Numerical solution for Benjamin-Bona-Mahony-Burgers equation with Strang time-splitting technique

MELİKE KARTA

Follow this and additional works at: https://journals.tubitak.gov.tr/math

Part of the Mathematics Commons

Recommended Citation
Available at: https://journals.tubitak.gov.tr/math/vol47/iss2/9

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.
Numerical solution for Benjamin-Bona-Mahony-Burgers equation with Strang time-splitting technique

Melike KARTA∗
Department of Mathematics, Faculty of Science and Arts, Ağrı İbrahim Çeçen University, Ağrı, Turkey

Received: 15.08.2022 • Accepted/Published Online: 04.01.2023 • Final Version: 09.03.2023

Abstract: In the present manuscript, the Benjamin-Bona-Mahony-Burgers (BBMB) equation will be handled numerically by Strang time-splitting technique. While applying this technique, collocation method based on quintic B-spline basis functions is applied. In line with our purpose, after splitting the BBM-Burgers equation given with appropriate initial boundary conditions into two subequations containing the derivative in terms of time, the quintic B-spline based collocation finite element method (FEM) for spatial discretization and the suitable finite difference approaches for time discretization is applied to each subequation and hereby two different systems of algebraic equations are obtained. Four test problems are utilized to test the efficiency and reliability of the presented method. The error norms $L_2$ and $L_\infty$ with mass, energy, and momentum conservation constants $I_1, I_2$ and $I_3$, respectively, are computed. To do a comparison with the other studies in the literature, the newly found approximate solutions are exhibited in both tabular and graphical formats. Also, stability analysis of numerical approach by the von Neumann method is researched.

Key words: BBMB equation, quintic B-splines, collocation method, Strang splitting

1. Introduction
Bruzon and Gandarias [7] considered a generalized BBM-Burgers equation given as

$$U_t - U_{xxt} - \alpha U_{xx} + \beta U_x + (g(U))_x = 0,$$

in which $\alpha$ is a given nonnegative real-number, $\beta \in R$ and $g(U)$ is a predefined nonlinear function. Peregrine [43] first proposed equation and Benjamin et al. [5] broadly explained it. In this article, by taking $g(U) = \frac{U^2}{2}$, we are going to handle the BBM-Burgers equation as follows:

$$U_t - U_{xxt} - \alpha U_{xx} + \beta U_x + UU_x = 0, \quad x \in [x_L, x_R], t \in [0, T],$$

with the condition given at initial time

$$U(x, 0) = g_0(x),$$

and the conditions given at the boundaries

$$U(x_L, t) = U(x_R, t) = 0,$$

$$U_x(x_L, t) = U_x(x_R, t) = 0,$$

$$U_{xx}(x_L, t) = U_{xx}(x_R, t) = 0 \quad t > 0.$$

∗Correspondence: mkarta@agri.edu.tr
2010 AMS Mathematics Subject Classification: 65D07, 65L60

This work is licensed under a Creative Commons Attribution 4.0 International License.
If \( \alpha = 0 \) is taken in Eq. (1.2), the (RLW) equation is obtained in form
\[
U_t - U_{xxx} + \beta U_x + U U_x = 0
\] (1.5)
and this is named as the Benjamin-Bona-Mahony (BBM) equation. The approximate solution of the BBM-Burgers equation is the subject of research by many authors using various methods. Among them, Jain et al. [30] solved BBM equation with a combination of the splitting method and cubic B-spline technique. Gardner et al. [22] applied A B-spline finite element method (FEM) involving Galerkin method based on quadratic B-splines. Gardner et al. [23] presented a least-squares techniques, Doğan [16] investigated equation using Galerkin method via linear space finite elements. Soliman and Raslan [46] and Soliman and Hussien [47] proposed quadratic and septic splines collocation method, respectively. Dağ et al. [17] studied with cubic B-spline collocation finite element method. Esen and Kutluay [20] used quadratic B-spline lumped Galerkin method and in the same year they [32] presented a linearized finite difference method for equation. Omrani and Ayadi [41] used finite difference method and also showed with the Brower’s fixed point theorem to existence of solutions. Moreover, Arora et al. [1] utilized collocation method with quartic B-spline for the BBMB equation. Yağmurlu et al. [53] solved by the Strang splitting technique using the finite element collocation method with cubic B-splines. Karakoç and Bhowmik [31] implemented a lumped Galerkin technique to the BBM-Burgers equation and founded to the existence and uniqueness of solutions. Zarebnia and Parvaz [54–57] proposed collocation method based on cubic, quadratic and quintic B-splines for the numerical algorithm of BBM-Burgers equation, respectively. Also, related studies for approximate solutions of the equation can be found as [3, 4, 6, 9, 21, 40, 45], [34–37] and [10–15]. Mehdi et al. [18] performed a finite difference formula for solving numerically of the generalized BBM-Burgers equation and then applied the energy method for the stability and convergence analyses. Zhang et al. [58] suggested two linearized finite difference schemes and proved convergence and unique solvability of the BBMB equation with the convergence order \( O(\tau^2 + h^2) \). Arora et al. [2] improved a hybrid technique including quintic Hermite splines collocation method (QHCM) for solving numerically of the BBM-Burgers equation.

We can say that the problem discussed throughout the article can be solved more effectively and faster numerically by Strang time-splitting technique combined with quintic B-spline collocation method. This article is designed as follows: In Section 2, a brief information about the Strang time-splitting technique is given. In Section 3 after explaining the collocation method with quintic B-spline in detail, Eq. (1.2) is split into two subequations and each of them is solved numerically with Strang time-splitting technique by applying the quintic B-spline collocation method. Also, In Section 4, the stability analysis for the numerical approach obtained with the present method is examined. In Section 5, four test problems are offered, one of which is inhomogeneous BBM-Burgers equation and the others are BBM-Burgers equations given by different initial condition and in this section the error norms \( L_2 \) and \( L_\infty \) with invariant values \( I_1 \), \( I_2 \) and \( I_3 \) are calculated to analyze the effectiveness and accuracy of the presented method and the computed values are given in tables and graphics to compare with existing studies in the literature. Lastly, Section 6 presents a brief conclusion as an overview.

2. Strang time-splitting technique

Before proceeding to the application of the method, it would be appropriate to give information about the Strang splitting technique, which is of great importance in the approximate solution for the given problem. To use this technique, the given complex problem is first divided into simpler two subproblems with smaller time
steps. If a Cauchy problem is assumed to have the following form

$$\frac{dU(t)}{dt} = \hat{A}U(t) + \hat{B}U(t), \quad U(0) = U_0, \quad t \in [0, T].$$  \hspace{1cm} (2.1)$$

Then, the problem is split into such that \( \frac{dU}{dt} = \hat{A}U, \quad \frac{dU}{dt} = \hat{B}U \) where \( U(x, t) \) is a semidiscretized function given on spatial direction and \( \hat{A} \) and \( \hat{B} \) are Lie operators. In this case, the proposed technique tries to solve Eq. (1.2) either numerically or analytically [8]. Let the exact or numerical solutions of the equations containing the operators \( \hat{A} \) and \( \hat{B} \) in \( \frac{dU}{dt} = \hat{A}U, \quad \frac{dU}{dt} = \hat{B}U \) be \( \rho_{\Delta t}^{[\hat{A}]} \) and \( \rho_{\Delta t}^{[\hat{B}]} \), and let \( \psi_{\Delta t} \) be the exact solution of the equation (2.1). In this case, first-order splitting methods are defined as [51]

$$\rho_{\Delta t}^{[\hat{B}]} \rho_{\Delta t}^{[\hat{A}]} = e^{\Delta t \hat{B}} e^{\Delta t \hat{A}} \quad \text{or} \quad \rho_{\Delta t}^{[\hat{A}]} \rho_{\Delta t}^{[\hat{B}]} = e^{\Delta t \hat{A}} e^{\Delta t \hat{B}}.$$ 

With the aid of the Taylor series, for an initial value \( U_0 \) it is seen that there is a first-order approximation to the solution of equation (2.1) as follows:

$$\psi_{\Delta t}(U_0) = (\rho_{\Delta t}^{[\hat{A}]} \rho_{\Delta t}^{[\hat{B}]}) (U_0) + O(\Delta t^2).$$

Exchanging the operators \( \hat{A} \) and \( \hat{B} \), combination for half time steps can be taken as follows:

$$U(t_{n+1}) = (e^{\frac{\Delta t}{2} \hat{A}} e^{\frac{\Delta t}{2} \hat{B}}) (e^{\frac{\Delta t}{2} \hat{B}} e^{\frac{\Delta t}{2} \hat{A}}) U(t_n) = e^{\frac{\Delta t}{2} \hat{A}} e^{\Delta t \hat{B}} e^{\frac{\Delta t}{2} \hat{A}} U(t_n).$$

Then we can characterize the approach mentioned as

$$S_{\Delta t} = e^{\frac{\Delta t}{2} \hat{A}} e^{\Delta t \hat{B}} e^{\frac{\Delta t}{2} \hat{A}} \quad \text{or} \quad S_{\Delta t}^* = e^{\frac{\Delta t}{2} \hat{B}} e^{\Delta t \hat{A}} e^{\frac{\Delta t}{2} \hat{B}}.$$ 

As it is widely known, this is referred to as Strang splitting technique [38] which have the "\( \hat{A} - \hat{B} - \hat{A} \)" and "\( \hat{B} - \hat{A} - \hat{B} \)" or the so-called symmetric Marchuk. The procedure for Strang splitting scheme can be presented as

$$\frac{dU^*(t)}{dt} = \hat{A}U^*(t), \quad U^*(0) = U_0, \quad t \in \left[ t_n, t_{n+\frac{1}{2}} \right]$$

$$\frac{dU^{**}(t)}{dt} = \hat{B}U^{**}(t), \quad U^{**}(0) = U^*(\frac{\Delta t}{2}), \quad t \in \left[ t_n, t_{n+1} \right]$$

$$\frac{dU^{***}(t)}{dt} = \hat{A}U^{***}(t), \quad U^{***}(0) = U^{**}(\Delta t), \quad t \in \left[ t_n, t_{n+\frac{1}{2}} \right].$$  \hspace{1cm} (2.2)$$

in which \( t_{n+1} = t_n + \Delta t, \quad t_{n+\frac{1}{2}} = t_n + \frac{\Delta t}{2} \). Here the desired solutions are easily obtained through the equation of \( U(t_{n+1}) = U^{***}(t_{n+1}) \). The formal solution of Equation (2.1) is written as \( U(t_{n+1}) = e^{\Delta t(\hat{A} + \hat{B})} U(t_n) \), where \( \Delta t = t_{n+1} - t_n \) is the time step. Taylor series expansion of this solution can be expressed as

$$U(t_{n+1}) = e^{\Delta t(\hat{A} + \hat{B})} U(t_n) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\hat{A}(u(t)) \frac{\partial}{\partial U} + \hat{B}(u(t)) \frac{\partial}{\partial U})^k U(t_n).$$

539
It is obtained in the following form such that $T_e$ is a local truncation error

$$
U(t_{n+1}) = e^{\Delta t(\hat{A}\hat{B})}U(t_n) \approx e^{\frac{\Delta t}{2} \hat{A}_e^{\Delta t} \hat{B}_e^{\Delta t}} + \Delta t T_e
$$

$$
T_e = \frac{1}{\Delta t} \left( e^{\Delta t(\hat{A}\hat{B})} - e^{\frac{\Delta t}{2} \hat{A}_e^{\Delta t} \hat{B}_e^{\Delta t}} \right) U(t_n).
$$

If the Taylor series expansion of $e^{\Delta t(\hat{A}\hat{B})}$ and $e^{\frac{\Delta t}{2} \hat{A}_e^{\Delta t} \hat{B}_e^{\Delta t}}$ exponential functions are substituted in (2.3), the local truncation error $T_e$ called as splitting error is obtained as follows:

$$
T_e = \frac{\Delta t^2}{24} \left( \hat{A}, \hat{B}, \hat{A} \right) - \left( \hat{B}, \hat{A}, \hat{B} \right) \right) U(t_n) + O(\Delta t^3)
$$

and this shows the fact that the proposed technique is of the second-order.

3. Collocation method via quintic B-spline

Before starting the numerical process, initially the interval $x_L \leq x \leq x_R$ is partitioned into uniformly such that $h = x_m - x_{m+1} = \frac{x_R - x_L}{N}$ by knots $x_m$ in which $(m = 0(1)N - 1)$ and $x_L = x_0 < x_1 < ... < x_N = x_R$. The quintic B-spline functions $\varphi_m(x)$ on $[x_L, x_R]$ stated as the solution region for $(m = -2(1)N + 2)$ at the nodes $x_m$ are introduced as follows [44]:

$$
\varphi_m(x) = \begin{cases} 
(x - x_{m+3})^5, & x \in [x_{m-3}, x_{m+2}], \\
(x - x_{m+3})^5 - 6(x - x_{m+2})^5, & x \in [x_{m-3}, x_{m-2}], \\
(x - x_{m+3})^5 - 6(x - x_{m+2})^5 + 15(x - x_{m-2})^5, & x \in [x_{m-3}, x_{m}], \\
(x_{m+3} - x)^5, & x \in [x_{m-1}, x_{m}]. \\
(x_{m+3} - x)^5 - 6(x_{m+2} - x)^5, & x \in [x_{m-1}, x_{m+1}], \\
(x_{m+3} - x)^5 - 6(x_{m+2} - x)^5 + 15(x_{m+1} - x)^5, & x \in [x_{m-1}, x_{m+2}], \\
0, & x \in [x_{m+1}, x_{m+3}], \\
\end{cases}
$$

(3.1)

For detailed information, the reader can refer to Ref. [19] related to B-splines. All of the quintic basis functions are zero on the element $[x_m, x_{m+1}]$ except those $\varphi_{m-2}(x), \varphi_{m-1}(x), \varphi_{m+3}(x), \varphi_{m+2}(x), \varphi_{m+3}(x)$. The approximate solution $U_N(x, t)$ of Eq. (1.2) on $[x_m, x_{m+1}]$ can be described as

$$
U_N(x, t) = \sum_{j=m-2}^{m+3} \varphi_j(x) \delta_j(t),
$$

(3.2)

where $\varphi_j(x)$ ($j = (m - 2)(1)(m + 3)$) are B-spline element functions and $\delta_j(t)$ ($j = m - 2, m - 1, ..., m + 3$) are time-dependent element parameters to be found out. The values of $U_N^r$ and its first and second order derivatives with respect to variable $x$ at the nodal points $x_m$ can be calculated in terms of $\delta_m$

$$
U_N^r(x_m, t) = \left( U_N^r \right)_m = (\delta_m - 2 + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2})
$$

$$
(U_N^r)^\prime_m = U_m^\prime = \frac{5}{h}(-\delta_m - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2})
$$

$$
(U_N^r)^\prime\prime_m = U_m^\prime\prime = \frac{20}{h^2}(\delta_{m+2} + 2\delta_{m+1} + 6\delta_m + 2\delta_{m+1} + \delta_{m+2}).
$$

(3.3)
Eq. (1.2) can be split as follows:

\[ U_t - U_{xxt} - \alpha U_{xx} + \beta U_x = 0 \]  
(3.4)
\[ U_t - U_{xxt} + UU_x = 0. \]  
(3.5)

When the values of \( U \) and its first and second-order derivatives are used in Eqs. (3.4) and (3.5), one obtains the following system of algebraic equations

\[
\begin{align*}
\dot{\delta}_{m-2} + 26\dot{\delta}_{m-1} + 66\dot{\delta}_m + 26\dot{\delta}_{m+1} + \dot{\delta}_{m+2} &- \frac{20}{h^2}(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}) \\
- \frac{20\alpha}{h^2}(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}) + \frac{5\beta}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) & = 0
\end{align*}
\]  
(3.6)

\[
\begin{align*}
\dot{\delta}_{m-2} + 26\dot{\delta}_{m-1} + 66\dot{\delta}_m + 26\dot{\delta}_{m+1} + \dot{\delta}_{m+2} &- \frac{20}{h^2}(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}) \\
+ \frac{5z_m}{h}(\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) & = 0.
\end{align*}
\]  
(3.7)

Here, "\( . \)" symbolizes the first derivative according to time \( t \) and \( z_m \) is considered as a linearization scheme. The values \( \delta_{-2}, \delta_{-1}, \delta_{N+1} \) and \( \delta_{N+2} \) are found from \( U \) and its first derivative. In this case, we use \( z_m \) as an approximate value at the nodal points \( m = 0(1)N \) until the until next time. That is;

\[
\begin{align*}
z_0 &= 0, \\
z_1 &= \delta_{-1} + 26\delta_0 + 66\delta_1 + 26\delta_2 + \delta_3, \\
& \vdots \\
z_{N-1} &= \delta_{N-3} + 26\delta_{N-2} + 66\delta_{N-1} + 26\delta_N + \delta_{N+1}, \\
z_N &= 0.
\end{align*}
\]  
(3.8)

By writing \( \frac{\delta_{m+2}^{n+1} - \delta_{m+1}^n}{\Delta t} \) as Crank-Nicolson approximation instead of the parameter \( \delta_m \) and \( \frac{\delta_{m+1}^{n+1} - \delta_{m+1}^n}{\Delta t} \) as forward difference approximation instead of the parameter \( \dot{\delta}_m \) in Eqs. (3.6) and (3.7), we can acquire a system of algebraic equations (3.9) and (3.10) given in the following

\[
c_1\delta_{m-2}^{n+1} + c_2\delta_{m-1}^{n+1} + c_3\delta_{m+1}^{n+1} + c_4\delta_{m+1}^{n+1} + c_5\delta_{m+2}^{n+1} = c_6\delta_{m-2}^n + c_7\delta_{m-1}^n + c_8\delta_{m}^n + c_9\delta_{m+1}^n + c_{10}\delta_{m+2}^n \]  
(3.9)

\[
d_1\delta_{m-2}^{n+1} + d_2\delta_{m-1}^{n+1} + d_3\delta_{m+1}^{n+1} + d_4\delta_{m+1}^{n+1} + d_5\delta_{m+2}^{n+1} = d_6\delta_{m-2}^n + d_7\delta_{m-1}^n + d_8\delta_{m}^n + d_9\delta_{m+1}^n + d_{10}\delta_{m+2}^n \]  
(3.10)

\[
c_1 = 1 - \frac{20}{h^2} - \frac{10\alpha\Delta t}{h^2} - \frac{5\beta\Delta t}{2h}, c_2 = 26 - \frac{40}{h^2} - \frac{20\alpha\Delta t}{h^2} - \frac{25\beta\Delta t}{h}, c_3 = 66 + \frac{120}{h^2} + \frac{60\alpha\Delta t}{h^2}, c_4 = 26 - \frac{40}{h^2} - \frac{20\alpha\Delta t}{h^2} + \frac{25\beta\Delta t}{h},
\]  
(541)
in the following form, found from the initial condition and its derivatives in Eq. (3)

Therefore, to explain in more detail, this parameter is computed from the system of algebraic equations presented in Eq. (3.12). Then we are able to obtain matrix system \((N+1)\times(N+1)\) for each of the above systems. The systems (3.9) and (3.10) are solved by Strang time-splitting technique and an inner iteration presented by form 

The systems (3.9) and (3.10) include \((N+5)\) unknowns and \((N+1)\) equations. Since each system has dummy parameters \(\delta_{-2}, \delta_{-1}\) and \(\delta_{N+1}, \delta_{N+2}\) which stand not inside the solution area, we first need to eliminate these illusory ones in order to obtain a unique solution of each system. For this reason, we have to use \(U\) and \(U'\) in Eq. (3.3) and the boundary conditions \(U(x_L,t) = U(x_R,t) = 0\) and \(U_x(x_L,t) = U_x(x_R,t) = 0\) in Eq. (1.4). Then we are able to obtain matrix system \((N+1)\times(N+1)\) for each of the above systems. The systems (3.9) and (3.10) are solved by Strang time-splitting technique and an inner iteration presented by form 

The initial vector \(\delta_0^m\) is needed to solve systems (3.9) and (3.10). To calculate this initial vector, we can use the condition given at initial time 

with the conditions given at the boundaries 

Therefore, to explain in more detail, this parameter is computed from the system of algebraic equations presented in the following form, found from the initial condition and its derivatives in Eq. (3)

\[
\begin{align*}
c_5 &= 1 - \frac{20}{h^2} \frac{\alpha \Delta t}{h^2} + \frac{5 \beta \Delta t}{2h}, \\
c_6 &= 1 - \frac{20}{h^2} \frac{\alpha \Delta t}{h^2} + \frac{5 \beta \Delta t}{2h}, \\
c_9 &= 26 - \frac{40}{h^2} + \frac{20 \alpha \Delta t}{h^2} + \frac{25 \beta \Delta t}{h}, \\
c_{10} &= 1 - \frac{20}{h^2} + \frac{10 \alpha \Delta t}{h^2} - \frac{5 \beta \Delta t}{2h}, \\

\delta_1 &= 1 - \frac{20}{h^2} - \frac{5 \delta_{m} \Delta t}{2h}, \\
d_2 &= 26 - \frac{40}{h^2} - \frac{25 \delta_{m} \Delta t}{h}, \\
d_3 &= 66 + \frac{120}{h^2}, \\
d_4 &= 26 - \frac{40}{h^2} + \frac{25 \delta_{m} \Delta t}{h}, \\
d_5 &= 1 - \frac{20}{h^2} + \frac{5 \delta_{m} \Delta t}{2h}, \\
d_6 &= 1 - \frac{20}{h^2} + \frac{5 \delta_{m} \Delta t}{2h}, \\
d_7 &= 26 - \frac{40}{h^2} + \frac{25 \delta_{m} \Delta t}{h}, \\
d_8 &= 66 + \frac{120}{h^2}, \\
d_9 &= 26 - \frac{40}{h^2} - \frac{25 \delta_{m} \Delta t}{h}, \\
d_{10} &= 1 - \frac{20}{h^2} - \frac{5 \delta_{m} \Delta t}{2h}.
\end{align*}
\]
Therefore, the matrix equation for the initial vector $\delta^0$ is finally obtained as

\[
\begin{bmatrix}
54 & 60 & 6 & 1 \\
25.25 & 67.5 & 26.25 & 1 \\
1 & 26 & 66 & 26 & 1 \\
1 & 26 & 66 & 26 & 1 \\
1 & 26.25 & 67.5 & 25.25 & 1 \\
6 & 60 & 54 & & \\
\end{bmatrix}
\begin{bmatrix}
\delta^0_0 \\
\delta^0_1 \\
\delta^0_2 \\
\vdots \\
\delta^0_{N-2} \\
\delta^0_{N-1} \\
\delta^0_N \\
\end{bmatrix}
= 
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_{N-2} \\
U_{N-1} \\
U_N \\
\end{bmatrix}.
\]

4. Stability analysis

Here, we are going to investigate the stability analysis of numerical schemes (3.9) and (3.10) by von Neumann method [49]. Firstly, for the half-time step, writing the expressions $c_1, c_2, ..., c_9, c_{10}$ that we find in Section 3 and the expression $\delta^0_n = e^{i\gamma_m h} \zeta^n$ in Eq. (3.9), we obtain equations in the following form after the necessary operations

\[
\sigma_1 = \sigma_A \left( \frac{n+2}{\zeta^n} \right) = \frac{A_1 + iB}{A_2 - iB},
\]

\[
A_1 = (c_1 + c_5)\cos(2\gamma h) + (c_2 + c_4)\cos(\gamma h) + c_3,
\]

\[
A_2 = (c_6 + c_{10})\cos(2\gamma h) + (c_7 + c_9)\cos(\gamma h) + c_8,
\]

\[
B = \frac{5\beta\Delta t}{h} \sin(2\gamma h) + \frac{50\beta\Delta t}{h} \sin(\gamma h),
\]

where

\[
(c_1 + c_5) = 2 - \frac{40}{h^2} - \frac{20\alpha\Delta t}{h^2}, \quad (c_2 + c_4) = 52 - \frac{80}{h^2} - \frac{40\alpha\Delta t}{h^2},
\]

\[
(c_6 + c_{10}) = 2 - \frac{40}{h^2} + \frac{20\alpha\Delta t}{h^2}, \quad (c_7 + c_9) = 52 - \frac{80}{h^2} + \frac{40\alpha\Delta t}{h^2},
\]

\[
c_3 = 66 + \frac{120}{h^2} + \frac{60\alpha\Delta t}{h^2}, \quad c_8 = 66 + \frac{120}{h^2} - \frac{60\alpha\Delta t}{h^2}.
\]

From Eq. (4.1), it can be clearly written that

\[
|\sigma_1|^2 = \sigma_1 \sigma_1^* = \frac{A_1^2 + B^2}{A_2^2 + B^2}
\]

and we can say that $A_1 \leq A_2$, hence $|\sigma_1| \leq 1$. This confirms that numerical scheme (3.9) is unconditionally stable. Secondly, to implement the von Neumann method to Eq. (3.10), first of all, when the $U$ in the nonlinear term $UU_x$ in system (16) is linearized, $z_m$ be going to behave as a local constant, that is, $z_m$ in system (3.10) will be a fixed number. Thus the von Neumann method will become applicable to research the stability of
numerical scheme (3.10). Writing the expressions \( d_1, d_2, \ldots, d_9, d_{10} \) that we find in section 3 and \( \delta_m^n = e^{i\gamma mh} \zeta^n \) in scheme (3.10), we obtain equations in the following form

\[
\sigma_2 = \sigma_B \left( \frac{\zeta^{n+1}}{\zeta^n} \right) = \frac{A_3 + iD}{A_4 - iD} \tag{4.2}
\]

\[
A_3 = (d_1 + d_5)\cos(2\gamma h) + (d_2 + d_4)\cos(\gamma h) + d_3,
\]

\[
A_4 = (d_6 + d_{10})\cos(2\gamma h) + (d_7 + d_9)\cos(\gamma h) + d_8,
\]

\[
D = \frac{5\zeta_m \Delta t}{h} \sin(2\gamma h) + \frac{50\zeta_m \Delta t}{h} \sin(\gamma h),
\]

where

\[
(d_6 + d_{10}) = (d_1 + d_5) = 2 - \frac{40}{h^2},
\]

\[
(d_7 + d_9) = (d_2 + d_4) = 52 - \frac{80}{h^2},
\]

\[
d_3 = d_8 = 66 + \frac{120}{h^2}.
\]

Here it is clear that \( A_3 = A_4 \). Hence, \( |\sigma_2| \leq 1 \) and as a result, we can state that the scheme (3.10) is unconditionally stable since the condition \( |\sigma_2| \leq 1 \) from Eq. (4.2) is satisfied. In that case, it can be clearly seen that

\[
|\rho(\zeta)| \leq |\sigma_1||\sigma_2||\sigma_1|
\]

and consequently schemes (3.9) and (3.10) obtained by Strang time-splitting technique are unconditionally stability.

5. Numerical examples and results

In the present section, we will handle four model problems to visually present the performance and effectiveness of the proposed method. The numerical solution of Equations (3.9) and (3.10) obtained by the second-order Strang splitting algorithms bring about the \((N + 1) \times (N + 1)\) matrix system that are easily and effectively calculated by means of Thomas algorithm. All computer calculations in the study have been carried out with the software Mathlab 2019b on a computer which has a memory 20GB and 64 bit. In order to determine how good the obtained results are, the error norms \(L_2\) and \(L_\infty\) characterized in the following are computed

\[
L_2 = ||U - U_N||_2 = \sqrt{\frac{1}{h} \sum_{j=0}^{N} (U - U_N)^2}
\]

\[
L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|.
\]
The BBMB equation given with appropriate initial and boundary conditions has the following three conservation constants

\[ I_1 = \int_{x_L}^{x_R} U(x,t) \, dx, \]
\[ I_2 = \int_{x_L}^{x_R} [U^2(x,t) + U_x(x,t)] \, dx, \]
\[ I_3 = \int_{x_L}^{x_R} [U^3(x,t) + 3U_x^2(x,t)] \, dx, \]

standing for mass, energy, momentum, respectively. These proportions can be watched in order to control the protected features of the numerical algorithm.

Example 5.1 For the first example, we are going to consider the present equation having the initial condition \( U(x,0) = \exp(-x^2) \) taking \( \alpha = \beta = 1 \) in Eq. (1.2) on the domain \([-10, 10]\). Figures 1 and 2 display graphically the numerical approach of Example 5.1 for \( \Delta t = 0.01 \), \( N = 100 \) at times \( t \leq 10 \) and these graphs exhibit the same physical behaviour as in those of [41], [1].

![Figure 1](image1.png)  
**Figure 1.** Approximate solutions of Example 5.1 at times \( t \in [0, 3] \) for parameters \( \alpha = \beta = 1 \).

![Figure 2](image2.png)  
**Figure 2.** Approximate solutions of Example 5.1 at times \( t \in [4, 10] \) for parameters \( \alpha = \beta = 1 \).

Example 5.2 For the second test problem, we are going to consider the nonhomogeneous BBM-Burgers equation in form

\[ U_t - U_{xxt} - \alpha U_{xx} + \beta U_x + UU_x = F, \quad x \in [0, \pi], t \in [0, T] \]  

in which \( F(x,t) = \exp(-t)[\cos x - \sin x + \frac{1}{2} e^{-t} \sin(2x)] \). The analytical solution of Eq. (5.2) is presented with \( U(x,t) = e^{-t} \sin x \) as in [57] and here IC and BCs can be gotten from the analytical solution. All numerical calculations are done for \( \Delta t = 0.01 \) and \( \alpha = \beta = 1 \) over the range \([0, \pi]\). Table 1 gives a comparison of the computed \( L_2 \) error norm for different values of \( N \) at \( t = 10 \) as in those of [41], [1] and [57]. Table 2 shows a comparison of the error norms \( L_2 \) and \( L_{\infty} \) for \( N = 121 \) and at various time-levels as given in those of [1] and [2]. It can be clearly said from Tables 1 and 2 that the approximate results obtained by the second-order
Strang scheme are quite small. That is, our results are the lowest of all of ones we compare in tables and hence much closer to the exact solutions. Figure 3 shows graphically the numerical approach of Example 5.2 at times \( t \leq 8 \) and this graph exhibits the same physical behaviour as in [1] and [2].

### Table 1. A comparison of the error norm results \( L_2 \) of Example 5.2 for various values of \( N \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>Present [1]</th>
<th>[1]</th>
<th>[57]</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0218e - 0</td>
<td>1.7147e - 4</td>
<td>2.9703e - 4</td>
</tr>
<tr>
<td>20</td>
<td>0.0053e - 0</td>
<td>5.6341e - 5</td>
<td>1.1446e - 4</td>
</tr>
<tr>
<td>40</td>
<td>0.0013e - 0</td>
<td>–</td>
<td>4.9603e - 5</td>
</tr>
<tr>
<td>80</td>
<td>0.3209e - 4</td>
<td>7.2635e - 6</td>
<td>2.327e - 5</td>
</tr>
<tr>
<td>160</td>
<td>0.8133e - 5</td>
<td>–</td>
<td>1.275e - 5</td>
</tr>
<tr>
<td>320</td>
<td>2.0706e - 5</td>
<td>8.1631e - 7</td>
<td>5.5619e - 6</td>
</tr>
<tr>
<td>640</td>
<td>5.1989e - 6</td>
<td>–</td>
<td>2.7632e - 6</td>
</tr>
</tbody>
</table>

### Table 2. A comparison of the error norm results \( L_2 \) and \( L_\infty \) of Example 5.2 at various time-levels.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( L_2 )</th>
<th>( L_\infty )</th>
<th>( L_2 )</th>
<th>( L_\infty )</th>
<th>( L_2 )</th>
<th>( L_\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.13E - 03</td>
<td>7.07E - 03</td>
<td>1.51E - 03</td>
<td>1.46E - 03</td>
<td>3.01e - 05</td>
<td>1.34e - 04</td>
</tr>
<tr>
<td>2</td>
<td>1.73E - 03</td>
<td>2.84E - 03</td>
<td>1.21E - 03</td>
<td>1.18E - 03</td>
<td>1.59e - 05</td>
<td>5.47e - 05</td>
</tr>
<tr>
<td>4</td>
<td>2.12E - 04</td>
<td>3.84E - 04</td>
<td>3.95E - 04</td>
<td>3.96E - 04</td>
<td>6.62e - 06</td>
<td>8.90e - 06</td>
</tr>
<tr>
<td>10</td>
<td>4.08E - 06</td>
<td>4.06E - 06</td>
<td>2.75E - 06</td>
<td>3.22E - 06</td>
<td>2.54e - 07</td>
<td>2.21e - 07</td>
</tr>
</tbody>
</table>

**Figure 3.** Approximate solutions of Example 5.2 at times \( t \leq 5 \) for parameters \( \alpha = \beta = 1 \).

**Example 5.3** For the third example, we are going to consider to the numerical approach of Eq. (1.2) presented with the initial condition \( U = (x, 0) = sech^2 \left[ \frac{x}{4} \right] \) for \( \alpha = 0, \beta = 1 \). The analytical solution of this problem is given as in [57]

\[
U(x, t) = sech^2 \left[ \frac{x}{4} - \frac{t}{3} \right].
\]
Here, we perform a comparison of the solutions of the presented method with the earlier studies in literature for various position and time steps up to $t = 40$ over the region $[-40, 100]$. For this, the error norm values $L_2$ and $L_\infty$ with invariants $I_1$, $I_2$, and $I_3$ are computed. The computed values are displayed in Tables 3 and 4. These tables clearly exhibit that the results achieved by our scheme are much smaller than those of the others and invariants $I_1$, $I_2$, and $I_3$ are almost the same at increasing time values. This situation shows consistency between the obtained invariants and their exact values. Figure 4 displays the numerical solutions obtained with Strang time-splitting technique at times $t = 0, 10, 20, 30$ and 40 for using parameters $h = 0.05$, $\Delta t = 0.025$. From this figure, it is clear that the single solitary waves go toward right having a constant velocity and protects its amplitude and shape as time progresses. At the beginning, we can say that the single solitary wave whose amplitude is 1 at time $t = 0$, position $x = 0$ and has the same amplitude at time $t = 40$, position $x = 53.35$. Also, the error distribution graph is shown in Figure 5 for the parameter $h = 0.05$, $\Delta t = 0.025$ at time $t = 40$.

Table 3. A comparison of the error norms $L_2$ and $L_\infty$ at various times on the range $[40, 100]$ of Example 5.3.

<table>
<thead>
<tr>
<th></th>
<th>$t = 10$</th>
<th>$t = 20$</th>
<th>$t = 30$</th>
<th>$t = 40$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0.2, \Delta t = 0.4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present $L_2$</td>
<td>0.00041695</td>
<td>0.00073104</td>
<td>0.00099458</td>
<td>0.01238753</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.00019018</td>
<td>0.00030898</td>
<td>0.00040563</td>
<td>0.00049613</td>
</tr>
<tr>
<td>[40] $L_\infty$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.10976282</td>
</tr>
<tr>
<td>[41] $L_2$</td>
<td>0.03195399</td>
<td>0.05446985</td>
<td>0.07306022</td>
<td>0.09025102</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.01477190</td>
<td>0.02321340</td>
<td>0.03003074</td>
<td>0.03638003</td>
</tr>
<tr>
<td>$h = \Delta t = 0.1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present $L_2$</td>
<td>0.00026149</td>
<td>0.00045867</td>
<td>0.00062439</td>
<td>0.00077763</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.00011930</td>
<td>0.00019395</td>
<td>0.00025496</td>
<td>0.00031177</td>
</tr>
<tr>
<td>[40] $L_\infty$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.00474257</td>
</tr>
<tr>
<td>[41] $L_2$</td>
<td>0.00204484</td>
<td>0.00341396</td>
<td>0.00457929</td>
<td>0.00571248</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.00095720</td>
<td>0.00147163</td>
<td>0.00189531</td>
<td>0.00231941</td>
</tr>
<tr>
<td>$h = 0.05, \Delta t = 0.025$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present $L_2$</td>
<td>0.00001635</td>
<td>0.00002867</td>
<td>0.00003904</td>
<td>0.00004862</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.00000746</td>
<td>0.00001213</td>
<td>0.00001594</td>
<td>0.00001949</td>
</tr>
<tr>
<td>[40] $L_\infty$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.00046983</td>
</tr>
<tr>
<td>[41] $L_2$</td>
<td>0.00012459</td>
<td>0.00025628</td>
<td>0.00040853</td>
<td>0.00055868</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.00005984</td>
<td>0.00010502</td>
<td>0.00016208</td>
<td>0.00021891</td>
</tr>
<tr>
<td>$h = 0.2, \Delta t = 0.01$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present $L_2$</td>
<td>0.00000304</td>
<td>0.00000513</td>
<td>0.00000689</td>
<td>0.00000851</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.00000145</td>
<td>0.00000223</td>
<td>0.00000288</td>
<td>0.00000349</td>
</tr>
<tr>
<td>[41] $L_2$</td>
<td>-</td>
<td>0.00060007</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>-</td>
<td>0.00031641</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>[41] $L_2$</td>
<td>0.00012676</td>
<td>0.00077372</td>
<td>0.00147163</td>
<td>0.00261489</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>0.00017784</td>
<td>0.00031687</td>
<td>0.00045923</td>
<td>0.00060343</td>
</tr>
</tbody>
</table>

Example 5.4 For the fourth example, we are going to consider for the approximate solution of Equation (1.2) with $\alpha = 0, \beta = 1$. The analytical solution for the test problem can be given as in [57]

$$U = (x, t) = 3\csech^2[k(x - x_0 - vt)]$$

in which $v = 1+c$ stand for the wave speed and $k = \frac{1}{2\sqrt{\frac{c}{1+c}}}$. This solution represent the movement of a single
solitary wave with amplitude $3c$ with width $k$ and initially centered at $x_0$. To be able to do a comparison with the earlier studies in literature, the error norm values $L_2$ and $L_\infty$ with invariants $I_1, I_2$ and $I_3$ are computed with the parameters $h = \Delta t = 0.1$ for $c = 0.1$ at the different times over the region $[-40, 60]$. The exact values

Table 4. A comparison of the invariants $I_1$, $I_2$ and $I_3$ at various times on the range $[40, 100]$ of Example 5.3.

<table>
<thead>
<tr>
<th>$h = 0.2, \Delta t = 0.4$</th>
<th>$t = 10$</th>
<th>$t = 20$</th>
<th>$t = 30$</th>
<th>$t = 40$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>$I_1$</td>
<td>$I_2$</td>
<td>$I_3$</td>
<td>$I_4$</td>
</tr>
<tr>
<td>$I_1$</td>
<td>7.99999991</td>
<td>7.99999981</td>
<td>7.99999972</td>
<td>7.99999963</td>
</tr>
<tr>
<td>$I_2$</td>
<td>5.59999982</td>
<td>5.59999965</td>
<td>5.59999947</td>
<td>5.59999930</td>
</tr>
<tr>
<td>$I_3$</td>
<td>20.2666611</td>
<td>20.2666560</td>
<td>20.2666530</td>
<td>20.2666512</td>
</tr>
<tr>
<td>[31]</td>
<td>$I_1$</td>
<td>$I_2$</td>
<td>$I_3$</td>
<td>$I_4$</td>
</tr>
<tr>
<td>$I_1$</td>
<td>8.0000005</td>
<td>8.0000001</td>
<td>7.9999998</td>
<td>7.9999995</td>
</tr>
<tr>
<td>$I_2$</td>
<td>5.6000015</td>
<td>5.6000036</td>
<td>5.6000063</td>
<td>5.6000071</td>
</tr>
<tr>
<td>$I_3$</td>
<td>20.2664360</td>
<td>20.2662535</td>
<td>20.2661684</td>
<td>20.2661288</td>
</tr>
</tbody>
</table>

$h = \Delta t = 0.1$

| Present                   | $I_1$   | $I_2$   | $I_3$   | $I_4$   |
| $I_1$                     | 8.00000007 | 8.00000013 | 8.00000018 | 8.00000024 |
| $I_2$                     | 5.60000000 | 5.60000000 | 5.60000000 | 5.60000000 |
| $I_3$                     | 20.2666667 | 20.2666667 | 20.2666666 | 20.2666666 |
| [31]                      | $I_1$   | $I_2$   | $I_3$   | $I_4$   |
| $I_1$                     | 7.99999964 | 7.99999964 | 7.99999964 | 7.99999964 |
| $I_2$                     | 5.6000010 | 5.6000010 | 5.6000010 | 5.6000010 |
| $I_3$                     | 20.2666706 | 20.2666706 | 20.2666706 | 20.2666706 |

$h = 0.05, \Delta t = 0.025$

| Present                   | $I_1$   | $I_2$   | $I_3$   | $I_4$   |
| $I_1$                     | 8.00000006 | 8.00000011 | 8.00000017 | 8.000000022 |
| $I_2$                     | 5.60000000 | 5.60000000 | 5.60000000 | 5.60000000 |
| $I_3$                     | 20.2666667 | 20.2666667 | 20.2666667 | 20.2666667 |
| [31]                      | $I_1$   | $I_2$   | $I_3$   | $I_4$   |
| $I_1$                     | 8.0000009 | 8.0000009 | 8.0000009 | 8.0000009 |
| $I_2$                     | 5.6000005 | 5.6000010 | 5.6000011 | 5.6000012 |
| $I_3$                     | 20.2666697 | 20.2666713 | 20.2666719 | 20.2666721 |

Figure 4. Approximate solutions of Example 5.3 at various times for $h = 0.05, \Delta t = 0.025$ on the solution domain $[-40, 100]$.

Figure 5. The error distribution graph of Example 5.3 at various times for $\Delta t = 0.025, h = 0.05$ over the region $[-40, 100]$. 


for the conservative constants are $I_1 = 3.9799497$, $I_2 = 0.81046249$ and $I_3 = 2.579007$ for $c = 0.1$. These calculated values are indicated in Tables 5 and 6. This tables clearly exhibit that the results achieved by our scheme are much smaller than those of the others and invariants $I_1, I_2$ and $I_3$ are almost the same at increasing time values. This situation shows consistency between the obtained invariants and their exact values. Figure 6 displays the numerical solutions obtained with Strang time-splitting technique at various times $t = 0, 5, 10, 15$ and $20$. From this figure, it is clear that the single solitary waves go to right with a fixed velocity and protects its amplitude and shape as time progresses. At the beginning, we can say that the single solitary wave whose amplitude is $0.3$ at time $t = 0$, position $x = 0$ and has the same amplitude at time $t = 20$, position $x = 22$. Also, the error distribution graph is shown in Figure 7 at time $t = 20$.

Table 5. A comparison of error norms $L_2$ and $L_\infty$ for $\Delta t = h = 0.1$, $c = 0.1$ at various times on the range $[-40, 60]$ for Example 5.4.

<table>
<thead>
<tr>
<th>Method</th>
<th>Error</th>
<th>$t$</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>$L_2 \times 10^3$</td>
<td>0.00829132</td>
<td>0.01652186</td>
<td>0.02400207</td>
<td>0.03060004</td>
<td>0.03710617</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.00436956</td>
<td>0.00556405</td>
<td>0.00662545</td>
<td>0.00875511</td>
<td>0.01268449</td>
<td></td>
</tr>
<tr>
<td>[22]</td>
<td>$L_2 \times 10^3$</td>
<td>39.82</td>
<td>79.46</td>
<td>118.8</td>
<td>157.7</td>
<td>196.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>13.74</td>
<td>27.66</td>
<td>41.35</td>
<td>54.60</td>
<td>67.35</td>
<td></td>
</tr>
<tr>
<td>[15]</td>
<td>$L_2 \times 10^3$</td>
<td>0.0096</td>
<td>0.0026</td>
<td>0.0064</td>
<td>0.0115</td>
<td>0.0184</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.1458</td>
<td>0.5786</td>
<td>0.9223</td>
<td>1.2148</td>
<td>1.5664</td>
<td></td>
</tr>
<tr>
<td>[16]</td>
<td>$L_2 \times 10^3$</td>
<td>0.116</td>
<td>0.224</td>
<td>0.325</td>
<td>0.417</td>
<td>0.511</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.054</td>
<td>0.100</td>
<td>0.139</td>
<td>0.171</td>
<td>0.198</td>
<td></td>
</tr>
<tr>
<td>[17]</td>
<td>$L_2 \times 10^3$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.116</td>
<td></td>
</tr>
<tr>
<td>[32]</td>
<td>$L_2 \times 10^3$</td>
<td>0.12</td>
<td>0.23</td>
<td>0.34</td>
<td>0.45</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.05</td>
<td>0.09</td>
<td>0.14</td>
<td>0.18</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>[20]</td>
<td>$L_2 \times 10^3$</td>
<td>0.048</td>
<td>0.094</td>
<td>0.138</td>
<td>0.180</td>
<td>0.219</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.019</td>
<td>0.038</td>
<td>0.056</td>
<td>0.071</td>
<td>0.086</td>
<td></td>
</tr>
<tr>
<td>[1]</td>
<td>$L_2 \times 10^3$</td>
<td>0.042</td>
<td>0.033</td>
<td>0.13</td>
<td>0.16</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.015</td>
<td>0.033</td>
<td>0.049</td>
<td>0.064</td>
<td>0.078</td>
<td></td>
</tr>
<tr>
<td>[50]</td>
<td>$L_2 \times 10^3$</td>
<td>0.0149</td>
<td>0.0271</td>
<td>0.0364</td>
<td>0.0429</td>
<td>0.0476</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.0060</td>
<td>0.0105</td>
<td>0.0132</td>
<td>0.0145</td>
<td>0.0150</td>
<td></td>
</tr>
<tr>
<td>[56]</td>
<td>$L_2 \times 10^3$</td>
<td>0.0646</td>
<td>0.1282</td>
<td>0.1901</td>
<td>0.2498</td>
<td>0.3072</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.0250</td>
<td>0.0505</td>
<td>0.0747</td>
<td>0.0971</td>
<td>0.1177</td>
<td></td>
</tr>
<tr>
<td>[53]</td>
<td>$L_2 \times 10^3$</td>
<td>0.0169</td>
<td>0.0329</td>
<td>0.0474</td>
<td>0.06039</td>
<td>0.0723</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.0072</td>
<td>0.0141</td>
<td>0.0199</td>
<td>0.0247</td>
<td>0.0288</td>
<td></td>
</tr>
<tr>
<td>[31]</td>
<td>$L_2 \times 10^3$</td>
<td>0.0475</td>
<td>0.0929</td>
<td>0.1365</td>
<td>0.1773</td>
<td>0.2162</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.0188</td>
<td>0.0379</td>
<td>0.0553</td>
<td>0.0706</td>
<td>0.0846</td>
<td></td>
</tr>
<tr>
<td>[57]</td>
<td>$L_2 \times 10^3$</td>
<td>0.0203</td>
<td>0.0383</td>
<td>0.0525</td>
<td>0.0630</td>
<td>0.0719</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_\infty \times 10^3$</td>
<td>0.0084</td>
<td>0.0160</td>
<td>0.0210</td>
<td>0.0241</td>
<td>0.0270</td>
<td></td>
</tr>
</tbody>
</table>

6. Conclusion
In this work, the approximate solution of the BBM-Burgers equation with convenient initial and boundary conditions is obtained using Strang time-splitting technique via finite element collocation method (FEM) with quintic B-spline. The efficiency and accuracy of the present method are demonstrated on four examples. The
The new results achieved from the numerical approach show that the error norms have improved. The results produced in the article have been compared with some existing studies in the literature in the table below.

Table 6. A comparison of $I_1, I_2$ and $I_3$ invariants for $c = 0.1$, $\Delta t = h = 0.1$ at various times on the range $[40, 60]$ for Example 5.4.

| $t$   | Present $I_1$ | $I_2$ | $I_3$ | $I_1$ | $I_2$ | $I_3$ | $I_1$ | $I_2$ | $I_3$ | $I_1$ | $I_2$ | $I_3$ | $I_1$ | $I_2$ | $I_3$ |
|-------|---------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|       | 3.97992702    | 0.81046249 | 2.57900744 | −     | 0.899873 | −     | 0.89839 | 3.97993 | 0.810610 | 2.57901 | 2.57901 | 3.97993 | 0.810465 | 2.57901 | 2.57901 |
|       | 3.97995533    | 0.81046249 | 2.57900744 | 4.24017 | 0.899236 | 2.86339 | 3.98039 | 0.810752 | 2.57996 | 2.57991 | 3.97995 | 0.810465 | 2.579901 | 2.579901 |
|       | 3.9797867     | 0.81046249 | 2.57900744 | 4.41822 | 0.889601 | 2.86106 | 3.98083 | 0.810884 | 2.58041 | 2.58041 | 3.97978 | 0.810465 | 2.58041 | 2.58041 |
|       | 3.97999973    | 0.81046249 | 2.57900744 | 4.41623 | 0.897967 | 2.85863 | 3.98125 | 0.810752 | 2.58041 | 2.58041 | 3.97999 | 0.810465 | 2.58041 | 2.58041 |
|       | 3.9801245     | 0.81046249 | 2.57900744 | 4.41423 | 0.897967 | 2.85613 | 3.98165 | 0.810752 | 2.58041 | 2.58041 | 3.98012 | 0.810465 | 2.58041 | 2.58041 |
|       | 3.9799912     | 0.81046249 | 2.57900744 | 4.41219 | 0.897967 | 2.85301 | 3.98206 | 0.810752 | 2.58041 | 2.58041 | 3.97999 | 0.810465 | 2.58041 | 2.58041 |

Figure 6. Approximate solutions of Example 5.4 for $h = \Delta t = c = 0.1$ over the region $[-40, 60]$.

Figure 7. The error distribution graph of Example 5.4 for $h = \Delta t = c = 0.1$ over region $[-40, 60]$.

The error norms $L_2$ and $L_\infty$ with invariants $I_1, I_2$ and $I_3$ are calculated and compared by other studies in the literature. The new results achieved from the numerical approach show that the error norms $L_2$ and $L_\infty$ are fairly small and invariants $I_1, I_2$ and $I_3$ reasonably protect and remain almost constant during computer run time. The results produced in the article have been compared with some existing studies in the literature in
the form of tables and graphics, and it can be said that the approach used in our study produces very good results. As a result, we can easily say that this technique is easy to apply and useful in obtaining more accurate numerical results.

References


