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Bi-periodic incomplete Horadam numbers

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Abstract: In this paper, we introduce bi-periodic incomplete Horadam numbers as a natural generalization of incomplete Horadam numbers. We study their basic properties and provide recurrence relations. In particular, we derive the generating function of these numbers.

Key words: Fibonacci sequence, Horadam sequence, bi-periodic Horadam sequence, bi-periodic incomplete Horadam sequence

1. Introduction

The Fibonacci sequence is one of the most famous and most studied sequences in mathematics. Its $n$th term $F_n$, also called as the $n$th Fibonacci number, is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ where $F_0 = 0$ and $F_1 = 1$ are the initial values. This recurrence relation also defines the Lucas sequence for the initial values $L_0 = 2$ and $L_1 = 1$. It is well known that $F_{n+1}$ counts the number of tilings of an $n$-board using either square tiles or two-square-wide dominoes [3]. It can be expressed as

$$F_{n+1} = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i}. $$

This expression gives rise to a fascinating class of integers called the incomplete Fibonacci numbers. They were introduced by Flipponi [7] for integers $n$ and $k$ with $0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ as

$$F_n(k) = \sum_{i=0}^{k} \binom{n-1-i}{i}. $$

Combinatorially, $F_{n+1}(k)$ counts the number of ways to tile an $n$-board with at most $k$ dominoes [2]. Flipponi [7] also defined the incomplete Lucas numbers for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ as

$$L_n(k) = \sum_{i=0}^{k} \frac{n}{n-i} \binom{n-i}{i}. $$

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Incomplete Fibonacci and Lucas numbers have many interesting properties. They generalize the Fibonacci and Lucas numbers. In other words, incomplete Fibonacci numbers reduce to Fibonacci numbers when \( k = \lfloor \frac{n-1}{2} \rfloor \), and incomplete Lucas numbers reduce to Lucas numbers when \( k = \lfloor \frac{n}{2} \rfloor \).

Horadam sequence \( \{W_n\} \) with arbitrary integer initial values \( W_0 \) and \( W_1 \) is defined by the recurrence relation \( W_n = pW_{n-1} + qw_{n-2} \) for \( n \geq 2 \). Its terms are called the Horadam numbers and they provide a generalization for Fibonacci numbers and Lucas numbers. Indeed, \( \{W_n\} \) reduces to \( \{F_n\} \) for \( p = q = 1 \) and \( W_0 \) = 0, \( W_1 \) = 1, and to \( \{L_n\} \) for \( p = q = 1 \) and \( W_0 = 2, W_1 = 1 \). With this in mind, a question arises whether or not incomplete Fibonacci and Lucas numbers extend to Horadam-like numbers. Belbachir and Belkhir [1] responded this question by introducing incomplete Horadam numbers for \( n \geq 2 \) and \( 0 \leq k \leq \lfloor \frac{n}{2} \rfloor \) as

\[
W_n(k) = \sum_{i=0}^{k} \frac{(n-2i)W_1 + pw_0}{n-i} \left( \binom{n-i}{i} \right) p^{n-2i-1} q^i,
\]

where \( p \) and \( q \) are integers. They also introduced hyper-Horadam numbers and provided a connection between incomplete Fibonacci and Lucas numbers, and hyper-Horadam numbers.

The bi-periodic Horadam sequence \( \{w_n\} \) is a natural generalization of the Horadam sequence. For arbitrary initial values \( w_0 \) and \( w_1 \), its terms are defined recursively for \( n \geq 2 \) by

\[
w_n = a^{\xi(n+1)}b^{\xi(n)}w_{n-1} + cw_{n-2},
\]

where \( a, b, \) and \( c \) are nonzero real numbers. Here, \( \xi(n) = \frac{1-(-1)^n}{2} \). It can easily be seen that the bi-periodic Fibonacci sequence, the generalized bi-periodic Fibonacci sequence, the bi-periodic Lucas sequence, the generalized bi-periodic Lucas sequence, and the classical Horadam sequence are special cases of the bi-periodic Horadam sequence. For example, \( \{w_n\} \) reduces to \( \{W_n\} \) when \( a = p, b = p, \) and \( c = q \). For details, we refer to [4–6, 12–14].

Ramírez [10] defined the bi-periodic incomplete Fibonacci numbers for \( n \geq 1 \) and \( 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor \) as

\[
q_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n+1}{2} \rfloor - i}.
\]

In this spirit, Tan and Ekin [12] introduced the bi-periodic incomplete Lucas numbers for \( 0 \leq k \leq \lfloor \frac{n}{2} \rfloor \) by

\[
p_n(k) = a^{\xi(n)} \sum_{i=0}^{k} \frac{n-i}{i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i}.
\]

Motivated by the above studies, we introduce in this paper the bi-periodic incomplete Horadam numbers. In particular, we give some recurrence relations and provide a connection between bi-periodic incomplete Fibonacci numbers and bi-periodic incomplete Horadam numbers. We then derive the generating function of these numbers. This new generalization shall give us a unified approach for many celebrated incomplete Fibonacci-like sequences such as bi-periodic incomplete Fibonacci and Lucas sequences, incomplete Fibonacci and Lucas sequences, incomplete balancing and Lucas-balancing sequences.
2. Main results
In this section, we shall introduce bi-periodic incomplete Horadam numbers. To this purpose, we begin with the following lemma. It provides a combinatorial expression for the bi-periodic Horadam numbers.

Lemma 2.1 For \( n \geq 1 \), the bi-periodic Horadam numbers satisfy

\[
w_n = a^{\xi(n-1)} \sum_{i=0}^{\frac{n}{2}} \frac{(n-2i)w_{1} + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-i}{2} \rfloor - i} c^i.
\]

Proof We will use induction on \( n \). Clearly, the equality holds for \( n = 1 \). Now suppose that the lemma is true for any integer \( m \) with \( 1 \leq m \leq n \). Then by the inductive hypothesis, we can write

\[
w_{n+1} = a^{\xi(n)}b^{\xi(n+1)}w_n + cw_{n-1}
\]

\[
= a^{\xi(n)}b^{\xi(n+1)}a^{\xi(n-1)} \sum_{i=0}^{\frac{n}{2}} \frac{(n-2i)w_{1} + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-i}{2} \rfloor - i} c^i +
\]

\[
a^{\xi(n)} \sum_{i=0}^{\frac{n}{2}+1} \frac{(n-1-2i)w_{1} + biw_0}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1-i}{2} \rfloor - i-1} c^{i+1}.
\]

Since \( \xi(n-1) = \xi(n+1) \), we have

\[
a^{-\xi(n)}w_{n+1} = \sum_{i=0}^{\frac{n}{2}} \frac{(n-2i)w_{1} + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-i}{2} \rfloor - i + \xi(n+1)} c^i +
\]

\[
\sum_{i=0}^{\frac{n}{2}+1} \frac{(n-1-2i)w_{1} + biw_0}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1-i}{2} \rfloor - i-1} c^{i+1}.
\]

\[
= \sum_{i=0}^{\frac{n}{2}} \frac{(n-2i)w_{1} + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-i}{2} \rfloor - i} c^i +
\]

\[
\sum_{i=0}^{\frac{n}{2}+1} \frac{(n-1-2i)w_{1} + biw_0}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1-i}{2} \rfloor - i-1} c^{i+1}.
\]

\[
= \sum_{i=0}^{\frac{n}{2}} \frac{(n-2i)w_{1} + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-i}{2} \rfloor - i} c^i +
\]

\[
\sum_{i=1}^{\frac{n}{2}+1} \frac{(n-2i+1)w_{1} + b(i-1)w_0}{n-i} \binom{n-i}{i-1} (ab)^{\lfloor \frac{n-i}{2} \rfloor - i} c^i.
\]
Thus, the given formula is true for any positive integer \( n \). \( \square \)

In the light of Lemma 2.1, we can define bi-periodic incomplete Horadam numbers as follows.

**Definition 2.2** Let \( n \) and \( k \) be positive integers such that \( 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \). We define the bi-periodic incomplete Horadam numbers as

\[
w_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \frac{(n-2i)w_1 + biw_0}{n-i} \left( \frac{n-i}{i} \right)^k (ab)^{\left\lfloor \frac{n-i}{2} \right\rfloor} c^i.
\]

Note that the incomplete Horadam numbers in (1.1) are a special case of this definition. They are obtained for \( a = p, b = p, \) and \( c = q \).

It can easily be seen that \( w_n(0) = a^{\xi(n-1)}w_1(ab)^{\left\lfloor \frac{n}{2} \right\rfloor} \) and \( w_n\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = w_n \) for \( n \geq 1 \). Similarly,

\[
w_n(1) = a^{\xi(n-1)} \left( w_1(ab)^{\left\lfloor \frac{n}{2} \right\rfloor} + (n-2)w_1 + bw_0 \right)(ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor} c,
\]

\[
w_n\left(\left\lfloor \frac{n-2}{2} \right\rfloor\right) = \begin{cases} w_n - w_0c^{\frac{n}{2}}, & \text{if } n \text{ is even}, \\ w_n - \left[ w_1 + \left(\frac{n-1}{2}\right)bw_0 \right] c^{\frac{n-1}{2}}, & \text{if } n \text{ is odd}, \end{cases}
\]

for \( n \geq 2 \).

**Example 2.3** For \( a = 3, b = 2, c = 1, w_0 = 4, w_1 = 2 \) and \( 1 \leq n \leq 10 \), all the values of \( w_n(k) \) are displayed in the table on the next page.

**Proposition 2.4** Consider the bi-periodic incomplete Horadam numbers \( w_n(k) \). For \( 0 \leq k \leq \frac{n-3}{2} \), they satisfy the nonlinear recurrence relation

\[
w_n(k) = a^{\xi(n+1)}b^{\xi(n)}w_{n-1}(k) + cw_{n-2}(k - 1).
\]

**Proof** Suppose \( n \) is even. Since \( \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil - 1 \), we have

\[
a^{\xi(n+1)}b^{\xi(n)}w_{n-1}(k) + cw_{n-2}(k - 1)
\]
The proof is similar when \( n \) is odd. This completes the proof. \( \square \)
Proposition 2.4 can be transformed into a nonhomogeneous recurrence relation as follows:

\[ w_n(k) = a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + cw_{n-2}(k-1) \]
\[ = a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + cw_{n-2}(k) + c(w_{n-2}(k-1) - w_{n-2}(k)) \]
\[ = a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + cw_{n-2}(k) - w_n(k) \]
\[ = a^{\xi(n+1)} \left( n - 2k - 2 \right) w_1 + bk w_0 \left( \frac{n - 2}{k} \right) \left( ab \right)^{\frac{n+1}{2} - k} c^{k+1} . \]  

(2.1)

Proposition 2.5 For \( 0 \leq k \leq \frac{n-s-1}{2} \), we have

\[ \sum_{i=0}^{s+1} \binom{s+1}{i} w_{n+i}(k+i) a^{i+\xi} b^{\frac{i}{2}} c^{s+1-i} = w_{n+2s}(k+s). \]  

(2.2)

**Proof** We proceed by induction on \( s \). The proof is clear for \( s = 0 \) and \( s = 1 \) from Proposition 2.4. So assume the relation in (2.2) holds for all positive \( j < s + 1 \). We will only verify it for \( j = s + 1 \) when \( n \) is even since the proof is similar when \( n \) is odd. Now,

\[ \sum_{i=0}^{s+1} \binom{s+1}{i} w_{n+i}(k+i) a^{i+\xi} b^{\frac{i}{2}} c^{s+1-i} \]
\[ = \sum_{i=0}^{s+1} \left[ \binom{s+1}{i} + \binom{s}{i-1} \right] w_{n+i}(k+i) a^{i+\xi} b^{\frac{i}{2}} c^{s+1-i} \]
\[ = \sum_{i=0}^{s+1} \binom{s}{i} w_{n+i}(k+i) a^{i+\xi} b^{\frac{i}{2}} c^{s+1-i} + \]
\[ \sum_{i=0}^{s+1} \binom{s}{i-1} w_{n+i}(k+i) a^{i+\xi} b^{\frac{i}{2}} c^{s+1-i} \]
\[ = \binom{s}{s+1} w_{n+s+1}(k+s+1) a^{s+\xi} b^{\frac{s+1}{2}} + \]
\[ c \sum_{i=0}^{s} \binom{s}{i} w_{n+i}(k+i) a^{i+\xi} b^{\frac{i}{2}} c^{s-i} + \]
\[ \sum_{i=0}^{s} \binom{s}{i} w_{n+i+1}(k+i+1) a^{i+\xi} b^{\frac{i+1}{2}} c^{s-i} \]
\[ = \binom{s}{s+1} w_{n+s+1}(k+s+1) a^{s+\xi} b^{\frac{s+1}{2}} + cw_{n+2s}(k+s) + \]
\[ \binom{s}{s+1} w_{n+k} c^{s+1} + a \sum_{i=0}^{s} \binom{s}{i} w_{n+i+1}(k+i+1) a^{i+\xi} b^{\frac{i+1}{2}} c^{s-i} \]
\[ = cw_{n+2s}(k+s) + aw_{n+2s+1}(k+s+1) = w_{n+2s+2}(k+s+1). \]

Hence the theorem holds for all \( j \). This completes the proof. \( \square \)
Proposition 2.6 For $s \geq 2k + 2$, we have

$$
\sum_{i=0}^{s-1} a^{\left\lfloor \frac{s-i-\frac{1-\xi(n+1)}{2}}{2} \right\rfloor n\xi(n) - \left\lfloor \frac{i-\xi(n)}{2} \right\rfloor b^{\left\lfloor \frac{s-i-\frac{1-\xi(n+1)}{2}}{2} \right\rfloor w_{n+1}(k)}
$$

$$
= w_{n+s+1}(k + 1) - a^{\left\lfloor \frac{s+i-\xi(n+1)}{2} \right\rfloor b^{\left\lfloor \frac{s+i-\xi(n+1)}{2} \right\rfloor w_{n+1}(k + 1)}.
$$

Proof We will use induction on $s$. We will only consider the case when $n$ is odd since the proof is similar when $n$ is even.

Suppose $n$ is odd. Then $\xi(n) = 1$ and $\xi(n + 1) = 0$. For $s = 2$, the right hand side of Equation 2.3 is $w_{n+3}(k + 1) - abw_{n+1}(k + 1)$, and it simplifies to $acw_n(k) + cw_{n+1}(k)$ by Proposition 2.4. This clearly equals the left hand side. Hence, the proposition is true for $s = 2$.

Now suppose that the proposition is true for all $2 < s$. We prove it for $s$. Since $\left\lfloor \frac{s+1}{2} \right\rfloor = \left\lfloor \frac{s}{2} \right\rfloor + \xi(s)$, we have

$$
\sum_{i=0}^{s} a^{\left\lfloor \frac{s-i-\frac{1-\xi(n+1)}{2}}{2} \right\rfloor n\xi(n) - \left\lfloor \frac{i-\xi(n)}{2} \right\rfloor b^{\left\lfloor \frac{s-i-\frac{1-\xi(n+1)}{2}}{2} \right\rfloor w_{n+1}(k)}
$$

$$
= \sum_{i=0}^{s} a^{\left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{i+1}{2} \right\rfloor} w_{n+1}(k)
$$

$$
= \sum_{i=0}^{s-1} a^{\left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{i+1}{2} \right\rfloor} w_{n+1}(k) + cw_{n+s}(k)
$$

$$
= \sum_{i=0}^{s-1} a^{\left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{i+1}{2} \right\rfloor} w_{n+1}(k) + cw_{n+s}(k)
$$

$$
= a^{\xi(s)} b^{\xi(s+1)} \sum_{i=0}^{s-1} a^{\left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{i+1}{2} \right\rfloor} w_{n+1}(k + 1) + cw_{n+s}(k)
$$

$$
= a^{\xi(s)} b^{\xi(s+1)} w_{n+s+1}(k + 1) + cw_{n+s}(k)
$$

$$
= a^{\xi(s)} b^{\xi(s+1)} w_{n+s+1}(k + 1) + cw_{n+s}(k) - a^{\xi(s)} b^{\xi(s+1)} w_{n+1}(k + 1)
$$

$$
= a^{\xi(s)} b^{\xi(s+1)} w_{n+s+1}(k + 1) + cw_{n+s}(k) - a^{\xi(s)} b^{\xi(s+1)} w_{n+1}(k + 1)
$$

$$
= w_{n+s+2}(k + 1) + a^{\left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{i+1}{2} \right\rfloor} w_{n+1}(k + 1)
$$

$$
= w_{n+s+2}(k + 1) - a^{\left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{i+1}{2} \right\rfloor} w_{n+1}(k + 1).
$$

This completes the proof.

We end this section by giving a connection between the generalized bi-periodic incomplete Fibonacci numbers $u_n(k)$ and the generalized bi-periodic incomplete Lucas numbers $v_n(k)$.

Proposition 2.7 For $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, we have

$$
v_n(k) = u_{n+1}(k) + cw_{n-1}(k - 1).
$$
Proof Recall that

\[ u_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \binom{n-1-i}{i} (ab)^{\frac{ab}{2}} \xi_i (1-ic^i), \quad 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \]

and

\[ v_n(k) = a^{\xi(n)} \sum_{i=0}^{k} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\frac{ab}{2}} \xi_i (1-ic^i), \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor. \]

So we have,

\[ u_{n+1}(k) + cu_{n-1}(k-1) = a^{\xi(n)} \sum_{i=0}^{k} \binom{n-i}{i} (ab)^{\frac{ab}{2}} \xi_i (1-ic^i) + ca^{\xi(n)} \sum_{i=0}^{k-1} \binom{n-2-i}{i} (ab)^{\frac{ab}{2}} \xi_i (1-ic^i) \]

\[ = a^{\xi(n)} \sum_{i=0}^{k} \binom{n-i}{i} (ab)^{\frac{ab}{2}} \xi_i (1-ic^i) + a^{\xi(n)} \sum_{i=1}^{k} \binom{n-1-i}{i-1} (ab)^{\frac{ab}{2}} \xi_i (1-ic^i) \]

\[ = a^{\xi(n)} \sum_{i=0}^{k} \left[ \binom{n-i}{i} + \binom{n-1-i}{i-1} \right] (ab)^{\frac{ab}{2}} \xi_i (1-ic^i) \]

\[ = a^{\xi(n)} \sum_{i=0}^{k} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\frac{ab}{2}} \xi_i (1-ic^i) = v_n(k). \]

\[ \Box \]

3. The generating function

In this section, we shall derive the generating function for the bi-periodic incomplete Horadam numbers. To this purpose, we need the following lemma which can be obtained from [9] and [10]. We refer to Srivastava and Monacha [11] for a general treatment of generating functions of special functions.

Lemma 3.1 Let \( \{r_n\}_{n=0}^{\infty} \) be a given complex sequence, and let \( a, b, \) and \( c \) be complex numbers. Suppose that a complex sequence \( \{s_n\}_{n=0}^{\infty} \) satisfies the nonhomogeneous and nonlinear recurrence relation

\[ s_n = \begin{cases} 
bs_{n-1} + cs_{n-2} + r_n, & \text{if } n \text{ is even,} \\
bs_{n-1} + cs_{n-2} + ar_n, & \text{if } n \text{ is odd,}
\end{cases} \]

for \( n > 1 \). Then the generating function \( U(t) \) of \( \{s_n\}_{n=0}^{\infty} \) is given by

\[ U(t) = \frac{aG(t) + (s_0 - r_0) + [s_1 - a(s_0 + r_1)]t + (b - a)tf(t) + (1 - a)G_1(t)}{1 - at - ct^2} \]

where \( f(t), G(t), \) and \( G_1(t) \) are the generating functions of \( \{s_{2n+1}\}_{n=0}^{\infty}, \{r_n\}_{n=0}^{\infty}, \) and \( \{r_{2n}\}_{n=0}^{\infty}, \) respectively, and

\[ f(t) = \frac{[s_1 - a(r_0 + r_1)]t + c[a(s_0 + r_1) - s_1]t^3 + atG_1(t) + a(1 - ct^2)G_2(t)}{1 - (ab + 2c)t^2 + c^2t^4} \]
where $G_2(t)$ denotes the generating function of $\{r_{2n-1}\}_{n=1}^{\infty}$.

**Proof** Let $U(t) = \sum_{n=0}^{\infty} s_n t^n$ and $G(t) = \sum_{n=0}^{\infty} r_n t^n$. Then,

$$(1 - at - ct^2)U(t) - aG(t) = (s_0 - ar_0) + [s_1 - a(s_0 + r_1)]t + \sum_{n=2}^{\infty} (s_n - as_{n-1} - cs_{n-2} - ar_n)t^n.$$

Let us simplify the summation above. Since $s_{2n+1} = as_{2n} + cs_{2n-1} + ar_{2n+1}$ and $s_{2m} = bs_{2m-1} + cs_{2m-2} + r_{2m}$, it follows that

$$\sum_{n=2}^{\infty} (s_n - as_{n-1} - cs_{n-2} - ar_n)t^n = \sum_{m=1}^{\infty} (s_{2m} - as_{2m-1} - cs_{2m-2} - ar_{2m})t^{2m}$$

$$= \sum_{m=1}^{\infty} [(b - a)s_{2m-1} + (1 - a)r_{2m}]t^{2m}$$

$$= (b - a)t \sum_{m=1}^{\infty} s_{2m-1}t^{2m-1} + (1 - a) \sum_{m=1}^{\infty} r_{2m}t^{2m}$$

$$= (b - a)t f(t) + (1 - a)G_1(t) - (1 - a)r_0.$$

Hence,

$$(1 - at - ct^2)U(t) - aG(t) = (s_0 - r_0) + [s_1 - a(s_0 + r_1)]t + (b - a)t f(t) + (1 - a)G_1(t).$$

Then the formula for the generating function follows by solving the above equation for $U(t)$.

Next, we calculate $f(t)$. For $m > 2$, it is easy to see that

$$s_{2m-1} = (ab + 2c)s_{2m-3} - c^2s_{2m-5} - a(cr_{2m-3} - r_{2m-2} - r_{2m-1}).$$

Moreover,

$$s_3 - (ab + 2c)s_1 + a(cr_1 - r_2 - r_3) = as_2 - cs_1 - abs_1 + acr_1 - ar_2$$

$$= acs_0 - cs_1 + acr_1.$$

Then we have,

$$[1 - (ab+2c)t^2 + c^2t^4]f(t) - atG_1(t) + a(ct^2 - 1)G_2(t)$$

$$= [s_1 - a(r_0 + r_1)]t + c[a(s_0 + r_1) - s_1])t^3.$$ The formula follows by solving the above equation for $f(t)$.

Now, we are ready to state the generating function for the bi-periodic incomplete Horadam numbers.
**Theorem 3.2** Consider the bi-periodic incomplete Horadam numbers $w_n(k)$. Let

$$G_1(t) = -\frac{c^{k+1}(w_0b - (w_0b - w_1)t)}{2} \left[ \frac{t^2}{(1 - (ab)^{\frac{1}{2}}t)^{k+1}} + \frac{t^2}{(1 + (ab)^{\frac{1}{2}}t)^{k+1}} \right]$$

$$G_2(t) = -\frac{c^{k+1}(w_0b - (w_0b - w_1)abt)}{2(ab)^{\frac{1}{2}}} \left[ \frac{t^2}{(1 - (ab)^{\frac{1}{2}}t)^{k+1}} - \frac{t^2}{(1 + (ab)^{\frac{1}{2}}t)^{k+1}} \right].$$

Then, the generating function $W_k(t)$ of $w_n(k)$ is given by

$$W_k(t) = \sum_{n=0}^{\infty} w_n(k)t^n = \frac{aG(t) + w_{2k} + w_{2k-1}t + (b - a)tG_1(t) + (1 - a)G_1(t)}{1 - at - ct^2}$$

where $G(t) = G_1(t) + G_2(t)$, and

$$f(t) = \frac{w_{2k+1}t - cw_{2k-1}t^3 + atG_1(t) + a(1 - ct^2)G_2(t)}{1 - (ab + 2c)t^2 + c^2t^4}.$$

**Proof** Let $k$ be a fixed positive integer. It is easy to see that $w_n(k) = 0$ for $0 \leq n < 2k$, and $w_{2k}(k) = w_{2k}$ and $w_{2k+1}(k) = w_{2k+1}$. When $n$ is even, it follows from the nonhomogeneous recurrence relation in Equation 2.1 that

$$w_n(k) = aw_{n-1}(k) + cw_{n-2}(k) - b^{-1}(n - 2k - 2)w_1 + bkw_0 \left( \frac{n - k - 2}{k} \right) (ab)^{\left\lfloor \frac{n - 2}{2} \right\rfloor} - k_c^{k+1}.$$

Similarly, when $n$ is odd

$$w_n(k) = bw_{n-1}(k) + cw_{n-2}(k) - (n - 2k - 2)w_1 + bkw_0 \left( \frac{n - k - 2}{k} \right) (ab)^{\left\lfloor \frac{n - 2}{2} \right\rfloor} - k_c^{k+1}.$$

Now let us define

$$s_0 = w_{2k}(k) = w_{2k}, \quad s_1 = w_{2k+1}(k) = w_{2k+1}, \quad s_n = w_{2k+n}(k),$$

and

$$r_0 = r_1 = 0, \quad r_n = -\frac{(n - 2)w_1 + bkw_0}{n + k - 2} \left( \frac{n - k - 2}{k} \right) (ab)^{\left\lfloor \frac{n - 2}{2} \right\rfloor} - 1 - c^{k+1}.$$

Then,

$$G(t) = G_1(t) + G_2(t)$$

$$= -\frac{c^{k+1}t^2}{2} \left[ \frac{w_0b - (w_0b - w_1)t}{(1 - (ab)^{\frac{1}{2}}t)^{k+1}} + \frac{w_0b - (w_0b - w_1)abt}{(1 + (ab)^{\frac{1}{2}}t)^{k+1}} \right] + \frac{[w_0b - (w_0b - w_1)t] - [w_0b - (w_0b - w_1)abt]}{(1 + (ab)^{\frac{1}{2}}t)^{k+1}}.$$
is the generating function of the sequence \( \{-r_n\} \). Thus the generating function of the sequence \( \{w_n(k)\}_{n=0}^{\infty} \) follows from [9, Lemma 3.1]. This completes the proof.

\[ \square \]

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References


