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## A note on $ss$ -supplement submodules

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**Abstract:** In this paper, we describe  $ss$ -supplement submodules in terms of a special class of endomorphisms. Let  $R$  be a ring with semisimple radical and  $P$  be a projective  $R$ -module. We show that there is a bijection between  $ss$ -supplement submodules of  $P$  and  $ss$ -supplement submodules of  $End_R(P)$ . Moreover, we define radical-s-projective modules as a generalization of projective modules. We prove that every  $ss$ -supplement submodule of a projective  $R$ -module is radical-s-projective over the ring  $R$  with semisimple radical. We show that over  $SSI$ -ring  $R$ , every radical-s-projective  $R$ -module is projective. We provide that over a ring  $R$  with semisimple radical, every  $ss$ -supplement submodule of a projective  $R$ -module is a direct summand if and only if every radical-s-projective  $R$ -module is projective.

**Key words:**  $ss$ -Supplement, radical-s-projective modules, endomorphism rings

### 1. Introduction

Throughout this study,  $R$  is associative ring with identity, all modules are unital left  $R$ -modules and homomorphisms operate on the right. Let  $M$  be an  $R$ -module. The notation  $U \leq M$  means that  $U$  is a submodule of  $M$ . A submodule  $U$  of  $M$  is called *small* in  $M$ , denoted as  $U \ll M$ , if  $M \neq U + K$  for every proper submodule  $K$  of  $M$ . An  $R$ -module homomorphism  $f : M \rightarrow N$  is called a *small epimorphism* if it is an epimorphism with  $Ker(f) \ll M$ . A projective module  $P$  together with a small epimorphism  $f : P \rightarrow M$  is called *projective cover* of  $M$ .  $End_R(M)$  indicates the endomorphism ring of an  $R$ -module  $M$ . By  $Rad(M)$ , we denote the intersection of all maximal submodules of  $M$ , equivalently the sum of all small submodules of  $M$ . Moreover,  $Soc(M)$  stands for the socle of a module  $M$ , i.e. the sum of all simple submodules of  $M$ . Note that  $Soc(M)$  is the largest semisimple submodule of a module  $M$  (see [13]).

Let  $M$  be a module and  $U, V$  be submodules of  $M$ .  $V$  is called a *supplement* of  $U$  in  $M$  if  $V$  is minimal element in the set of submodules  $L \leq M$  with  $M = U + L$ , equivalently  $M = U + V$  and  $U \cap V \ll V$  (see [13]).

Following [14], the sum of all simple submodules of  $M$  which are small in  $M$  is named with  $Soc_s(M)$  for a module  $M$ , that is,  $Soc_s(M) = \sum\{U \ll M \mid U \text{ is simple}\}$ . Denote that  $Soc_s(M) \leq Soc(M)$  and  $Soc_s(M) \leq Rad(M)$ . In [7], a submodule  $V$  is called an  *$ss$ -supplement* of  $U$  in a module  $M$  if  $M = U + V$  and  $U \cap V \leq Soc_s(V)$ . In the same paper, it is proved that  $V$  is an  $ss$ -supplement of  $U$  in  $M$  if and only if  $M = U + V$ ,  $U \cap V \ll V$  and  $U \cap V$  is semisimple if and only if  $M = U + V$ ,  $U \cap V \leq Rad(V)$  and  $U \cap V$  is semisimple. Now it is clear that the following implication on submodules of a module holds:

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direct summands  $\implies$   $ss$ -supplement submodules  $\implies$  supplement submodules

In [4], an element  $x$  of a ring  $R$  is called *left (CE) element* if there exists  $a \in R$  such that  $ax^2 = x$  and  $x - x^2 \in \text{Rad}(R)$ . An element  $x$  of a ring  $R$  is called *weak left (CE) element* if there exists  $b \in R$  such that  $bx^2 = x$  and for every  $b' \in R$  with  $b'x^2 = x$ , there exists  $c \in R$  with  $cb'x = x$ . Note that every left (CE) element is a weak left (CE) element. The author showed that there is a relationship between left (CE) elements of endomorphism ring of a projective module  $M$  and supplement submodules of  $M$ . In addition, he described radical-projective modules as a generalization of projective modules. A module  $M$  is called *radical-projective* if for every  $R$ -module epimorphism  $g : A \rightarrow B$  and a homomorphism  $f : M \rightarrow B$ , there exists a homomorphism  $h : M \rightarrow A$  such that  $(M)(f - hg) \ll B$ . The obtained results in [4] allow to extend some of the characterizations of the rings for which every supplement submodule of a finitely generated projective module is a direct summand to those in which every supplement submodule of a (nonnecessarily finitely generated) projective module is a direct summand.

In this paper, we start by searching what kind of connection there is between  $ss$ -supplements of a projective module  $M$  over the ring with semisimple radical and left (CE) elements of the ring  $\text{End}_R(M)$ . We show that  $ss$ -supplement submodules in a projective module  $M$  over the ring with semisimple radical are of the form  $\text{Im}(f)$  for a left (CE) endomorphism  $f$  of  $M$ .

In Section 3, we define radical-s-projective modules as a generalization of projective modules. We prove that every  $ss$ -supplement submodule of a projective module is radical-projective. We also provide that  $ss$ -supplements in projective modules are radical-s-projective over the rings with semisimple radical. Over a ring with semisimple radical, we show that for a radical-s-projective module  $M$ ,  $\text{Rad}(M) = \text{Rad}(R)M$ ,  $\text{Rad}(M)$  is semisimple and  $ss$ -supplements in  $M$  are radical-s-projective. We give some examples of projective modules whose  $ss$ -supplements are direct summands. Moreover, we give an example of a ring whose every radical-s-projective module is projective.

## 2. $ss$ -Supplement submodules in projective modules

First of all, we state the rings whose every submodule is an  $ss$ -supplement in the following proposition.

**Proposition 2.1** *The following statements are equivalent for a ring  $R$ :*

1.  $R$  is a semisimple ring.
2. Every submodule of  ${}_R R$  is an  $ss$ -supplement in  ${}_R R$ .
3. Every submodule of  ${}_R R$  is a supplement in  ${}_R R$ .

**Proof** (1)  $\implies$  (2) and (2)  $\implies$  (3) are clear.

(3)  $\implies$  (1) follows from [9, Lemma 2.13]. □

In general, every supplement submodule of a projective module is not an  $ss$ -supplement submodule. For example; consider the ring  $R = \mathbb{Z}_8$ . The projective module  ${}_R R$  is a supplement of  $\text{Rad}({}_R R)$  in  ${}_R R$ ; but  ${}_R R$  is not an  $ss$ -supplement.

**Lemma 2.2** *Let  $R$  be a ring with semisimple  $\text{Rad}(R)$ ,  $P$  be a projective left  $R$ -module and  $K, L \leq P$ . Then  $K$  is an  $ss$ -supplement of  $L$  in  $P$  if and only if  $K$  is a supplement of  $L$  in  $P$ .*

**Proof** The necessity is clear.

Conversely, suppose that  $K$  is a supplement of  $L$  in  $P$ . Then  $P = K + L$  and  $K \cap L \ll K$ . Thus, we have that  $K \cap L \leq \text{Rad}(P) = \text{Rad}(R)P \leq \text{Soc}({}_R R)P = \text{Soc}(P)$ , since  $P$  is projective and by the assumption. Hence,  $K$  is an  $ss$ -supplement of  $L$  in  $P$  from [7, Lemma 3].  $\square$

Left  $(CE)$  elements were used by Zöschinger in [15] and their relationship with supplements in projective modules was given by Izurdiaga in [4]. Now we shall give the relationship of left  $(CE)$  elements with  $ss$ -supplements in projective modules over the rings whose radical is semisimple.

**Proposition 2.3** *Let  $P$  be a projective  $R$ -module where  $\text{Rad}(R)$  is semisimple and  $f \in \text{End}_R(P)$ . Then the following statements are equivalent:*

1.  $f$  is a left  $(CE)$  element in  $\text{End}_R(P)$ .
2.  $\text{Im}(f)$  is an  $ss$ -supplement of  $\text{Im}(1_P - f)$  in  $P$ .

**Proof** (1)  $\implies$  (2) Since  $f \in \text{End}_R(P)$  is left  $(CE)$ , then by [4, Proposition 1.4]  $\text{Im}(f)$  is a supplement of  $\text{Im}(1_P - f)$  in  $P$ . Thus,  $\text{Im}(f)$  is an  $ss$ -supplement of  $\text{Im}(1_P - f)$  in  $P$  from Lemma 2.2.

(2)  $\implies$  (1)  $\text{Im}(f)$  is a supplement of  $\text{Im}(1_P - f)$  in  $P$ , by the hypothesis. Thus,  $f$  is a left  $(CE)$  element in  $\text{End}_R(P)$  by [4, Proposition 1.4].  $\square$

**Proposition 2.4** *Let  $P$  be a projective  $R$ -module where  $\text{Rad}(R)$  is semisimple and  $f \in \text{End}_R(P)$ . Then the following statements are equivalent:*

1.  $f$  is weak left  $(CE)$  element in  $\text{End}_R(P)$ .
2.  $\text{Im}(f)$  is an  $ss$ -supplement of  $\text{Ker}(f)$  in  $P$ .

**Proof** (1)  $\implies$  (2) Let  $f$  be weak left  $(CE)$  endomorphism of  $P$ . Then by [4, Proposition 1.3],  $\text{Im}(f)$  is a supplement of  $\text{Ker}(f)$  in  $P$ . Thus,  $\text{Im}(f)$  is an  $ss$ -supplement of  $\text{Ker}(f)$  in  $P$  from Lemma 2.2.

(2)  $\implies$  (1) Let  $\text{Im}(f)$  be an  $ss$ -supplement of  $\text{Ker}(f)$  in  $P$ . Then  $\text{Im}(f)$  is a supplement of  $\text{Ker}(f)$  in  $P$ . Hence,  $f$  is a weak left  $(CE)$  element of  $\text{End}_R(P)$  by [4, Proposition 1.3].  $\square$

It is well known that if  $M$  is a module and  $f$  is an endomorphism of  $M$ , then  $M = \text{Im}(f) \oplus \text{Im}(1_M - f)$  if and only if  $f$  is idempotent. The above propositions generalize this one in the case of projective modules over the rings whose radical is semisimple,  $ss$ -supplements and left  $(CE)$  endomorphisms.

The following result shows that when the module is projective over a ring with semisimple radical, every  $ss$ -supplement is of the form  $\text{Im}(f)$  for a left  $(CE)$  endomorphism  $f$ . This is a generalization of the well known result about direct summands (see, for example, [1, 5.7]).

**Theorem 2.5** *Let  $R$  be a ring with semisimple  $\text{Rad}(R)$ ,  $P$  be projective  $R$ -module and  $K, L \leq P$ . Then the following statements are equivalent:*

1.  $K$  is an  $ss$ -supplement of  $L$  in  $P$ .
2. There exists  $f \in \text{End}_R(P)$  left  $(CE)$  such that:

- a)  $Im(f) = K$ .
- b)  $Im(1_P - f) \leq L$
- c)  $(L)f \ll K$  and  $(L)f$  is semisimple.

3. There exists  $f \in End_R(P)$  weak left (CE) such that:

- a)  $Im(f) = K$ .
- b)  $Ker(f) \leq L$
- c)  $(L)f \ll K$ .

**Proof** (1)  $\implies$  (2) We can easily write the decomposition  $\frac{P}{K \cap L} = \frac{K}{K \cap L} \oplus \frac{L}{K \cap L}$ , by the hypothesis. By using this decomposition and the projectivity of  $P$ , we can construct the following commutative diagram with canonical projections  $\pi_1$  and  $\pi_2$

$$\begin{array}{ccc} & P & \\ & \swarrow f & \downarrow \pi_2 \\ K & \xrightarrow{\pi_1} & \frac{K}{K \cap L} \end{array}$$

and there exists a homomorphism  $f : P \rightarrow K$  such that  $f\pi_1 = \pi_2$ . Since  $(K)f\pi_1 = \frac{K}{K \cap L}$ ,  $f|_K$  is an epimorphism by [1, 5.15], and so (a) holds. Since  $P$  is projective, then we get from the following diagram that there is a  $g \in End_R(P)$  such that  $gf^2 = f$ .

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow f & & \\ P & \xleftarrow{g} & K & \xrightarrow{f|_K} & Im(f) \end{array}$$

Now for every  $x \in P$ ,  $x = k + l$  where  $k \in K$  and  $l \in L$ . Then we have that  $(x)f\pi_1 = (k + l)f\pi_1 = (k + l)\pi_2$ . Thus, the equality  $(x)f + K \cap L = k + K \cap L$  implies that  $k - (x)f \in K \cap L$ . Since  $k + l - (x)f - l \in K \cap L \leq L$ , we deduce that  $x - (x)f \in L$ . Thus, (b) holds; and so we get that  $(P)(1_P - f)f\pi_1 = (P)(f - f^2)\pi_1 = 0$ . It follows immediately that  $(P)(f - f^2) \leq Ker(\pi_1) = K \cap L \ll K \leq P$ ; and thus  $f - f^2 \in Rad(End_R(P))$  by [1, 17.11]. On the other hand, since  $(L)f\pi_1 = (L)f + K \cap L = (L)\pi_2 = 0$ , then we get that  $(L)f \leq K \cap L \ll K$  and  $(L)f$  is semisimple as a submodule of  $K \cap L$  from [6, 8.1.5]; hence, (c) holds.

(2)  $\implies$  (3) is clear.

(3)  $\implies$  (1)  $Im(f)$  is an *ss*-supplement of  $Ker(f)$  in  $P$  from Proposition 2.4. Thus, we have that  $P = K + L$ . Now let  $X$  be a submodule of  $K$  such that  $P = X + L$ . Since  $(L)f \ll K$ , then from the equality  $(X)f + (L)f = (P)f = K$ , we get that  $K = (X)f$ . Therefore,  $P = X + Ker(f)$ . Since  $K = Im(f)$  is a supplement of  $Ker(f)$  in  $P$ , we obtain that  $X = K$ . Consequently,  $K$  is an *ss*-supplement of  $L$  in  $P$  by Lemma 2.2.  $\square$

**Corollary 2.6** Let  $R$  be a ring with semisimple  $Rad(R)$  and  $K, L$  be left ideals of  $R$ . Then the following statements are equivalent:

1.  $K$  is an  $ss$ -supplement of  $L$  in  ${}_R R$ .
2. There exists left  $(CE)$  element  $x$  of  $R$  such that:
  - a)  $Rx = K$ .
  - b)  $R(1 - x) \subseteq L$
  - c)  $Lx \ll K$  and  $Lx$  is semisimple.
3. There exists a weak left  $(CE)$  element  $x$  of  $R$  such that:
  - a)  $Rx = K$ .
  - b)  $\{r \in R \mid rx = 0\} \subseteq L$
  - c)  $Lx \ll K$ .

**Proof** It follows from Theorem 2.5. □

We should note that the left  $(CE)$  endomorphisms which are obtained in Theorem 2.5 is not uniquely determined by  $K$  and  $L$ . Now we give an example of this case.

**Example 2.7** Let consider the ring  $R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ . Let  $K = \begin{bmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ ,  $L = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & 0 \end{bmatrix}$  and  $a, b \in \mathbb{Z}_3$  distinct and nonzero. Note that  $L = Soc({}_R R)$ .  $K$  is an  $ss$ -supplement of  $L$  in  ${}_R R$ . Also, since  $f = \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix}$  and  $g = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$  are idempotent elements of  $R$ , they are distinct left  $(CE)$  endomorphisms of  $R$  that satisfy the conditions of Theorem 2.5.

Now we shall give the relationship between  $ss$ -supplement submodules of a projective module  $M$  over a ring with semisimple radical and  $ss$ -supplement submodules of endomorphism ring of  $M$ . For a module  $M$ , we denote by  $S_s(M)$  the set of  $ss$ -supplement submodules of  $M$ .

**Proposition 2.8** Let  $R$  be a ring with semisimple  $Rad(R)$ . For every projective  $R$ -module  $P$  with endomorphism ring  $E = End_R(P)$ , there is a bijective function between the sets  $S_s(P)$  and  $S_s({}_E E)$ .

**Proof** Let take any  $ss$ -supplement submodule of  $P$ , say  $K$ , we have left  $(CE)$  endomorphism  $f_K$  of  $P$  such that  $Im(f_K) = K$  by Theorem 2.5. Now consider the left ideal  $E.f_K$  of  $E$ . By Corollary 2.6, the left ideal  $E.f_K$  is an  $ss$ -supplement submodule of  ${}_E E$ . We claim that the left ideal  $E.f_K$  does not depend on the election of the endomorphism  $f_K$ . For every left  $(CE)$  endomorphism  $f'$  with  $Im(f') = K$ , we can construct the following commutative diagram

$$\begin{array}{ccc}
 & & P \\
 & \nearrow h' & \downarrow f' \\
 P & \xrightarrow{h} & K \\
 & \searrow f_K & \\
 & & 
 \end{array}$$

that means  $h'f' = f_K$  and  $hf_K = f'$  as  $P$  is projective. Thus, for every  $g \in E$ , we obtain that  $gf_K = g(h'f') = (gh')f' \in E.f'$ . Conversely, for every  $g' \in E$ , we get that  $g'f' = g'(hf_K) = (g'h)f_K \in E.f_K$ . Hence, we deduce that  $E.f_K = E.f'$ .

Analogously, if we take an  $ss$ -supplement submodule of  ${}_E E$ , say  $L$ , then there exists a left  $(CE)$  element  $x_L$  in  $E$  such that  $L = E.x_L$  by Corollary 2.6. In this case,  $Im(x_L)$  is an  $ss$ -supplement submodule in  $P$  by Proposition 2.3 and this submodule does not depend on the election of  $x_L$ , as the reason can be checked as above with the help of the projectivity of  $P$ .

Now we define the maps  $\Phi : S_s(P) \rightarrow S_s({}_E E)$  via  $\Phi(K) = E.f_K$  where  $Im(f_K) = K$  for all  $K \in S_s(P)$  and  $\Psi : S_s({}_E E) \rightarrow S_s(P)$  via  $\Psi(L) = Im(x_L)$  where  $L = E.x_L$  for all  $L \in S_s({}_E E)$ . It can be easily checked that  $\Phi$  and  $\Psi$  are mutually inverse.  $\square$

### 3. Radical-s-projective modules

We shall define radical-s-projective modules as a generalization of radical-projective modules.

**Definition 3.1** *We call a module  $M$  is radical-s-projective if for every  $R$ -module epimorphism  $g : A \rightarrow B$  and every homomorphism  $f : M \rightarrow B$ , there exists a homomorphism  $h : M \rightarrow A$  such that  $(M)(f - hg) \ll B$  and  $(M)(f - hg)$  is semisimple.*

Now we have the following implications on modules under given definitions:

$$\text{projective modules} \implies \text{radical-s-projective modules} \implies \text{radical-projective modules}$$

**Proposition 3.2** *Let  $P$  be a projective module and  $K$  be an  $ss$ -supplement submodule in  $P$ . Then  $K$  is radical-projective.*

**Proof** If  $K$  is an  $ss$ -supplement in  $P$ , then  $K$  is a supplement submodule in  $P$ . Hence,  $K$  is radical-projective from [4, Lemma 2.2].  $\square$

**Proposition 3.3** *Let  $R$  be a ring with semisimple  $Rad(R)$ . Then every  $ss$ -supplement submodule of a projective  $R$ -module is radical-s-projective.*

**Proof** Let  $P$  be a projective  $R$ -module and  $K$  be an  $ss$ -supplement submodule of  $P$ . Then there exists  $t \in End_R(P)$  left  $(CE)$  such that  $Im(t) = K$  by Proposition 2.3. Let  $g : A \rightarrow B$  be an  $R$ -module epimorphism and  $f : K \rightarrow B$  be a homomorphism. Since  $P$  is projective, there exists a commutative diagram

$$\begin{array}{ccc} & P & \\ h' \swarrow & \downarrow tf & \\ A & \xrightarrow{g} & B \end{array}$$

such that  $h'g = tf$ . Note that, since  $t$  is left  $(CE)$  endomorphism, then  $(P)(t - t^2) \ll P$  by [1, 17.11]. Now put  $h = h' |_K$ , then we deduce that

$$(K)(f - hg) = (K)t(f - hg) = (K)(1_B - t)hg = (K)(t - t^2)hg \ll B$$

by [13, 19.3(4)], since  $(K)(t - t^2) \ll K$ . Moreover,  $(K)(t - t^2) \leq Rad(K) = Rad(R)K$  from [15, Lemma 2.1]. Therefore, we get that  $(K)(f - hg) = (K)(t - t^2)hg \leq (Rad(R)K)hg = Rad(R)(K)hg \leq Soc({}_R R)(K)hg \leq Soc((K)hg)$  by the assumption. It completes the proof.  $\square$

**Remark 3.4** Note that the homomorphism  $h$  that is obtained in Proposition 3.3 satisfies  $(K)(f - hg) \leq \text{Rad}(R)B$  since  $\text{Rad}(K) = \text{Rad}(R)K$  and  $(K)(t - t^2) \leq \text{Rad}(R)K$  by [15, Lemma 2.1].

The following lemma is a slight generalization of [8, Lemma 2.3] for  $ss$ -supplement submodules.

**Lemma 3.5** Let  $R$  be a ring with semisimple  $\text{Rad}(R)$ ,  $P$  be a projective  $R$ -module and  $K \leq P$ . Then the following statements are equivalent:

1.  $K$  is an  $ss$ -supplement in  $P$ .
2. There exist a small  $R$ -module epimorphism  $q : M \rightarrow K$  and a homomorphism  $p : P \rightarrow M$  such that  $qp$  is small epimorphism and  $\text{Ker}(qp)$  is semisimple.

Moreover, when these conditions are verified, for every small epimorphism  $q' : M \rightarrow K$  there exists  $p' : P \rightarrow M$  such that  $q'p'$  is a small epimorphism with semisimple  $\text{Ker}(q'p')$ . In addition, the homomorphism  $p'$  can be obtained so that  $p'q'$  is left (CE) endomorphism of  $P$  with image  $K$ .

**Proof** (1)  $\implies$  (2) If  $K$  is an  $ss$ -supplement in  $P$ , then there exists a left (CE) endomorphism  $t$  of  $P$  such that  $\text{Im}(t) = K$ , by Theorem 2.5. Thus,  $\text{Im}(t)$  is an  $ss$ -supplement of  $\text{Ker}(t)$  in  $P$  from Proposition 2.4. Thus, we can see  $q = t|_K : K \rightarrow P$  is a small epimorphism because  $\text{Im}(q) = K$  and  $\text{Ker}(q) = K \cap \text{Ker}(t) \ll K$ . Hence, the first part of (2) is verified. Note that  $K \cap \text{Ker}(t)$  is semisimple, by the hypothesis. Now let  $q' : M \rightarrow K$  be a small epimorphism. Since  $P$  is projective, we can construct the following commutative diagram

$$\begin{array}{ccc}
 & P & \\
 p' \swarrow & & \downarrow t \\
 M & \xrightarrow{q'} & K
 \end{array}$$

from which we infer that there exists a homomorphism  $p' : P \rightarrow M$  such that  $p'q' = t$ . Since  $(K)p'q' = (K)t = K$  and  $q'$  is small epimorphism, then  $p'|_K$  is an epimorphism by [1, 5.15]. Since  $q$  is small epimorphism, i.e  $\text{Ker}(q) = \text{Ker}(t|_K) \ll K$ , it is obtained that  $\text{Ker}(p'|_K) \ll K$ . This means that  $p'|_K$  is small epimorphism. Therefore,  $q'p' = q'(p'|_K)$  is a small epimorphism as a composition of two such epimorphism from [13, 19.3(1)]. Moreover, since  $P$  is projective,  $\text{Ker}(q'p') \leq \text{Rad}(P) = \text{Rad}(R)P \leq \text{Soc}(R)P = \text{Soc}(P)$  by the assumption, and so  $\text{Ker}(q'p')$  is semisimple.

(2)  $\implies$  (1) Let  $q : M \rightarrow K$  be small epimorphism and  $p : P \rightarrow M$  be a homomorphism such that  $qp$  is small epimorphism with semisimple  $\text{Ker}(qp)$ . We may assume that  $M$  is a submodule of  $K$ , and of  $P$ , too. In this case, since  $qp$  is an epimorphism by the hypothesis, then we get that  $P = K + \text{Ker}(p)$ . Since  $K \cap \text{Ker}(p) \leq \text{Ker}(qp)$  is small in  $K$ , we obtain that  $K \cap \text{Ker}(p) \ll K$ . Hence,  $K$  is an  $ss$ -supplement in  $P$  by Lemma 2.2. □

Now we shall determine  $ss$ -supplement submodules of free modules in terms of radical-s-projective modules using left (CE) morphisms.

**Theorem 3.6** Let  $R$  be a ring with semisimple  $\text{Rad}(R)$  and  $M$  be an  $R$ -module. Then the following statements are equivalent:



1.  $M$  is radical- $s$ -projective.
2. There exist a free  $R$ -module  $F$ , an  $ss$ -supplement submodule  $K$  of  $F$ , and a small epimorphism  $\varphi : M \rightarrow K$  with semisimple  $Ker(\varphi)$ .

**Proof** (1)  $\implies$  (2) Let  $F$  be a convenient free module such that there exists an epimorphism  $\psi : F \rightarrow M$ . Since  $M$  is radical- $s$ -projective, there exists a homomorphism  $h : M \rightarrow F$  such that  $(M)(1_M - h\psi) \ll M$  and  $(M)(1_M - h\psi)$  is semisimple. Since the equality  $M = (M)h\psi + (M)(1_M - h\psi)$  implies that  $M = (M)h\psi$ ,  $h\psi$  is an epimorphism. Moreover, since  $Ker(h\psi) \leq Im(1_M - h\psi) \ll M$  and  $(M)(1_M - h\psi)$  is semisimple,  $h\psi$  is a small epimorphism with semisimple  $Ker(h\psi)$ . By Lemma 3.5,  $Im(h)$  is an  $ss$ -supplement submodule of  $F$ . However, since  $Ker(h) \leq Ker(h\psi)$ ,  $h : M \rightarrow Im(h)$  is a small epimorphism with semisimple  $Ker(h)$ . Hence, the result follows, if we take  $K = Im(h)$  and  $\varphi = h$ .

(2)  $\implies$  (1) By Lemma 3.5, there exists a homomorphism  $p : F \rightarrow M$  such that  $\varphi p$  is small epimorphism with semisimple  $Ker(\varphi p)$ , and in addition  $p\varphi$  is a left  $(CE)$  element in  $End_R(F)$  with  $Im(p\varphi) = K$ . Let  $g : A \rightarrow B$  be an  $R$ -module epimorphism and  $f : M \rightarrow B$  be a homomorphism. By Proposition 3.3,  $K$  is radical- $s$ -projective, and so we can construct the following diagram

$$\begin{array}{ccc}
 & K & \\
 h' \swarrow & \downarrow p|_K f & \\
 A & \xrightarrow{g} & B
 \end{array}$$

and  $(K)(pf - h'g) \ll B$  and  $(K)(pf - h'g)$  is semisimple. Put  $h = \varphi h'$ . Since  $p\varphi$  is left  $(CE)$  with image  $K$ ,  $p|_K \varphi$  is an epimorphism, and so  $p|_K$  is too, by [1, 5.15]. Since  $\varphi p$  and  $\varphi$  are small epimorphisms, we obtain that

$$(M)(f - hg) = (M)\varphi p(f - hg) = (M)\varphi p(f - \varphi h'g) = (K)p(f - \varphi h'g) = (K)(pf - h'g + h'g - p\varphi h'g) \leq (K)(pf - h'g) + (K)(1_F - p\varphi)h'g. \tag{1}$$

There exists  $t \in End_R(F)$  with  $t(p\varphi)^2 = p\varphi$ , since  $p\varphi$  is left  $(CE)$ . Note that  $t(p\varphi - (p\varphi)^2) \in Rad(End_R(F))$  since  $Rad(End_R(F))$  is a left ideal of  $End_R(F)$ . So that, by using  $(K)p\varphi = K$  we obtain from [1, 17.11] that

$$(K)(1_F - p\varphi)h'g = (K)(p\varphi - (p\varphi)^2)h'g = (K)t(p\varphi - (p\varphi)^2)p\varphi h'g \ll B. \tag{2}$$

Therefore, by [13, 19.3(3)], we deduce from (1) and (2) that  $(M)(f - hg) \leq (K)(pf - h'g) + (K)(1_F - p\varphi)h'g \ll B$ . Moreover, since  $(K)t(p\varphi - (p\varphi)^2)p\varphi \leq Rad(K) = Rad(R)K$  by [15, Lemma 2.1], then we get that  $(K)t(p\varphi - (p\varphi)^2)p\varphi h'g \leq (Rad(R)K)h'g = Rad(R)(K)h'g \leq Soc({}_R R)(K)h'g \leq Soc((K)h'g)$ . Consequently,  $(M)(f - hg)$  is semisimple by [6, 8.1.5].  $\square$

**Corollary 3.7** *Let  $R$  be a ring where  $Rad(R)$  is semisimple and  $M$  be radical- $s$ -projective module. Then;*

1.  $Rad(M) = Rad(R)M$  and  $Rad(M)$  is semisimple.
2.  $ss$ -Supplement submodules of  $M$  are radical- $s$ -projective.

**Proof** (1) By Theorem 3.6, we have  $K$  that is an  $ss$ -supplement submodule of a free module and  $\varphi : M \rightarrow K$  that is a small epimorphism with semisimple  $Ker(\varphi)$ , since  $M$  is radical-s-projective. By Lemma 3.5, there exists a homomorphism  $p : K \rightarrow M$  such that  $\varphi p$  is small epimorphism with semisimple  $Ker(\varphi p)$ . Since  $\varphi$  is small epimorphism,  $p$  is too, from [8, Lemma 2.1]. Thus, we obtain that  $Rad(M) = (Rad(K))p = (Rad(R)K)p = Rad(R)(K)p = Rad(R)M$  by [1, 9.15] and [15, Lemma 2.1], as desired. On the other hand, since  $Rad(R)$  is semisimple,  $Rad(M) \leq Soc({}_R R)M \leq Soc(M)$ .

(2) Let  $L$  be an  $ss$ -supplement submodule of  $M$ . Since  $M$  is radical-s-projective, there are free module  $F$ , an  $ss$ -supplement submodule  $K$  of  $F$  and a small epimorphism  $\varphi : M \rightarrow K$  with semisimple  $Ker(\varphi)$ , by Theorem 3.6. Note that  $\frac{M}{Ker(\varphi)} \cong \varphi(M) = K$ . Consider the canonical projection  $\pi : M \rightarrow \frac{M}{Ker(\varphi)}$ . By the way, if  $L$  is an  $ss$ -supplement in  $M$ , then  $\varphi(L)$  is an  $ss$ -supplement submodule of  $K$ , because  $\frac{L+Ker(\varphi)}{Ker(\varphi)}$  is an  $ss$ -supplement in  $\frac{M}{Ker(\varphi)}$ . Thus,  $\varphi(L)$  is a supplement submodule of  $K$ . Since  $K$  is a supplement submodule of  $F$ , by [3, 20.6(2)]  $\varphi(L)$  is a supplement submodule of  $F$ . Note that  $\varphi(L)$  is an  $ss$ -supplement submodule of  $F$  from Lemma 2.2. Since  $Ker(\varphi) \ll M$ , then we get that  $L \cap Ker(\varphi) \ll L$  from [13, 41.1(5)]. Therefore,  $\varphi|_L : L \rightarrow \varphi(L)$  is a small epimorphism with semisimple kernel by [6, 8.1.5]. Hence,  $L$  is radical-s-projective from Theorem 3.6.  $\square$

Recall from [13] that an ideal  $I$  of a ring  $R$  is *left t-nilpotent* if for every sequence  $a_1, a_2, \dots$  of elements in  $I$ , there is a  $k \in \mathbb{Z}^+$  with  $a_k a_{k-1} \dots a_1 = 0$ .

In [8] a module  $M$  is said to be  $J(R)$ -projective if for every  $R$ -module epimorphism  $g : A \rightarrow B$  with  $Rad(R)B = 0$  and every homomorphism  $f : M \rightarrow B$ , there exists a homomorphism  $h : M \rightarrow A$  such that  $f = hg$ .

If a module  $M$  is radical-s-projective, then Remark 3.4 follows that the homomorphism  $h$  that is obtained verifies  $(M)(f - hg) \leq Rad(R)B$ . Thus if  $Rad(R)B = 0$ , then  $f = hg$ . Hence,  $M$  is  $J(R)$ -projective, but the converse is not true in general, as the following example shows:

**Example 3.8** Let  $R = \mathbb{Z}_2[[x]]$  the ring of formal power series in  $x$  with coefficients in  $\mathbb{Z}_2$ . Then  $R$  is left noetherian ring and  $Rad(R)$  is not left t-nilpotent. Thus, there exists a left  $R$ -module  $M$  such that  $Rad(R)M = M$  from [1, 28.3]. Therefore,  $M$  is  $J(R)$ -projective, but  $M$  is not radical-s-projective module from [4, 3.11].

Now we give necessary conditions for  $J(R)$ -projective modules to be radical-s-projective.

**Proposition 3.9** Let  $R$  be a ring with semisimple  $Rad(R)$  and  $M$  be a finitely generated  $R$ -module. Then  $M$  is radical-s-projective if and only if  $M$  is  $J(R)$ -projective.

**Proof** Let  $M$  be  $J(R)$ -projective module. Let  $g : A \rightarrow B$  be an  $R$ -module epimorphism and  $f : M \rightarrow B$  be a homomorphism. Consider the canonical projection  $\pi : B \rightarrow \frac{B}{Rad(R)B}$ . Since  $M$  is  $J(R)$ -projective, then there exists a commutative diagram

$$\begin{array}{ccc}
 & M & \\
 h \swarrow & \downarrow f\pi & \\
 A & \xrightarrow{g\pi} & \frac{B}{Rad(R)B}
 \end{array}$$

such that there exists a homomorphism  $h : M \rightarrow A$  with  $hg\pi = f\pi$ . Thus, we obtain that  $0 = (M)(f\pi - hg\pi) = (M)(f - hg)\pi$ , and so  $(M)(f - hg) \leq \text{Rad}(R)B \leq \text{Rad}(B)$ . Since  $M$  is finitely generated,  $(M)(f - hg)$  is finitely generated. Since finitely generated submodules that are contained in radical are small,  $(M)(f - hg) \ll B$ . Moreover, we have that  $(M)(f - hg) \leq \text{Rad}(R)B \leq \text{Soc}({}_R R)B \leq \text{Soc}(B)$  by the assumption. Hence,  $M$  is radical-s-projective.  $\square$

Rings for which every supplement submodule of a finitely generated (in [4] not necessarily finitely generated) projective module is a direct summand have been widely studied (see, for example, [8], [15]). Now we shall give the analogous fact for *ss*-supplement submodules of projective modules.

**Lemma 3.10** *Let  $R$  be a ring with semisimple  $\text{Rad}(R)$ ,  $P$  be a projective  $R$ -module, and  $K$  be an *ss*-supplement submodule of  $P$ . Then the following statements are equivalent:*

1.  $K$  is a direct summand.
2.  $K$  is projective.
3. For every  $K' \leq P$  such that  $K$  is an *ss*-supplement of  $K'$ , the factor module  $\frac{P}{K'}$  has a projective cover whose kernel is semisimple.
4.  $K$  has a projective cover whose kernel is semisimple.

**Proof** (1)  $\implies$  (2) follows from [13, 18.1]

(2)  $\implies$  (1) By Theorem 2.5 there exists a left (CE) element  $f$  in  $\text{End}_R(P)$  with  $\text{Im}(f) = K$ . Since  $K \cong \frac{P}{\text{Ker}(f)} \cong \frac{K}{K \cap \text{Ker}(f)}$ , we deduce that  $K \cap \text{Ker}(f)$  is a direct summand of  $K$ . On the other hand,  $K \cap \text{Ker}(f)$  is small in  $K$ , by Proposition 2.4. Therefore,  $K \cap \text{Ker}(f) = 0$ . Again from Proposition 2.4, we get that  $P = K \oplus \text{Ker}(f)$ .

(2)  $\implies$  (3) Let  $K'$  be a submodule of  $P$  such that  $K$  is an *ss*-supplement of  $K'$ . Thus, we can write that  $\frac{P}{K'} \cong \frac{K}{K \cap K'}$ . Then the canonical projection  $\pi : K \rightarrow \frac{K}{K \cap K'}$  is a projective cover with semisimple  $\text{Ker}(\pi) = K \cap K'$ , as required.

(3)  $\implies$  (4) Let  $K$  be an *ss*-supplement of  $K'$  in  $P$ . Note that  $\frac{P}{K'} \cong \frac{K}{K \cap K'}$ . Let  $\varphi : Q \rightarrow \frac{K}{K \cap K'}$  be the projective cover of  $\frac{K}{K \cap K'}$  whose kernel is semisimple. Since  $Q$  is projective, there exists a homomorphism  $\psi : Q \rightarrow K$  such that  $\psi\pi = \varphi$  where  $\pi : K \rightarrow \frac{K}{K \cap K'}$  is the canonical projection. Then we have that  $\text{Ker}(\psi) \leq \text{Ker}(\varphi) \ll Q$ , and also  $\text{Ker}(\psi)$  is semisimple. Hence  $\psi$  is the required projective cover.

(4)  $\implies$  (2) Let  $\varphi : Q \rightarrow K$  be a projective cover of  $K$  whose kernel is semisimple. By Proposition 3.3,  $K$  is radical-s-projective. Thus, there exists a homomorphism  $h : K \rightarrow Q$  such that  $(K)(1_K - h\varphi) \ll K$  and  $(K)(1_K - h\varphi)$  is semisimple. Since  $K = (K)(1_K - h\varphi) + (K)h\varphi$ ,  $h\varphi$  is an epimorphism. Since  $\varphi$  is a small epimorphism, then  $h$  is an epimorphism by [1, 5.15]. Moreover, since  $Q$  is projective, there exists a homomorphism  $f : Q \rightarrow K$  such that  $fh = 1_Q$ , and so  $K = \text{Im}(f) \oplus \text{Ker}(h)$ . On the other hand, since  $\text{Ker}(h) \leq (K)(1_K - h\varphi)$ ,  $\text{Ker}(h)$  is small in  $K$ . Therefore,  $\text{Ker}(h)$  has to be zero. Hence,  $h$  is an isomorphism and  $K$  is projective.  $\square$

For the modules  $M$  and  $N$ ,  $N$  is said to be *M-cyclic* if there exists an epimorphism  $\varphi : M \rightarrow N$ .

**Corollary 3.11** *Let  $R$  be a ring with semisimple  $\text{Rad}(R)$ ,  $P$  be a projective  $R$ -module with endomorphism ring  $E = \text{End}_R(P)$ . Then the following statements are equivalent:*

1. *Every  $ss$ -supplement submodule of  $P$  is a direct summand.*
2. *Every  $ss$ -supplement submodule of  ${}_E E$  is a direct summand.*
3. *Every  $P$ -cyclic radical- $s$ -projective  $R$ -module is projective.*
4. *Every  $P$ -cyclic radical- $s$ -projective  $R$ -module has a projective cover whose kernel is semisimple.*

**Proof** (1)  $\implies$  (2) Let  $L$  be a left ideal of  $E$  and suppose that  $L$  is an  $ss$ -supplement in  $E$ . Since there is a bijective function between  $ss$ -supplement submodules of  $P$  and  $ss$ -supplement submodules of  ${}_E E$  from Proposition 2.8, by following the notation of the proof of Proposition 2.8,  $\Psi(L)$  is a direct summand of  $P$  by assumption. Thus, there exists an idempotent element  $e$  in  $E$  with  $\Psi(L) = \text{Im}(e)$  by [1, 5.8]. Again applying Proposition 2.8, we have that  $L = \Phi\Psi(L) = E.e$ . Hence,  $L$  is a direct summand of  $E$ .

(2)  $\implies$  (1) Let  $K$  be an  $ss$ -supplement submodule of  $P$ . By following the notation of the proof of Proposition 2.8, since there is a bijective function  $\Phi$  from the set of  $ss$ -supplement submodules of  $P$  to the set of  $ss$ -supplement submodules of  ${}_E E$  by Proposition 2.8, then  $\Phi(K)$  is an  $ss$ -supplement submodule of  ${}_E E$ . By assumption,  $\Phi(K)$  is a direct summand in  ${}_E E$ . Thus, we get that  $\Phi(K) = E.f$  for some idempotent element  $f$  of  $E$  by [1, 5.8]. Now we deduce that  $K = \Psi\Phi(K) = \text{Im}(f)$  is a direct summand in  $P$ .

(1)  $\implies$  (3) Let  $M$  be a  $P$ -cyclic radical- $s$ -projective module. Then there exist an  $ss$ -supplement  $K$  of  $P$  and a small epimorphism  $\varphi : M \rightarrow K$  with semisimple  $\text{Ker}(\varphi)$  by Theorem 3.6. By assumption,  $K$  is a direct summand of  $P$ , and so  $K$  is projective from [13, 18.1]. Thus, there exists a homomorphism  $f : K \rightarrow M$  such that  $f\varphi = 1_K$ . Therefore,  $M = \text{Ker}(\varphi) \oplus \text{Im}(f)$ , but since  $\text{Ker}(\varphi)$  is small in  $M$ , then  $\text{Ker}(\varphi)$  has to be zero. Hence,  $\varphi$  is an isomorphism and  $M$  is projective.

(3)  $\implies$  (4) Let  $M$  be a  $P$ -cyclic radical- $s$ -projective module. By hypothesis,  $M$  is projective. Hence,  $1_M$  is the desired projective cover of  $M$ .

(4)  $\implies$  (1) Let  $K$  be an  $ss$ -supplement submodule of  $P$ . From Theorem 2.5, there exists a left  $(CE)$  endomorphism  $f$  of  $P$  with  $\text{Im}(f) = K$ . Then  $K$  is  $P$ -cyclic. Also, from Proposition 3.3,  $K$  is radical- $s$ -projective. Thus,  $K$  has a projective cover whose kernel is semisimple, by the assumption. Hence,  $K$  is a direct summand of  $P$  from Lemma 3.10.  $\square$

The following result gives a characterization of rings with semisimple radical for which every  $ss$ -supplement submodule of a projective module is a direct summand. For a set  $\Gamma$ ,  $\mathcal{RFM}_\Gamma(R)$  will indicate the ring of row finite  $\Gamma$ -matrices with entries in  $R$ .

Recall that a ring  $R$  is called *von-Neumann regular* if every element  $x$  can be written in the form  $xax$ , for some  $a \in R$ . A ring  $R$  is von-Neumann regular if and only if for every  $x \in R$ ,  $Rx$  is a direct summand of  ${}_R R$  (see [6, p. 38]).

**Corollary 3.12** *Let  $R$  be a ring with semisimple  $\text{Rad}(R)$ . Then the following statements are equivalent:*

1. *Every  $ss$ -supplement submodule of a projective  $R$ -module is a direct summand.*
2. *Every supplement submodule of a projective  $R$ -module is a direct summand.*
3. *For any set  $\Gamma$ , every weak left  $(CE)$  matrix  $A \in \mathcal{RFM}_\Gamma(R)$  is (von-Neumann) regular.*

4. For any set  $\Gamma$ , every left (CE) matrix  $A \in \mathcal{RFM}_\Gamma(R)$  is (von-Neumann) regular.
5. For every set  $\Gamma$ , left (CE) matrix  $A \in \mathcal{RFM}_\Gamma(R)$  and  $B \in \mathcal{RFM}_\Gamma(R)$  with  $BA^2 = A$  it is provided that  $ABA = A$ .
6. Every radical-s-projective  $R$ -module has a projective cover whose kernel is semisimple.
7. Every radical-s-projective  $R$ -module is projective.

**Proof** It is clear from Lemma 2.2, Corollary 3.11, and [4, Corollary 3.4]. □

Recall that a module  $M$  is called *hereditary* (respectively, *semihereditary*) if every (respectively, finitely generated) submodule of  $M$  is projective. A ring  $R$  is called *left hereditary* (respectively, *left semihereditary*) if  ${}_R R$  is hereditary (respectively, semihereditary) (see [13]).

**Example 3.13** Every (respectively, finitely generated) *ss-supplement* submodule of a hereditary (respectively, semihereditary) projective  $R$ -module is a direct summand where  $\text{Rad}(R)$  is semisimple by Lemma 3.10. Moreover, if  $R$  is a hereditary (respectively, semihereditary) ring with semisimple  $\text{Rad}(R)$ , then every *ss-supplement* submodule of a (respectively, finitely generated) projective  $R$ -module is a direct summand by Lemma 3.10.

In [5], a ring  $R$  is said to be a *left V-ring* if every simple left  $R$ -module is injective. It is well known that a ring  $R$  is a left  $V$ -ring if and only if  $\text{Rad}(M) = 0$  for every left  $R$ -module  $M$ .

**Example 3.14** Consider the commutative ring  $\prod_{i \geq 1}^\infty F_i$  where  $F_i = F$  is any field. Let  $R$  be the subring of this ring of the formed by all sequences  $(r_n)_{n \in \mathbb{N}}$  such that there exist  $r \in F$ ,  $m \in \mathbb{N}$  with  $r_n = r$  for all  $n \geq m$ . Then  $R$  is a left  $V$ -ring such that  $\text{Soc}({}_R R)$  is a maximal submodule of  ${}_R R$  by [11, Example 2.5]. Clearly, every *ss-supplement* submodule of  ${}_R R$  is a direct summand, since  $\text{Rad}(R) = 0$ . Thus, every radical-s-projective  $R$ -module is projective by Corollary 3.12.

In [2], a ring  $R$  is called *SSI-ring* if every semisimple left  $R$ -module is injective.

**Proposition 3.15** Let  $R$  be SSI-ring and  $M$  be an  $R$ -module. If  $M$  is radical-s-projective, then  $M$  is projective.

**Proof** Let  $g : A \rightarrow B$  be an  $R$ -module epimorphism and  $f : M \rightarrow B$  be any homomorphism. Then by the hypothesis, there exists a homomorphism  $h : M \rightarrow A$  such that  $(M)(f - hg) \ll B$  and  $(M)(f - hg)$  is semisimple. Since  $R$  is SSI-ring, then  $(M)(f - hg)$  is injective. Thus,  $(M)(f - hg)$  is a direct summand of  $B$  by [10, Theorem 2.15], but since  $(M)(f - hg)$  is small in  $B$ , it has to be zero. Hence, we obtain that  $f = hg$ , so  $M$  is a projective module. □

Recall from [12] that a module  $M$  is said to have *the exchange property* if for any module  $K$  and any two decompositions

$$K = M' \oplus L = \bigoplus_{i \in I} A_i$$

where  $M' \cong M$ , there are submodules  $A'_i \leq A_i$  such that

$$K = M' \oplus \left(\bigoplus_{i \in I} A'_i\right).$$

Note that  $M$  has finite exchange property if this holds whenever the index set  $I$  is finite. A ring  $R$  is called an *exchange ring* if  ${}_R R$  has the exchange property. For finitely generated modules, the exchange and finite exchange properties coincide.

**Example 3.16** (1) Let  $P$  be a projective module such that  $E = \text{End}_R(P)$  is an exchange ring. Since every *ss-supplement* submodule of  $P$  is a supplement, then it is a direct summand of  $P$  from [4, Example 3.10(iii)] as idempotents lift modulo  $\text{Rad}(E)$  by [15, Example 2].

(2) Let  $R$  be the ring of eventually constant sequences  $r = (s_1, s_2, \dots, s_n, t, t, t, \dots)$  where  $s_1, s_2, \dots, s_n \in \mathbb{Q}$ ,  $t$  is in the set of integers localised at the prime ideal  $2\mathbb{Z}$  and  $n$  depends on  $r$ . Then  $R$  is an exchange ring with  $\text{Rad}(R) = 0$  from [3, 11.42(3)]. Since  ${}_R R$  has finite exchange property, by [12, Theorem 2]  $\text{End}_R({}_R R)$  is an exchange ring for all  $0 \neq n \in \mathbb{N}$ . So that every *ss-supplement* in a finitely generated projective  $R$ -module is a direct summand by [4, Example 3.10(iii)].

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