\in X}\min_{y\in Y}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\widehat{a}_{ij}x_iy_j + \sum_{j=1}^{n}y_jc\right) \\ &= \max_{x\in X}\min_{y\in Y}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\widehat{a}_{ij}x_i + c\right)y_j \\ &= \max_{x\in X}\min_{j\in J}\left(\sum_{i=1}^{m}\widehat{a}_{ij}x_i + c\right) \end{aligned} \tag{11}$$

The strategy x^* satisfying (11) is optimal solution of the mathematical problem (10). □

It can be demonstrated similarly for the linear membership function.

Now, we consider Player 2's minimax solution.

If $\overline{\mu}(xAy)$ is a linear function:

$$\overline{\mu}(xAy) = \left\{ \begin{array}{ccc} 1 & if & xAy \leq \underline{a} \\ 1 - \frac{xAy - \underline{a}}{\overline{a} - \underline{a}} & if & \underline{a} \leq xAy \leq \overline{a} \\ 0 & if & \overline{a} \leq xAy \end{array} \right\}, \tag{12}$$

the membership function for $\overline{\mu}(xAy)$ is an exponential function:

$$\overline{\mu}(xAy) = \left\{ \begin{array}{ccc} 0 & if & xAy \leq \underline{a} \\ \frac{e^{-s\left(\frac{\overline{a} - xAy}{\overline{a} - \underline{a}}\right)} - e^{-s}}{1 - e^{-s}} & if & \underline{a} \leq xAy \leq \overline{a} \\ 1 & if & \overline{a} \leq xAy \end{array} \right\}, \tag{13}$$

where the parameter $\overline{a}$ is the payoff giving the worst to Player 2 and the parameter $\underline{a}$ is the payoff giving the best to Player 2.

The parameter $\underline{a}$ is:

$$\underline{a} = \min_{x \in X} \min_{y \in Y} xAy = \min_{i \in I} \min_{j \in J} a_{ij} \tag{14}$$

and the parameter $\overline{a}$ is:

$$\overline{a} = \max_{x \in X} \max_{y \in Y} xAy = \max_{i \in I} \max_{j \in J} a_{ij} \tag{15}$$

Theorem 3.2 *If membership function is an exponential function, Player 2's minimax solution is equal to an optimal solution of (16) mathematical problem:*

$$\begin{array}{l} \min imize \quad \lambda \\ subject\ to \\ \qquad \widehat{a}_{i1} y_1 + ... + \widehat{a}_{in} y_n + c \leq \lambda, \ \ i = 1, ..., m \\ \qquad y_1 + ... + y_n = 1 \\ \qquad y_j \geq 0, \ \ j = 1, ..., n. \end{array} \tag{16}$$

Proof

$$\begin{aligned}
\max_{y\in Y}\min_{x\in X}\overline{\mu}(x,y) &= \max_{y\in Y}\min_{x\in X}\left(\frac{e^{-\left(\frac{\overline{a}-xAy}{\overline{a}-\underline{a}}\right)}-e^{-1}}{1-e^{-1}}\right)\\
&= \max_{y\in Y}\min_{x\in X}\left(-\sum_{i=1}^{m}\sum_{j=1}^{n}\widehat{a}_{ij}x_iy_j+1-c\right)\\
&= \max_{y\in Y}\min_{x\in X}\left(-\sum_{i=1}^{m}\sum_{j=1}^{n}\widehat{a}_{ij}x_iy_j+1-\sum_{i=1}^{m}x_ic\right) \qquad (17)\\
&= \max_{y\in Y}\min_{x\in X}\sum_{i=1}^{m}\left(-\sum_{j=1}^{n}\widehat{a}_{ij}y_j+1-c\right)x_i\\
&= \max_{y\in Y}\min_{i\in I}\left(-\sum_{j=1}^{n}\widehat{a}_{ij}y_j+1-c\right)
\end{aligned}$$

The strategy y^* satisfying (17) is optimal solution of the mathematical problem (18):

$$\begin{aligned}
&\max imize \;\; \lambda\\
&subject\ to\\
&-\widehat{a}_{i1}y_1-...-\widehat{a}_{in}y_n+1-c\geq\lambda,\;\; i=1,...,m\\
&y_1+...+y_n=1 \qquad (18)\\
&y_j\geq 0,\;\; j=1,...,n
\end{aligned}$$

the problem is equivalent to the mathematical problem (16). □

Example 3.3 *We assume that each player has three pure strategies and s=1. The payoff matrix is:*

$$A=\begin{bmatrix} -3 & 7 & 2\\ 0 & -2 & 0\\ 3 & -1 & -6 \end{bmatrix}$$

from (7) and (10):

$$\begin{aligned}
&\max imize\ \lambda\\
&subject\ to\\
&\tfrac{e^{-\left(\frac{3}{13}\right)}}{1-e^{-1}}x_1+\tfrac{e^{-\left(\frac{6}{13}\right)}}{1-e^{-1}}x_2+\tfrac{e^{-\left(\frac{9}{13}\right)}}{1-e^{-1}}x_3-\tfrac{e^{-(1)}}{1-e^{-1}} \geq \lambda\\
&\tfrac{e^{-(1)}}{1-e^{-1}}x_1+\tfrac{e^{-\left(\frac{4}{13}\right)}}{1-e^{-1}}x_2+\tfrac{e^{-\left(\frac{5}{13}\right)}}{1-e^{-1}}x_3-\tfrac{e^{-(1)}}{1-e^{-1}} \geq \lambda\\
&\tfrac{e^{-\left(\frac{8}{13}\right)}}{1-e^{-1}}x_1+\tfrac{e^{-\left(\frac{6}{13}\right)}}{1-e^{-1}}x_2+\tfrac{e^{-(0)}}{1-e^{-1}}x_3-\tfrac{e^{-(1)}}{1-e^{-1}} \geq \lambda\\
&x_1+x_2+x_3 = 1\\
&x_1,x_2,x_3 \geq 0.
\end{aligned}$$

An optimal solution of this problem is:

$$x_1 = 0.2128, \quad x_2 = 0.6792, \quad x_3 = 0.1080 \text{ and } \lambda = 0.4481$$

If we solve the same example with the linear membership function as (6), an optimal solution is:

$$x_1 = 0.1837, \quad x_2 = 0.7143, \quad x_3 = 0.1020 \text{ and } \lambda = 0.4427$$

By identifying the membership function as (13), for the minimax strategy of Player 2, the mathematical problem (16) is formulated:

$$\begin{array}{ll} \min imize \ \lambda & \\ subject\ to & \\ \frac{e^{-\left(\frac{3}{13}\right)}}{1-e^{-1}} y_1 + \frac{e^{-(1)}}{1-e^{-1}} y_2 + \frac{e^{-\left(\frac{8}{13}\right)}}{1-e^{-1}} y_3 - \frac{e^{-(1)}}{1-e^{-1}} & \leq \lambda \\ \frac{e^{-\left(\frac{6}{13}\right)}}{1-e^{-1}} y_1 + \frac{e^{-\left(\frac{4}{13}\right)}}{1-e^{-1}} y_2 + \frac{e^{-\left(\frac{6}{13}\right)}}{1-e^{-1}} y_3 - \frac{e^{-(1)}}{1-e^{-1}} & \leq \lambda \\ \frac{e^{-\left(\frac{9}{13}\right)}}{1-e^{-1}} y_1 + \frac{e^{-\left(\frac{5}{13}\right)}}{1-e^{-1}} y_2 + y_3 - \frac{e^{-(1)}}{1-e^{-1}} & \leq \lambda \\ y_1 + y_2 + y_3 & = 1 \\ y_1, y_2, y_3 & \geq 0. \end{array}$$

An optimal solution of this problem is for minimax strategy of Player 2:

$$y_1 = 0.5716, \quad y_2 = 0.1984, \quad y_3 = 0.2300 \text{ and } \lambda = 0.4481.$$

If we solve the same example with the linear membership function as (12), an optimal solution is:

$$y_1 = 0.5714, \quad y_2 = 0.1225, \quad y_3 = 0.3061 \text{ and } \lambda = 0.4427.$$

4. Conclusion

In this paper, we have considered two-person zero sum games with fuzzy goals. We proved players who are playing a zero sum game with fuzzy goals. We investigated the cases where the membership functions of the players are not linear. We examined how the solutions should be if the membership functions of players were exponential functions. In case players' membership functions are exponential, we developed a new method for the maximin solution according to a degree of attainment of the fuzzy goals. An application was made to show the effectiveness of the method.

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