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Twisted Dirac operators and the Kastler-Kalau-Walze type theorem for five dimensional manifolds with boundary

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Abstract: In this paper, we prove the Kastler-Kalau-Walze type theorems for twisted Dirac operators on 5-dimensional manifolds with boundary.

Key words: Twisted Dirac operators, noncommutative residue Kastler-Kalau-Walze type theorems

1. Introduction

The noncommutative residue found in [4, 14] plays a prominent role in noncommutative geometry. For arbitrary closed compact n -dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [14] using the theory of zeta functions of elliptic pseudodifferential operators. In [1], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogue. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in [2]. In [6], Kastler gave a brute-force proof of this theorem. In [5], Kaulau and Walze proved this theorem in the normal coordinates system simultaneously, which is called the Kastler-Kalau-Walze theorem now.

An important application of Riemannian geometry is to allow us to define the volume element of a Riemannian manifold (M_n, g) . The noncommutative residue of Wodzicki [14] and Guillemin [4] is a trace on the algebra of (integer order) ΨDOs on M . An important feature is that it allows us to extend to all ΨDOs by the Dixmier trace, which plays the role of the integral in the framework of noncommutative geometry. Fedosov etc. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [3]. In [7], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. For a spin manifold M with boundary ∂M , by the composition formula in Boutet de Monvel's algebra and the definition of \widetilde{Wres} [11], $\widetilde{Wres}[(\pi^+ D^{-1})^2]$ should be the sum of two terms from interior and boundary of M , where $\pi^+ D^{-1}$ is an element in Boutet de Monvel's algebra [12].

In [8], Wang and Wang computed the lower volume $Vol_5^{(1,1)}$ and proved a Kastler-Kalau-Walze type theorem for the Dirac operator on 5-dimensional manifolds with boundary. In [9], Wang and Wang gave two Lichnerowicz type formulas and the proof of the Kastler-Kalau-Walze type theorem for twisted Dirac operators

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and twisted signature operators on four dimensional manifolds with boundary. In the present paper, we shall restrict our attention to the cases $\widetilde{\text{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+(\widetilde{D}_H^{-1})]$ and $\widetilde{\text{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+(\widetilde{D}_H^*)^{-1}]$ for 5-dimensional manifolds with boundary, where twisted Dirac operators $\widetilde{D}_H, \widetilde{D}_H^*$ are defined in (2.9), (2.10).

The paper is organized in the following way. In Section 2, we recall the definition of twisted Dirac operators and define lower dimensional volumes of compact Riemannian manifolds with boundary. In Section 3, we compute $\widetilde{\text{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+(\widetilde{D}_H^{-1})], \widetilde{\text{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+(\widetilde{D}_H^*)^{-1}]$ and get the Kastler-Kalau-Walze type theorem for twisted Dirac operators on 5-dimensional spin manifolds with boundary.

2. Twisted Dirac operators and lower-dimensional volumes of spin manifolds with boundary

Firstly, we give the expression of the Dirac operator. Let ∇^L denote the Levi-civita connection about g^M . In the local coordinates $x_i; 1 \leq i \leq n$ and the fixed orthonormal frame $\{\widetilde{e}_1, \dots, \widetilde{e}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e}_1, \dots, \widetilde{e}_n)^t = (\omega_{s,t})(\widetilde{e}_1, \dots, \widetilde{e}_n)^t. \tag{2.1}$$

The classical Dirac operator is defined on $S(TM)$ by

$$D = \sum_{j=1}^n c(\widetilde{e}_j)[\widetilde{e}_j + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e}_j)c(\widetilde{e}_s)c(\widetilde{e}_t)]. \tag{2.2}$$

Next, we consider an n -dimensional oriented Riemannian manifold (M, g^M) equipped with a fixed spin structure. We recall twisted Dirac operators. Let $S(TM)$ be the spinors bundle and H be an additional smooth vector bundle equipped with a nonunitary connection $\widetilde{\nabla}^H$. Let $\widetilde{\nabla}^{H,*}$ be the dual connection on H , and define

$$\nabla^H = \frac{\widetilde{\nabla}^H + \widetilde{\nabla}^{H,*}}{2}, \quad \Phi = \frac{\widetilde{\nabla}^H - \widetilde{\nabla}^{H,*}}{2}; \tag{2.3}$$

then ∇^H is a metric connection and Φ is an endomorphism of H with a 1-form coefficient. We consider the tensor product vector bundle $S(TM) \otimes H$, which becomes a Clifford module via the definition;

$$c(a) = c(a) \otimes id_H, a \in TM, \tag{2.4}$$

and which we equip with the compund connection:

$$\widetilde{\nabla}^{S(TM) \otimes H} = \nabla^{S(TM)} \otimes id_H + id_{S(TM)} \otimes \widetilde{\nabla}^H, \tag{2.5}$$

$$\nabla^{S(TM) \otimes H} = \nabla^{S(TM)} \otimes id_H + id_{S(TM)} \otimes \nabla^H, \tag{2.6}$$

then the spinor connection $\widetilde{\nabla}$ induced by $\widetilde{\nabla}^{S(TM) \otimes H}$ is locally given by

$$\widetilde{\nabla}^{S(TM) \otimes H} = \nabla^{S(TM)} \otimes id_H + id_{S(TM)} \otimes \nabla^H + id_{S(TM)} \otimes \Phi. \tag{2.7}$$

Let

$$D_H = \sum_{i=1}^n c(\widetilde{e}_i) \nabla_{\widetilde{e}_i}^{S(TM) \otimes H}. \tag{2.8}$$

Then we can obtain twisted Dirac operators \tilde{D}_H and \tilde{D}_H^* associated to the connection $\tilde{\nabla}$ as follows:

$$\begin{aligned} \tilde{D}_H &= D_H + \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi(\tilde{e}_j) \\ &= \sum_{j=1}^n c(\tilde{e}_j) \tilde{e}_j + \frac{1}{4} \sum_{j=1}^n \omega_{st}(\tilde{e}_j) c(\tilde{e}_j) c(\tilde{e}_s) c(\tilde{e}_t) + \sum_{j=1}^n c(\tilde{e}_j) \otimes \sigma^H(\tilde{e}_j) + \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi(\tilde{e}_j). \end{aligned} \quad (2.9)$$

$$\begin{aligned} \tilde{D}_H^* &= D_H - \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi^*(\tilde{e}_j) \\ &= \sum_{j=1}^n c(\tilde{e}_j) \tilde{e}_j + \frac{1}{4} \sum_{j=1}^n \omega_{st}(\tilde{e}_j) c(\tilde{e}_j) c(\tilde{e}_s) c(\tilde{e}_t) + \sum_{j=1}^n c(\tilde{e}_j) \otimes \sigma^H(\tilde{e}_j) - \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi^*(\tilde{e}_j). \end{aligned} \quad (2.10)$$

where σ^H is the connection matrix of ∇^H and Φ^* is the transport matrix of Φ .

Lemma 2.1 *Let $\{\tilde{e}_i\}(1 \leq i, j \leq n)$ ($\{\partial_i\}$) be the orthonormal frames (natural frames respectively) on TM ,*

$$D_H = \sum_{i,j} g^{ij} c(\partial_i) \nabla_{\partial_i}^{S(TM)} \otimes H = \sum_{j=1}^n c(\tilde{e}_j) \nabla_{\tilde{e}_j}^{S(TM)} \otimes H, \quad (2.11)$$

where $\nabla_{\partial_i}^{S(TM)} \otimes H = \partial_j + \sigma_j^s + \sigma_j^H$ and $\sigma_j^s = \frac{1}{4} \sum_{j,k} \langle \nabla_{\partial_i}^L \tilde{e}_j, \tilde{e}_k \rangle c(\tilde{e}_j) c(\tilde{e}_k)$, σ_j^H is the connection matrix of ∇^H .

Theorem 2.2 [3] **(Fedosov-Golse-Leichtnam-Schrohe)** *Let M and ∂M be connected, $\dim M = n \geq 3$,*

$A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by p , b , and s the local symbols of P, G , and S respectively. Define:

$$\begin{aligned} \widetilde{\text{Wres}}(A) &= \int_X \int_{\mathbf{S}} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx \\ &\quad + 2\pi \int_{\partial X} \int_{\mathbf{S}'} \{ \text{tr}_E [(\text{tr} b_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx', \end{aligned} \quad (2.12)$$

where $\widetilde{\text{Wres}}$ denotes the noncommutative residue of an operator in the Boutet de Monvel's algebra.

Then a) $\widetilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Definition 2.3 [12] *Lower dimensional volumes of spin manifolds with boundary are defined by*

$$\text{Vol}_n^{(p_1, p_2)} M := \widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}]. \quad (2.13)$$

By [12], we get

$$\widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M \otimes \mathbb{C}} [\sigma_{-n}(D^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Psi, \quad (2.14)$$

and

$$\begin{aligned} \Psi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\Lambda^* T^* M} \otimes_{\mathbb{C}} [\partial_{x_n}^j \partial_{\xi'}^{\alpha} \partial_{\xi_n}^k \sigma_r^+(D^{-p_1})(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{x'}^{\alpha} \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D^{-p_2})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \tag{2.15}$$

where the sum is taken over $r + l - k + |\alpha| - j - 1 = -n, \quad r \leq -p_1, l \leq -p_2$.

3. A Kastler-Kalau-Walze type theorem for 5-dimensional spin manifolds with boundary

In this section, we compute the lower dimensional volume for \tilde{D}_H and \tilde{D}_H^* on five dimensional compact spin manifolds with boundary and get a Kastler-Kalau-Walze type theorem in this case.

Proposition 3.1 [13] *The following identity holds:*

$$1) \text{When } p_1 + p_2 = n, \text{Vol}_n^{(p_1, p_2)} M = c_0 \text{Vol} M; \tag{3.1}$$

$$2) \text{When } p_1 + p_2 \equiv n \pmod{1}, \text{Vol}_n^{(p_1, p_2)} M = \int_{\partial M} \Psi. \tag{3.2}$$

Next, we compute $\text{Vol}_5^{(1,1)}$ for \tilde{D}_H on 5-dimensional spin manifolds with boundary, then we have

$$\widetilde{\text{Wres}}[\pi^+(\tilde{D}_H^{-1}) \circ \pi^+(\tilde{D}_H^{-1})] = \int_{\partial M} \Psi. \tag{3.3}$$

Write

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha; \sigma(\tilde{D}_H) = p_1 + p_0; \sigma(\tilde{D}_H^{-1}) = \sum_{j=1}^{\infty} q_{-j}. \tag{3.4}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned} 1 &= \sigma(\tilde{D}_H \circ \tilde{D}_H^{-1}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma(\tilde{D}_H)] D_x^{\alpha} [\sigma(\tilde{D}_H^{-1})] \\ &= (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \dots) \\ &\quad + \sum_j (\partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_{-1} + D_{x_j} q_{-2} + D_{x_j} q_{-3} + \dots) \\ &= p_1 q_{-1} + (p_1 q_{-2} + p_0 q_{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} q_{-1}) + \dots, \end{aligned} \tag{3.5}$$

so

$$q_{-1} = p_1^{-1}; \tag{3.6}$$

$$q_{-2} = -p_1^{-1} [p_0 p_1^{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} (p_1^{-1})]; \tag{3.7}$$

$$q_{-3} = -p_1^{-1} [p_0 q_{-2} + \sum_{j=1}^{n-1} c(dx_j) \partial_{\xi_j} q_{-2} + c(dx_n) \partial_{x_n} q_{-2}]. \tag{3.8}$$

Since Ψ is a global form on ∂M , for any fixed point $x_0 \in \partial M$, we choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Psi(x_0)$ in the coordinates $\tilde{U} = U \times [0, 1) \subset M$ and the metric $\frac{1}{h(x_n)}g^{\partial M} + dx_n^2$. The dual metric of g^M on \tilde{U} is $h(x_n)g^{\partial M} + dx_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $g_M^{ij} = g^M(dx_i, dx_j)$, then

$$[g_{ij}^M] = \begin{bmatrix} \frac{1}{h(x_n)}[g_{ij}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_M^{ij}] = \begin{bmatrix} h(x_n)[g_{\partial M}^{ij}] & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.9}$$

and

$$\partial_{x_s} g_{ij}^{\partial M}(x_0) = 0, 1 \leq i, j \leq n-1; \quad g_{ij}^M(x_0) = \delta_{ij}. \tag{3.10}$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal frame field in U about $g^{\partial M}$ which is parallel along geodesics and $e_i = \frac{\partial}{\partial x_i}(x_0)$, then $\{\tilde{e}_1 = \sqrt{h(x_n)}e_1, \dots, \tilde{e}_{n-1} = \sqrt{h(x_n)}e_{n-1}, \tilde{e}_n = \frac{\partial}{\partial x_n}\}$ is the orthonormal frame field in \tilde{U} about g^M . Locally $S(TM)|_{\tilde{U}} \cong \tilde{U} \times \wedge_C^*(\frac{n}{2})$. Let $\{f_1, \dots, f_n\}$ be the orthonormal basis of $\wedge_C^*(\frac{n}{2})$. Take a spin frame field $\sigma : \tilde{U} \rightarrow Spin(M)$ such that $\pi\sigma = \{\tilde{e}_1, \dots, \tilde{e}_n\}$ where $\pi : Spin(M) \rightarrow O(M)$ is a double covering, then $\{[\sigma, f_i], 1 \leq i \leq 4\}$ is an orthonormal frame of $S(TM)|_{\tilde{U}}$. In the following, since the global form Ψ is independent of the choice of the local frame, so we can compute $tr_{S(TM)}$ in the frame $\{[\sigma, f_i], 1 \leq i \leq 4\}$. Let $\{\hat{e}_1, \dots, \hat{e}_n\}$ be the canonical basis of R^n and $c(\hat{e}_i) \in Hom(\wedge_C^*(\frac{n-1}{2}), \wedge_C^*(\frac{n-1}{2}))$ be the Clifford action. By [15], then

$$c(\tilde{e}_i) = [\sigma, c(\hat{e}_i)]; \quad c(\tilde{e}_i)[\sigma, f_i] = [\sigma, c(\hat{e}_i)f_i]; \quad \frac{\partial}{\partial x_i} = [\sigma, \frac{\partial}{\partial x_i}]. \tag{3.11}$$

then we have $\frac{\partial}{\partial x_i}c(\tilde{e}_i) = 0$ in the above frame. By Lemma 2.2 in [12], we have:

Lemma 3.2 [12] *With the metric g^M on M near the boundary*

$$\partial_{x_j}(|\xi|_{g^M}^2)(x_0) = \begin{cases} 0, & \text{if } j < n, \\ h'(0)|\xi'|_{g^{\partial M}}^2, & \text{if } j = n; \end{cases} \tag{3.12}$$

$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases} 0, & \text{if } j < n, \\ \partial_{x_n}(c(\xi'))(x_0), & \text{if } j = n, \end{cases} \tag{3.13}$$

where $\xi = \xi' + \xi_n dx_n$.

By Lemma 2.1 in [12], we have:

Lemma 3.3 *The symbols of twisted Dirac operators:*

$$\sigma_{-1}(\tilde{D}_H^{-1}) = \sigma_{-1}((\tilde{D}_H^*)^{-1}) = \frac{ic(\xi)}{|\xi|^2}; \tag{3.14}$$

$$\sigma_{-2}(\tilde{D}_H^{-1}) = \frac{c(\xi)\sigma_0(\tilde{D}_H)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]; \tag{3.15}$$

$$\sigma_{-2}((\tilde{D}_H^*)^{-1}) = \frac{c(\xi)\sigma_0(\tilde{D}_H^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]; \tag{3.16}$$

$$\sigma_{-3}(\tilde{D}_H^{-1}) = -\frac{1}{p_1} [p_0q_{-2} + \sum_{j=1}^{n-1} c(dx_j)\partial_{x_j}q_{-2} + c(dx_n)\partial_{x_n}q_{-2}]; \tag{3.17}$$

$$\sigma_{-3}((\tilde{D}_H^*)^{-1}) = -\frac{1}{p_1} [p_0^*q_{-2}^* + \sum_{j=1}^{n-1} c(dx_j)\partial_{x_j}q_{-2}^* + c(dx_n)\partial_{x_n}q_{-2}^*]; \tag{3.18}$$

$$\tag{3.19}$$

where

$$\sigma_0(\tilde{D}_H)(x_0) = -h'(0)c(dx_n) + \sum_{j=1}^n c(\tilde{e}_j)(\sigma_j^H + \Phi(\tilde{e}_j))(x_0); \tag{3.20}$$

$$\sigma_0(\tilde{D}_H^*)(x_0) = -h'(0)c(dx_n) + \sum_{j=1}^n c(\tilde{e}_j)(\sigma_j^H - \Phi^*(\tilde{e}_j))(x_0). \tag{3.21}$$

Let us consider $\sigma_{-3}(\tilde{D}_H^{-1})$ of twisted Dirac operators. From Lemma 3.7 in [8], we get:

Lemma 3.4 [8] *For the Dirac operators, the following identity holds:*

$$\begin{aligned}
 \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} &= \frac{-ih'^2}{(1+\xi_n^2)^3}c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
 &+ \frac{ih'}{(1+\xi_n^2)^3}c(\xi)c(dx_n)c(\xi)c(dx_n)\partial_{x_n}[c(\xi)](x_0) \\
 &+ \frac{-ih'^2}{(1+\xi_n^2)^4}c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
 &+ \frac{1}{8}c(\xi)\sum_{\beta i s \alpha}R_{\beta i s \alpha}^{\partial M}(x_0)\frac{-i}{(1+\xi_n^2)^3}c(\xi)c(\tilde{e}_i)c(\xi)c(\tilde{e}_\beta)c(\tilde{e}_s)c(\tilde{e}_\alpha)c(\xi) \\
 &+ \frac{1}{6}\sum_{l,t < n}\left(R_{tilj}^{\partial M}(x_0)+R_{ijli}^{\partial M}(x_0)\right)\frac{-i}{(1+\xi_n^2)^3}c(\xi)c(\tilde{e}_i)c(\xi)c(dx_j)c(\tilde{e}_i) \\
 &+ \frac{1}{3}\sum_{\alpha,\beta < n}\left(R_{i\alpha j\beta}^{\partial M}(x_0)+R_{i\beta j\alpha}^{\partial M}(x_0)\right)\xi_\alpha\xi_\beta\frac{i}{(1+\xi_n^2)^4}c(\xi)c(\tilde{e}_i)c(\xi)c(dx_j)c(\xi) \\
 &+ i\left(\frac{h'}{(1+\xi_n^2)^3}+\frac{h'}{(1+\xi_n^2)^4}\right)c(\xi)c(dx_n)\partial_{x_n}[c(\xi)](x_0)c(dx_n)c(\xi) \\
 &- i\left(\frac{h'^2-h''}{(1+\xi_n^2)^3}+\frac{2h'^2-h''}{(1+\xi_n^2)^4}+\frac{3h'^2}{(1+\xi_n^2)^5}\right)c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
 &+ i\left(\frac{h'}{(1+\xi_n^2)^3}+\frac{3h'}{(1+\xi_n^2)^4}\right)c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
 &+ \frac{-i}{(1+\xi_n^2)^3}c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]c(dx_n)\partial_{x_n}[c(\xi)](x_0) \\
 &+ \left(\frac{3h'^2}{4}-\frac{h''}{2}\right)\frac{-i}{(1+\xi_n^2)^4}c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi'). \tag{3.22}
 \end{aligned}$$

Lemma 3.5 *For twisted Dirac operators, the following identity holds:*

$$\begin{aligned}
 \sigma_{-3}(\tilde{D}_H^{-1})(x_0)|_{|\xi'|=1} &= \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + \frac{ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^6} + \frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^6} \\
 &+ \frac{-ic(\xi)\gamma c(\xi)\gamma c(\xi)}{|\xi|^6} + \frac{-ih'c(\xi)\gamma c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6} + \frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^8} \\
 &+ \frac{-ic(\xi)}{|\xi|^6}\sum_{j=1}^{n-1}c(\tilde{e}_j)c(\xi)\partial_{x_i}(\gamma)c(\xi) + \frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\gamma c(\xi)}{|\xi|^6} \\
 &+ \frac{2ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^8} + \frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^6} \\
 &+ \frac{-ic(\xi)c(dx_n)c(\xi)\gamma\partial_{x_n}[c(\xi)]}{|\xi|^6} \\
 &:= \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + Q_{-3}(x_0)|_{|\xi'|=1}, \tag{3.23}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{-3}((\tilde{D}_H^*)^{-1})(x_0)|_{|\xi'|=1} &= \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + \frac{ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^6} + \frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^6} \\
 &+ \frac{-ic(\xi)\tau c(\xi)\tau c(\xi)}{|\xi|^6} + \frac{-ih'c(\xi)\tau c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6} + \frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^8} \\
 &+ \frac{-ic(\xi)}{|\xi|^6} \sum_{j=1}^{n-1} c(\tilde{e}_j)c(\xi)\partial_{x_i}(\tau)c(\xi) + \frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\tau c(\xi)}{|\xi|^6} \\
 &+ \frac{2ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^8} + \frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\tau)c(\xi)}{|\xi|^6} \\
 &+ \frac{-ic(\xi)c(dx_n)c(\xi)\tau\partial_{x_n}[c(\xi)]}{|\xi|^6} \\
 &:= \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + R_{-3}(x_0)|_{|\xi'|=1}, \tag{3.24}
 \end{aligned}$$

where $\gamma = \sum_{j=1}^n c(\tilde{e}_j)(\sigma_j^H + \Phi(\tilde{e}_j))$, and $\tau = \sum_{j=1}^n c(\tilde{e}_j)(\sigma_j^H - \Phi^*(\tilde{e}_j))$.

When $n = 5$, then $\text{tr}_{S(TM)} \otimes_H [\text{id}] = 4 \dim H$, where tr as shorthand of trace, the sum is taken over $-r - l + k + j + |\alpha| = 4$, $r \leq -1, l \leq -1$, then we have the following fifteen cases:

case (1) $r = -1, l = -1, k = 0, j = |\alpha| = 1$.

By (2.15), we get

$$\Psi_1 = \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{x_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(\tilde{D}_H^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.25}$$

By Lemma 3.3, for $i < n$, then

$$\partial_{x_i} \left(\sigma_{-1}(\tilde{D}_H^{-1})(x_0) \right) (x_0) = \frac{i \partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{ic(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \tag{3.26}$$

so $\Psi_1 = 0$.

case (2) $r = -1, l = -1, k = |\alpha| = 0, j = 2$.

By (2.15), we get

$$\Psi_2 = \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j=2} \text{tr}[\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n}^3 \sigma_{-1}(\tilde{D}_H^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.27}$$

By Lemma 3.2, we have

$$\partial_{\xi_n}^3 \sigma_{-1}((\tilde{D}_H)^{-1})(x_0) |_{|\xi'|=1} = ic(\xi') \frac{24\xi_n - 24\xi_n^3}{(1 + \xi_n^2)^4} + ic(dx_n) \frac{-64\xi_n^4 + 36\xi_n^2 - 6}{(1 + \xi_n^2)^4}; \tag{3.28}$$

From the Lemma 3.2-3.4, we get

$$\begin{aligned} \partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1})|_{|\xi'|=1} &= \left(\frac{3}{4}(h'(0))^2 - \frac{1}{2}h''(0) \right) \frac{c(\xi')}{2(\xi_n - i)} - h'(0) \frac{\xi_n - 2i}{2(\xi_n - i)} \partial_{x_n} [c(\xi')] \\ &\quad - h''(0) \left[\frac{\xi_n - 2i}{4(\xi_n - i)} c(\xi') + \frac{1}{4(\xi_n - i)^2} c(d_{x_n}) \right] \\ &\quad + 2i(h'(0))^2 \left[\frac{-3i\xi_n^2 - 9\xi_n + 8i}{16(\xi_n - i)^3} c(\xi') + \frac{-i\xi_n - 3}{16(\xi_n - i)^3} c(d_{x_n}) \right]. \end{aligned} \tag{3.29}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we have the equalities:

$$\begin{aligned} \text{tr}[c(\xi')c(dx_n)] &= 0; \quad \text{tr}[c(dx_n)^2] = -4\dim H; \quad \text{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4\dim H; \\ \text{tr}[\partial_{x_n} c(\xi')c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n} c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0)\dim H. \end{aligned} \tag{3.30}$$

Similarly, we have

$$\begin{aligned} &\text{tr} \left[\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \partial_{\xi_n}^3 \sigma_{-1}(\tilde{D}_H^{-1}) \right] (x_0)|_{|\xi'|=1} \\ &= h'(0)^2 \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} \\ &\quad + h''(0) \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4}. \end{aligned} \tag{3.31}$$

Then,

$$\begin{aligned} \Psi_2 &= -\frac{1}{6}(h'(0))^2 \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{6}h''(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{6}(h'(0))^2 \Omega_3 \int_{\Gamma^+} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} d\xi_n dx' \\ &\quad - \frac{1}{6}h''(0) \Omega_3 \int_{\Gamma^+} \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} d\xi_n dx' \\ &= -\frac{1}{6}(h'(0))^2 \frac{2\pi i}{6!} \left[\frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{(1 + \xi_n^2)^4} \right]^{(6)} \Omega_3 dx' \\ &\quad - \frac{1}{6}h''(0) \frac{2\pi i}{5!} \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} \Omega_3 dx' \\ &= \left(\frac{29}{64}(h'(0))^2 - \frac{3}{8}h''(0) \right) \pi \Omega_3 dx'. \end{aligned} \tag{3.32}$$

where Ω_3 is the canonical volume of S^3 .

case (3) $r = -1, l = -1, j = 0, |\alpha| = 2, k = 0$.

$$\begin{aligned} \Psi_3 &= \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=2} \text{tr}[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^{\alpha} \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{4} s \partial_M \pi^3 dx', \end{aligned} \tag{3.33}$$

where $\Sigma_{t,l < n} R_{tll}^{\partial M}(x_0)$ is the scalar curvature $s_{\partial M}$ (see [8]).

case (4) $r = -1, l = -1, k = 1, j = 1, |\alpha| = 0$

$$\begin{aligned} \Psi_4 &= \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n}^3 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{5}{16} (h'(0))^2 \pi \Omega_3 dx'. \end{aligned} \tag{3.34}$$

case (5) $r = -1, l = -1, k = 1, j = 0, |\alpha| = 1$.

$$\begin{aligned} \Psi_5 &= \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi'}^{\alpha} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^{\alpha} \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= 0. \end{aligned} \tag{3.35}$$

case (6) $r = -1, l = -1, k = 2, j = |\alpha| = 0$.

$$\begin{aligned} \Psi_6 &= \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{k=2} \text{tr}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \partial_{x_n}^2 \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \left(\frac{29}{64} (h'(0))^2 - \frac{3}{8} h''(0) \right) \pi \Omega_3 dx'. \end{aligned} \tag{3.36}$$

case (7) $r = -1, l = -2, k = 0, j = 1, |\alpha| = 0$.

$$\begin{aligned} \Psi_7 &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-2}((D)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \frac{c(\xi) \gamma^c(\xi)}{|\xi|^4}](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.37}$$

By Lemma 3.3, we have

$$\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1})(x_0)|_{|\xi'|=1} = \frac{1}{(\xi_n - i)^3} \partial_{x_n}[c(\xi')](x_0) + h'(0) \left[\frac{4i - \xi_n}{2(\xi_n - i)^4} c(\xi') - \frac{3}{2(\xi_n - i)^4} c(dx_n) \right]. \quad (3.38)$$

From Lemma 3.2 and Lemma 3.3. we have

$$\begin{aligned} \sigma_{-2}(\tilde{D}_H^{-2}) &= \frac{c(\xi)\sigma_0(D_H)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j)[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)] \\ &= \sigma_{-2}(D^{-1}) + \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}. \end{aligned} \quad (3.39)$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\begin{aligned} \text{tr}[\partial_{x_n}[c(\xi')]c(\xi)\gamma c(\xi)] &= 4\xi_n h'(0)\text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]; \quad \text{tr}[c(\xi')c(\xi)\gamma c(\xi)] = 8\xi_n \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]; \\ \text{tr}[c(dx_n)c(\xi)\gamma c(\xi)] &= -4(1 - \xi_n^2)\text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]; \quad \text{tr}[c(\xi')\gamma] = \text{tr}\left[-\sum_{j=1}^{n-1} \xi_j(\sigma_j^H + \Phi(\tilde{e}_j))\right]; \\ \text{tr}[c(dx_n)\gamma] &= \text{tr}[-id \otimes (\sigma_n^H + \Phi(\tilde{e}_n))] = -4\text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]. \end{aligned} \quad (3.40)$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$, we get:

$$\begin{aligned} &\text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0) \\ &= \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[\partial_{x_n}[c(\xi')]c(\xi)\gamma c(\xi)](x_0) + \frac{h'(0)(4i - \xi_n)}{2(\xi_n - i)^4 |\xi|^4} \text{tr}[c(\xi')c(\xi)\gamma c(\xi)](x_0) \\ &\quad - \frac{3h'(0)}{2(\xi_n - i)^4 |\xi|^4} \text{tr}[c(dx_n)c(\xi)\gamma c(\xi)](x_0) \\ &= \left\{ \frac{4\xi_n h'(0)}{(\xi_n - i)^3 |\xi|^4} + \frac{(16\xi_n i - 10\xi_n^2 + 6)h'(0)}{(\xi_n - i)^4 |\xi|^4} \right\} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]. \end{aligned} \quad (3.41)$$

So we have:

$$\begin{aligned} \Psi_7 &= \frac{39}{32} (h'(0))^2 \pi \Omega_3 dx' \\ &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{39}{32} (h'(0))^2 \pi \Omega_3 dx' + \frac{(5i - 2)h'}{8} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx'. \end{aligned} \quad (3.42)$$

case (8) $r = -1$, $l = -2$, $k = j = 0$, $|\alpha| = 1$.

From (2.15) in [12] and the Leibniz rule, we get:

$$\begin{aligned}
 \Psi_8 &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &+ \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.43}
 \end{aligned}$$

By Lemmas 3.2 and 3.4, computations show:

$$\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1})(x_0)|_{|\xi'|=1} = \frac{-1}{2(\xi_n - i)^2} c(dx_i) - \xi_i \frac{3i - \xi_n}{2(\xi_n - i)^3} c(\xi') + \xi_i \frac{1}{(\xi_n - i)^3} c(dx_n). \tag{3.44}$$

From Lemma 3.2, we have:

$$\partial_{x'}^\alpha \frac{c(\xi)\gamma c(\xi)}{|\xi|^4} |_{|\xi'|=1} = \frac{c(\xi)}{|\xi|^4} \partial_{x_i}(\gamma) c(\xi). \tag{3.45}$$

By the relation of the Clifford action and $trAB = trBA$, then we have the equalities at a fixed point x_0 :

$$\begin{aligned}
 \text{tr}[c(dx_i)c(\xi)\partial_{x_i}(\gamma)c(\xi)](x_0) &= -4(1 - \xi_n^2 + 2\xi_i\xi_j)\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}; \\
 \xi_i\text{tr}[c(\xi')c(\xi)\partial_{x_i}(\gamma)c(\xi)](x_0) &= 4(1 - \xi_n^2)\xi_i\xi_j\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}; \\
 \xi_i\text{tr}[c(dx_n)c(\xi)\partial_{x_i}(\gamma)c(\xi)](x_0) &= 8\xi_n\xi_i\xi_j\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}; \tag{3.46}
 \end{aligned}$$

where $\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}(x_0) = \sum_{j=1}^{n-1} \text{tr}[\nabla_{\partial_{x_j}}^H(\sigma_j^H + \Phi(\tilde{e}_j))](x_0)$.

Combining (3.44)-(3.46) and direct computations, we obtain:

$$\begin{aligned}
 &\sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0) \\
 &= \frac{-1}{2(\xi_n - i)^2|\xi|^4} \text{tr}[c(dx_i)c(\xi)\partial_{x_i}(\gamma)c(\xi)] - \xi_i \frac{3i - \xi_n}{2(\xi_n - i)^3|\xi|^4} \text{tr}[c(\xi')c(\xi)\partial_{x_i}(\gamma)c(\xi)] \\
 &+ \xi_i \frac{1}{(\xi_n - i)^3|\xi|^4} \text{tr}[c(dx_n)c(\xi)\partial_{x_i}(\gamma)c(\xi)] \\
 &= \left(\frac{2}{(\xi_n - i)^3(\xi_n + i)} - \xi_i\xi_j \frac{4}{(\xi_n - i)^4(\xi_n + i)^2} + 2\xi_i\xi_j \frac{3\xi_n^2i - \xi_n^3 - 3i + 5\xi_n}{(\xi_n - i)^5(\xi_n + i)^2} \right) \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}. \tag{3.47}
 \end{aligned}$$

Considering for $i < n$, $\int_{|\xi'|=1} \int_{i_1} \int_{i_2} \cdots \int_{i_{2q+1}} \sigma(\xi') = 0$, and $\int_{S^3} \xi_i \xi_j = \frac{\pi^2}{2} \delta^{ij}$ see ([6]). Therefore,

$$\begin{aligned} \Psi_8 &= \left(\frac{3}{16} + \frac{3i}{32}\right) s_{\partial M} \pi \Omega_3 dx' \\ &+ \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left(\frac{2}{(\xi_n - i)^3 (\xi_n + i)} - \xi_i \xi_j \frac{4}{(\xi_n - i)^4 (\xi_n + i)^2} + 2\xi_i \xi_j \frac{3\xi_n^2 i - \xi_n^3 - 3i + 5\xi_n}{(\xi_n - i)^5 (\xi_n + i)^2} \right) \\ &\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}(x_0) d\xi_n \sigma(\xi') dx' \\ &= \left(\frac{3}{16} + \frac{3i}{32}\right) s_{\partial M} \pi \Omega_3 dx' - \left(4 + \frac{\pi^2}{2}\right) \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \pi \Omega_3 dx'. \end{aligned} \tag{3.48}$$

case (9) $r = -1, l = -2, k = 1, j = |\alpha| = 0$.

From (2.15) and the Leibniz rule, we get:

$$\begin{aligned} \Psi_9 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &+ \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr} \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \left(\frac{c(\xi) \gamma c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.49}$$

By (2.2.29) in [12], we have:

$$\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) |_{|\xi'|=1} = \frac{1}{(\xi_n - i)^3} c(\xi') + \frac{i}{2(\xi_n - i)^3} c(dx_n). \tag{3.50}$$

From Lemma 3.2, a simple computation shows:

$$\partial_{x_n} \left(\frac{c(\xi) \gamma c(\xi)}{|\xi|^4} \right) = \frac{\partial_{x_n}[c(\xi')] \gamma c(\xi)}{|\xi|^4} - \frac{2h' c(\xi) \gamma c(\xi)}{|\xi|^6} + \frac{c(\xi) \partial_{x_n}(\gamma) c(\xi)}{|\xi|^4} + \frac{c(\xi) \gamma \partial_{x_n}[c(\xi')]}{|\xi|^4}. \tag{3.51}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\begin{aligned} \text{tr}[c(\xi) \partial_{x_n}(\gamma) c(\xi) c(\xi')](x_0) &= 8\xi_n \text{tr}[\partial_{x_n}(\sigma_n^H + \Phi(\tilde{e}_n))]; \\ \text{tr}[c(\xi) \partial_{x_n}(\gamma) c(\xi) c(dx_n)](x_0) &= -4(1 - \xi_n^2) \text{tr}[\partial_{x_n}(\sigma_n^H + \Phi(\tilde{e}_n))]. \end{aligned} \tag{3.52}$$

Combining (3.50)-(3.52) and direct computations, we obtain:

$$\begin{aligned}
 & \sum_{|\alpha|=1} \operatorname{tr} \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \right] \\
 &= \frac{1}{(\xi_n - i)^3 |\xi|^4} \operatorname{tr}[\partial_{x_n}[c(\xi')] \gamma c(\xi) c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \operatorname{tr}[\partial_{x_n}[c(\xi')] \gamma c(\xi) c(dx_n)] \\
 &\quad - \frac{2h'}{(\xi_n - i)^3 |\xi|^6} \operatorname{tr}[c(\xi) \gamma c(\xi) c(\xi')] - \frac{ih'}{(\xi_n - i)^3 |\xi|^6} \operatorname{tr}[c(\xi) \gamma c(\xi) c(dx_n)] \\
 &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} \operatorname{tr}[c(\xi) \partial_{x_n}(\gamma) c(\xi) c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \operatorname{tr}[c(\xi) \partial_{x_n}(\gamma) c(\xi) dx_n] \\
 &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} \operatorname{tr}[c(\xi) \gamma \partial_{x_n}[c(\xi')] c(\xi)] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \operatorname{tr}[c(\xi) \gamma \partial_{x_n}[c(\xi')] c(dx_n)] \\
 &= \left(\frac{(4\xi_n - 2i)h'(0)}{(\xi_n - i)^5 (\xi_n + i)^2} + \frac{h'(0)(4i - 16\xi_n - 4i\xi_n^2)}{(\xi_n - i)^6 (\xi_n + i)^3} \right) \operatorname{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \\
 &\quad + \frac{8\xi_n + 2\xi_n^2 i - 2i}{(\xi_n - i)^5 (\xi_n + i)^2} \operatorname{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))]. \tag{3.53}
 \end{aligned}$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. Therefore,

$$\begin{aligned}
 \Psi_9 &= \left(\frac{-367}{128} (h'(0))^2 + \frac{103}{64} h''(0) \right) \pi \Omega_3 dx' \\
 &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left(\frac{(4\xi_n - 2i)h'(0)}{(\xi_n - i)^5 (\xi_n + i)^2} + \frac{h'(4i - 16\xi_n - 4i\xi_n^2)}{(\xi_n - i)^6 (\xi_n + i)^3} \right) \operatorname{tr}[\sigma_n^H + \Phi(\tilde{e}_n)](x_0) d\xi_n \sigma(\xi') dx' \\
 &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{8\xi_n + 2\xi_n^2 i - 2i}{(\xi_n - i)^5 (\xi_n + i)^2} \operatorname{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))](x_0) d\xi_n \sigma(\xi') dx' \\
 &= \left(\frac{-367}{128} (h'(0))^2 + \frac{103}{64} h''(0) \right) \pi \Omega_3 dx' - \frac{153h'}{128} \operatorname{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' \\
 &\quad - \frac{9}{16} \operatorname{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx'. \tag{3.54}
 \end{aligned}$$

case (10) $r = -2, l = -1, k = 0, j = 1, |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned}
 \Psi_{10} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\partial_{x_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.55}
 \end{aligned}$$

By the Leibniz rule, trace property and "++" and "--" vanishing after the integration over ξ_n in [3], then

$$\begin{aligned} & \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_H^{-1}) \right] d\xi_n \\ &= \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_H^{-1}) \right] d\xi_n - \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \right] d\xi_n. \end{aligned} \quad (3.56)$$

By Lemma 3.2, a simple computation shows:

$$\partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H)^{-1})(x_0)|_{|\xi'|=1} = \frac{6\xi_n^2 - 2}{(1 + \xi_n^2)^3} ic(\xi') + \frac{2\xi_n^3 - 6\xi_n}{(1 + \xi_n^2)^3} ic(dx_n). \quad (3.57)$$

Similarly,

$$\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) = \frac{\partial_{x_n}[c(\xi')]\gamma c(\xi)}{|\xi|^4} - \frac{2h'c(\xi)\gamma c(\xi)}{|\xi|^6} + \frac{c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^4} + \frac{c(\xi)\gamma\partial_{x_n}[c(\xi')]}{|\xi|^4}. \quad (3.58)$$

Combining (3.57), (3.58) and direct computations, we obtain

$$\begin{aligned} & \text{tr} \left[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H)^{-1}) \right] \\ &= \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \text{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(dx_n)] \\ &\quad - \frac{2h'}{(\xi_n - i)^3 |\xi|^6} \text{tr}[c(\xi)\gamma c(\xi)c(\xi')] - \frac{ih'}{(\xi_n - i)^3 |\xi|^6} \text{tr}[c(\xi)\gamma c(\xi)c(dx_n)] \\ &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(dx_n)] \\ &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(\xi)] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(dx_n)] \\ &= \frac{16h'i(\xi_n^3 - 2\xi_n^2 - \xi_n^5)}{(\xi_n - i)^5 (\xi_n + i)^5} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))] + \frac{8i(2\xi_n^3 + \xi_n^5 + \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^5} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))]. \end{aligned} \quad (3.59)$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. Therefore,

$$\begin{aligned} \Psi_{10} &= \left(-\frac{367}{128} h'^2 + \frac{103}{64} h'' \right) \pi \Omega_3 dx' \\ &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \left(-\frac{367}{128} h'^2 + \frac{103}{64} h'' \right) \pi \Omega_3 dx' - \frac{129h'}{128} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' \\ &\quad - \frac{9}{16} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx'. \end{aligned} \quad (3.60)$$

case (11) $r = -2, l = -1, k = j = |\alpha| = 0$.

From (2.15), we have:

$$\Psi_{11} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi'}^{\alpha} \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.61}$$

By Lemma 3.2 and Lemma 3.3, for $i < n$, we have:

$$\partial_{x_i} \sigma_{-1}((\tilde{D}_H)^{-1})(x_0) = \partial_{x_i} \left(\frac{ic(\xi)}{|\xi|^2} \right) (x_0) = \frac{i \partial_{x_i} [c(\xi)](x_0)}{|\xi|^2} - \frac{ic(\xi) \partial_{x_i} (|\xi|)^2(x_0)}{|\xi|^4} = 0. \tag{3.62}$$

So $\Psi_{11} = 0$.

case (12) $r = -2, l = -1, k = 1, j = |\alpha| = 0$

From (2.15), we have:

$$\begin{aligned} \Psi_{12} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \Psi_7 + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\sigma_{-2}(D^{-1}) \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}\left[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})\right](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.63}$$

From Lemma 3.2-Lemma 3.3 and direct computations, we obtain:

$$\partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})(x_0)|_{|\xi'|=1} = \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^3} \partial_{x_n} [c(\xi')](x_0) + ih'(0) \left[\frac{4(1 - 5\xi_n^2)}{(1 + \xi_n^2)^4} c(\xi') - \frac{12\xi_n(\xi_n^{-1})}{(1 + \xi_n^2)^4} c(dx_n) \right]. \tag{3.64}$$

Combining (3.64) and direct computations, we obtain:

$$\begin{aligned} &\text{tr} \left[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1}) \right] \\ &= \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^5} \text{tr}[c(\xi)\gamma c(\xi) \partial_{x_n} [c(\xi')]] + ih'(0) \frac{4(1 - 5\xi_n^2)}{(1 + \xi_n^2)^6} \text{tr}[c(\xi)\gamma c(\xi) c(\xi')] - \frac{12\xi_n(\xi_n^{-1})}{(1 + \xi_n^2)^6} \text{tr}[c(\xi)\gamma c(\xi) c(dx_n)] \\ &= \left(\frac{4\xi_n i(6\xi_n^2 - 2)h'(0)}{(1 + \xi_n^2)^5} + \frac{-16i(\xi_n + 4\xi_n^3 + 3\xi_n^5)h'(0)}{(1 + \xi_n^2)^6} \right) \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]. \end{aligned} \tag{3.65}$$

Combining these assertions, we get:

$$\begin{aligned} \Psi_{12} &= \Psi_7 + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}\left[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H)^{-1})\right](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{39}{32} (h'(0)^2) \pi \Omega_3 dx' + \frac{(5i + 6)h'}{8} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx'. \end{aligned} \tag{3.66}$$

case (13) $r = -2, l = -2, k = j = |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned} \Psi_{13} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &+ i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)](x_0) d\xi_n \sigma(\xi') dx' \\ &+ i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right) \times \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &+ i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right) \times \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.67}$$

By [9], we obtain:

$$\pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right) = \frac{(-i\xi_n - 2)c(\xi')\gamma c(\xi') - i[c(dx_n)\gamma c(\xi') + c(\xi')\gamma c(dx_n)] - i\xi_n c(dx_n)\gamma c(dx_n)}{4(\xi_n - i)^2}. \tag{3.68}$$

$$\sigma_{-2}(D^{-1}) = -h' \frac{\xi_n^2 + 2}{(1 + \xi_n^2)^3} c(\xi)c(dx_n)c(\xi) + \frac{1}{(1 + \xi_n^2)^3} c(\xi)c(dx_n)\partial_{x_n}[c(\xi')]. \tag{3.69}$$

Then we have:

$$\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right) = \frac{(\xi_n i + 3)c(\xi')\gamma c(\xi')}{4(\xi_n - i)^3} + \frac{(\xi_n i - 1)c(dx_n)\gamma c(\xi')}{4(\xi_n - i)^3} + \frac{ic(dx_n)\gamma c(\xi')}{2(\xi_n - i)^3} \frac{ic(\xi')\gamma c(dx_n)}{2(\xi_n - i)^3}. \tag{3.70}$$

$$\begin{aligned} \partial_{\xi_n} \pi_{\xi_n}^+ (\sigma_{-2}(D^{-1}))(x_0)|_{|\xi'|=1} &= h'(0) \frac{-i\xi_n - 3}{4(\xi_n - i)^3} c(\xi')c(dx_n)c(\xi') + h'(0) \frac{i}{(\xi_n - i)^3} c(\xi') \\ &+ h'(0) \frac{i\xi_n - 1}{4(\xi_n - i)^3} c(dx_n) + \frac{-i}{2(\xi_n - i)^3} \partial_{x_n}[c(\xi')](x_0) \\ &+ \frac{i\xi_n + 3}{4(\xi_n - i)^3} c(\xi')c(dx_n)\partial_{x_n}[c(\xi')](x_0) \\ &+ h'(0) \frac{-2i\xi_n - 8}{8(\xi_n - i)^4} c(\xi') + h'(0) \frac{i\xi_n^2 + 4\xi_n - 9i}{8(\xi_n - i)^4} c(dx_n). \end{aligned} \tag{3.71}$$

So

$$\begin{aligned} &i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{13h'}{16} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]\pi\Omega_3 dx'. \end{aligned} \tag{3.72}$$

Similarly, we obtain:

$$\begin{aligned} & i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{73h'}{32} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx'. \end{aligned} \tag{3.73}$$

And

$$\begin{aligned} & i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right)](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{16} \sum_{i=1}^n \text{tr}[(\sigma_i^H + \Phi(\tilde{e}_i))^2] \pi \Omega_3 dx'. \end{aligned} \tag{3.74}$$

Combining these assertions, we see:

$$\Psi_{13} = -\frac{821}{256} \pi \Omega_3 dx' + \frac{h'}{4} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' - \frac{1}{16} \sum_{i=1}^n \text{tr}[(\sigma_i^H + \Phi(\tilde{e}_i))^2] \pi \Omega_3 dx'. \tag{3.75}$$

case (14) $r = -1, l = -3, k = j = |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned} \Psi_{14} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-3}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-3}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-3}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &+ i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times Q_{-3}|_{|\xi'|=1}](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.76}$$

By (2.2.29) in [12], we have:

$$\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}. \tag{3.77}$$

In the orthonormal frame filed, we have:

$$\begin{aligned} Q_{-3}|_{|\xi'|=1} &= \frac{ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^6} + \frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^6} + \frac{-ic(\xi)\gamma c(\xi)\gamma c(\xi)}{|\xi|^6} \\ &+ \frac{-ih'c(\xi)\gamma c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6} + \frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^8} + \frac{-ic(\xi)}{|\xi|^6} \sum_{j=1}^{n-1} c(\tilde{e}_j)c(\xi)\partial_{x_i}(\gamma)c(\xi) \\ &+ \frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\gamma c(\xi)}{|\xi|^6} + \frac{2ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^8} + \frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^6} \\ &+ \frac{-ic(\xi)c(dx_n)c(\xi)\gamma\partial_{x_n}[c(\xi)]}{|\xi|^6}. \end{aligned} \tag{3.78}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\begin{aligned} \text{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\gamma c(\xi)] &= (\xi_n^3 - 3\xi_n)\text{tr}[c(dx_n)\gamma] = -4(\xi_n^3 - 3\xi_n)\text{tr}[\sigma_n^H + \Phi(\widetilde{e}_n)]; \\ \text{tr}[c(dx_n)c(\xi)\gamma c(\xi)\gamma c(\xi)] &= 4(3\xi_n - \xi_n^3)\{\text{tr}[(\sigma_n^H + \Phi(\widetilde{e}_n))^2] - 4(3\xi_n - \xi_n^3) \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\widetilde{e}_j))^2]\}; \\ \text{tr}[c(dx_n)c(\xi) \sum_{j=1}^{n-1} c(\widetilde{e}_j)c(\xi)\partial_{x_i}(\gamma)c(\xi)] &= 4(\xi_n^3 - 3\xi_n)\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}; \\ \text{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\partial_{x_n}(\gamma)c(\xi)] &= 4(3\xi_n - \xi_n^3)\text{tr}[\nabla_{\partial_{x_n}}^H(\sigma_n^H + \Phi(\widetilde{e}_n))]. \end{aligned} \tag{3.79}$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1}\xi_{i_2} \cdots \xi_{i_{2q+1}}\sigma(\xi') = 0$. From Lemma 3.4, combining (3.77)-(3.79) and direct computations, we obtain:

$$\begin{aligned} &i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\widetilde{D}_H^{-1}) \times Q_{-3}|_{|\xi'|=1}](x_0)d\xi_n\sigma(\xi')dx' \\ &= -\frac{51h'}{16}\text{tr}[\sigma_n^H + \Phi(\widetilde{e}_n)]\pi\Omega_3dx' + \frac{3}{4}\text{tr}[\nabla_{\partial_{x_n}}^H(\sigma_n^H + \Phi(\widetilde{e}_n))]\pi\Omega_3dx' \\ &- \frac{3}{4}\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}\pi\Omega_3dx' - \frac{3}{4}\sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\widetilde{e}_j))^2] + \frac{3}{4}\text{tr}[(\sigma_n^H + \Phi(\widetilde{e}_n))^2]. \end{aligned} \tag{3.80}$$

Therefore,

$$\begin{aligned} \Psi_{14} &= \left(\frac{239}{64}(h'(0))^2 - \frac{27}{16}h''(0) - \frac{11}{192}s_{\partial M}\right)\pi\Omega_3dx' \\ &- \frac{51h'}{16}\text{tr}[\sigma_n^H + \Phi(\widetilde{e}_n)]\pi\Omega_3dx' + \frac{3}{4}\text{tr}[\nabla_{\partial_{x_n}}^H(\sigma_n^H + \Phi(\widetilde{e}_n))]\pi\Omega_3dx' \\ &- \frac{3}{4}\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\}\pi\Omega_3dx' - \frac{3}{4}\sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\widetilde{e}_j))^2]\pi\Omega_3dx' \\ &+ \frac{3}{4}\text{tr}[(\sigma_n^H + \Phi(\widetilde{e}_n))^2]\pi\Omega_3dx'. \end{aligned} \tag{3.81}$$

case (15) $r = -3$, $l = -1$, $k = j = |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned} \Psi_{15} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-3}(\widetilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx' \\ &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-3}(D^{-1}) \times \partial_{\xi_n} \sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx' \\ &- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ Q_{-3}|_{|\xi'|=1} \times \partial_{\xi_n} \sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \end{aligned} \tag{3.82}$$

By the Leibniz rule, trace property and "++" and "--" vanishing after the integration over ξ_n in [3], then

$$\begin{aligned} & \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ Q_{-3}|_{|\xi'|=1} \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_H^{-1})](x_0) d\xi_n \\ &= \int_{-\infty}^{+\infty} \text{tr}[Q_{-3}|_{|\xi'|=1} \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_H^{-1})](x_0) d\xi_n - \int_{-\infty}^{+\infty} \text{tr}[Q_{-3}|_{|\xi'|=1} \times \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1})](x_0) d\xi_n. \end{aligned} \tag{3.83}$$

By Lemma 3.2, a simple computation shows:

$$\partial_{\xi_n} \sigma_{-1}(\tilde{D}_H^{-1}) = \frac{-2i\xi_n c(\xi') + i(1 - \xi_n^2) c(dx_n)}{(\xi_n^2 + 1)^2}. \tag{3.84}$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. From Lemma 3.4, combining (3.84) and direct computations, we obtain:

$$\begin{aligned} & -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[Q_{-3}|_{|\xi'|=1} \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_H^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{17h'}{48} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' - \frac{11}{24} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx' \\ &+ \frac{11}{24} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \pi \Omega_3 dx' - \frac{11}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \pi \Omega_3 dx' \\ &+ \frac{11}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \pi \Omega_3 dx'. \end{aligned} \tag{3.85}$$

Therefore, we obtain:

$$\begin{aligned} \Psi_{15} &= \left(\frac{239}{64} (h'(0))^2 - \frac{27}{16} h''(0) - \frac{11}{192} s_{\partial M} \right) \pi \Omega_3 dx' \\ &- \frac{17h'}{6} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' + \frac{7}{24} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx' \\ &- \frac{7}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \pi \Omega_3 dx' + \frac{7}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \pi \Omega_3 dx' \\ &- \frac{7}{24} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \pi \Omega_3 dx'. \end{aligned} \tag{3.86}$$

Now Ψ is the sum of the $\Psi_{(1,2,\dots,15)}$, then we obtain:

$$\begin{aligned} \sum_{i=1}^{15} \Psi_i &= \left(\frac{399}{256} (h'(0))^2 - \frac{29}{32} h''(0) + \left(\frac{71}{96} + \frac{3i}{32} \right) s_{\partial M} \right) \pi \Omega_3 dx' \\ &+ \frac{(240i - 529)h'}{192} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' + \frac{23}{48} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx' \\ &- \frac{8\pi^2 + 89}{16} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \pi \Omega_3 dx' - \frac{1}{16} \sum_{j=1}^n \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \pi \Omega_3 dx' \\ &- \frac{25}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \pi \Omega_3 dx' + \frac{25}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \pi \Omega_3 dx'. \end{aligned} \tag{3.87}$$

Hence,

Theorem 3.6 *Let M be a 5-dimensional compact spin manifold with the boundary ∂M , and twisted Dirac operator $\tilde{D}_H = \sum_{j=1}^n c(\tilde{e}_j) \nabla_{\tilde{e}_j}^{S(TM)} \otimes^H + \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi(\tilde{e}_j)$, then*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+(\tilde{D}_H^{-1}) \circ \pi^+(\tilde{D}_H^{-1})] &= \int_{\partial M} \left[\left(\frac{399}{256} (h'(0))^2 - \frac{29}{32} h''(0) + \left(\frac{71}{96} + \frac{3i}{32} \right) s_{\partial M} \right) \right. \\ &\quad + \frac{(240i - 529)h'}{192} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] + \frac{23}{48} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \\ &\quad - \frac{8\pi^2 + 89}{16} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} - \frac{1}{16} \sum_{j=1}^n \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \\ &\quad \left. - \frac{25}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] + \frac{25}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \right] \pi \Omega_3 \text{dvol}_{\partial M}. \end{aligned} \tag{3.88}$$

Next we recall the Einstein-Hilbert action for manifolds with boundary (see [12] or [13]),

$$I_{Gr} = \frac{1}{16\pi} \int_M s \text{dvol}_M + 2 \int_{\partial M} K s \text{dvol}_{\partial M} := I_{Gr,i} + I_{Gr,b}, \tag{3.89}$$

where

$$K = \sum_{1 \leq i, j \leq n-1} K_{ij} g_{\partial M}^{ij}; \quad K_{ij} = -\Gamma_{ij}^n, \tag{3.90}$$

and K_{ij} is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma A.2 in [12], $K_{ij}(x_0) = \sum_{ij} K_{ij}(x_0) g_{\partial M}^{ij}(x_0) = \sum_{i=1}^4 K_{ii}(x_0) = -2h'(0)$ Then

$$I_{Gr,b} = -4h'(0) \text{vol}_{\partial M}. \tag{3.91}$$

Then, we have:

Theorem 3.7 *Let M be a 5-dimensional compact spin manifold with the boundary ∂M , and twisted Dirac operator $\tilde{D}_H = \sum_{j=1}^n c(\tilde{e}_j) \nabla_{\tilde{e}_j}^{S(TM)} \otimes^H + \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi(\tilde{e}_j)$. The following identity holds:*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+(\tilde{D}_H^{-1}) \circ \pi^+(\tilde{D}_H^{-1})] &= \int_{\partial M} \left[\left(\frac{225}{64} K^2 + \frac{29}{8} s_M|_{\partial M} + \left(\frac{197}{24} + \frac{3i}{2} \right) s_{\partial M} \right) \right. \\ &\quad + \frac{(529 - 240i)}{384} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] K + \frac{23}{48} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \\ &\quad - \frac{8\pi^2 + 89}{16} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} - \frac{1}{16} \sum_{j=1}^n \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \\ &\quad \left. - \frac{25}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] + \frac{25}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \right] \pi \Omega_3 \text{dvol}_{\partial M}. \end{aligned} \tag{3.92}$$

where $s_M, s_{\partial M}$ are respectively scalar curvatures on M and ∂M .

On the other hand, we compute $\widetilde{\text{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+(\widetilde{D}_H^*)^{-1}]$ for 5-dimensional spin manifolds with boundary, then we have:

$$\widetilde{\text{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+(\widetilde{D}_H^*)^{-1}] = \int_{\partial M} \widetilde{\Psi}. \tag{3.93}$$

When $n = 5$, then $\text{tr}_{S(TM)} \otimes_H [\text{id}] = 4\dim H$, where tr as shorthand of trace, the sum is taken over $-r - l + k + j + |\alpha| = 4$, $r \leq -1, l \leq -1$, then we have the following fifteen cases:

case (1) $r = -1, l = -1, k = 0, j = |\alpha| = 1$.

By (2.15), we get:

$$\widetilde{\Psi}_1 = \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{x_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\widetilde{D}_H^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.94}$$

By Lemma 3.3, for $i < n$, then

$$\partial_{x_i} \left(\sigma_{-1}((\widetilde{D}_H^*)^{-1})(x_0) \right) (x_0) = \frac{i \partial_{x_i} [c(\xi)](x_0)}{|\xi|^2} - \frac{ic(\xi) \partial_{x_i} (|\xi|^2)(x_0)}{|\xi|^4} = 0, \tag{3.95}$$

so $\widetilde{\Psi}_1 = 0$.

case (2) $r = -1, l = -1, k = |\alpha| = 0, j = 2$.

By (2.15), we get:

$$\widetilde{\Psi}_2 = \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j=2} \text{tr}[\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\widetilde{D}_H^{-1}) \times \partial_{\xi_n}^3 \sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.96}$$

By Lemma 3.2, we have:

$$\partial_{\xi_n}^3 \sigma_{-1}((\widetilde{D}_H^*)^{-1})(x_0) |_{|\xi'|=1} = ic(\xi') \frac{24\xi_n - 24\xi_n^3}{(1 + \xi_n^2)^4} + ic(dx_n) \frac{-64\xi_n^4 + 36\xi_n^2 - 6}{(1 + \xi_n^2)^4}; \tag{3.97}$$

Similarly, we get:

$$\begin{aligned} \partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\widetilde{D}_H^{-1}) |_{|\xi'|=1} &= \left(\frac{3}{4} (h'(0))^2 - \frac{1}{2} h''(0) \right) \frac{c(\xi')}{2(\xi_n - i)} - h'(0) \frac{\xi_n - 2i}{2(\xi_n - i)} \partial_{x_n} [c(\xi')] \\ &\quad - h''(0) \left[\frac{\xi_n - 2i}{4(\xi_n - i)} c(\xi') + \frac{1}{4(\xi_n - i)^2} c(dx_n) \right] \\ &\quad + 2i(h'(0))^2 \left[\frac{-3i\xi_n^2 - 9\xi_n + 8i}{16(\xi_n - i)^3} c(\xi') + \frac{-i\xi_n - 3}{16(\xi_n - i)^3} c(dx_n) \right]. \end{aligned} \tag{3.98}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we have the equalities:

$$\begin{aligned} \text{tr}[c(\xi')c(dx_n)] &= 0; \quad \text{tr}[c(dx_n)^2] = -4\dim H; \quad \text{tr}[c(\xi')^2](x_0) |_{|\xi'|=1} = -4\dim H; \\ \text{tr}[\partial_{x_n} c(\xi')c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n} c(\xi')c(\xi')](x_0) |_{|\xi'|=1} = -2h'(0)\dim H. \end{aligned} \tag{3.99}$$

Similarly, we have:

$$\begin{aligned} & \operatorname{tr} \left[\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1} (\tilde{D}_H^{-1}) \partial_{\xi_n}^3 \sigma_{-1} ((\tilde{D}_H^*)^{-1}) \right] (x_0) |_{|\xi'|=1} \\ &= h'(0)^2 \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} \\ &+ h''(0) \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4}. \end{aligned} \tag{3.100}$$

Then

$$\begin{aligned} \tilde{\Psi}_2 &= -\frac{1}{6}(h'(0))^2 \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\ &- \frac{1}{6}h''(0) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') dx' \\ &= \left(\frac{29}{64}(h'(0))^2 - \frac{3}{8}h''(0) \right) \pi \Omega_3 dx'. \end{aligned} \tag{3.101}$$

where Ω_3 is the canonical volume of S^3 .

case (3) $r = -1, l = -1, j = 0, |\alpha| = 2, k = 0$.

$$\begin{aligned} \tilde{\Psi}_3 &= \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=2} \operatorname{tr} [\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1} (\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1} ((\tilde{D}_H^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{4} s \partial_M \pi^3 dx', \end{aligned} \tag{3.102}$$

where $\Sigma_{t,l<n} R_{tll}^{\partial M}(x_0)$ is the scalar curvature $s_{\partial M}$ (see [8]).

case (4) $r = -1, l = -1, k = 1, j = 1, |\alpha| = 0$.

$$\begin{aligned} \tilde{\Psi}_4 &= \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr} [\partial_{x_n} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (\tilde{D}_H^{-1}) \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1} ((\tilde{D}_H^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{5}{16}(h'(0))^2 \pi \Omega_3 dx'. \end{aligned} \tag{3.103}$$

case (5) $r = -1, l = -1, k = 1, j = 0, |\alpha| = 1$.

$$\begin{aligned} \tilde{\Psi}_5 &= \frac{i}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \operatorname{tr} [\partial_{\xi'}^\alpha \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \partial_{x_n} \sigma_{-1} ((\tilde{D}_H^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &= 0. \end{aligned} \tag{3.104}$$

case (6) $r = -1, l = -1, k = 2, j = |\alpha| = 0$.

$$\begin{aligned} \tilde{\Psi}_6 &= \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{k=2} \text{tr}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \partial_{x_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \left(\frac{29}{64} (h'(0))^2 - \frac{3}{8} h''(0) \right) \pi \Omega_3 dx'. \end{aligned} \tag{3.105}$$

case (7) $r = -1, l = -2, k = 0, j = 1, |\alpha| = 0$.

$$\begin{aligned} \tilde{\Psi}_7 &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-2}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-2}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-2}((D)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \frac{c(\xi) \tau c(\xi)}{|\xi|^4}](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.106}$$

By Lemma 3.3, we have:

$$\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1})(x_0)|_{|\xi'|=1} = \frac{1}{(\xi_n - i)^3} \partial_{x_n} [c(\xi')](x_0) + h'(0) \left[\frac{4i - \xi_n}{2(\xi_n - i)^4} c(\xi') - \frac{3}{2(\xi_n - i)^4} c(dx_n) \right]. \tag{3.107}$$

From Lemma 3.2 and Lemma 3.3, we have:

$$\begin{aligned} \sigma_{-2}((\tilde{D}_H^*)^{-2}) &= \frac{c(\xi) \sigma_0(D_H) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) [\partial_{x_j}(c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j} (|\xi|^2)] \\ &= \sigma_{-2}(D^{-1}) + \frac{c(\xi) \tau c(\xi)}{|\xi|^4}. \end{aligned} \tag{3.108}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\begin{aligned} \text{tr}[\partial_{x_n} [c(\xi')] c(\xi) \tau c(\xi)] &= 4\xi_n h'(0) \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)]; \quad \text{tr}[c(\xi') c(\xi) \tau c(\xi)] = 8\xi_n \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)]; \\ \text{tr}[c(dx_n) c(\xi) \tau c(\xi)] &= -4(1 - \xi_n^2) \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]; \quad \text{tr}[c(\xi') \tau] = \text{tr}[-\sum_{j=1}^{n-1} \xi_j (\sigma_j^H - \Phi^*(\tilde{e}_j))]; \\ \text{tr}[c(dx_n) \tau] &= \text{tr}[-id \otimes (\sigma_n^H - \Phi^*(\tilde{e}_n))] = -4\text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)]. \end{aligned} \tag{3.109}$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. Then, we get:

$$\begin{aligned} & \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \frac{c(\xi)\tau c(\xi)}{|\xi|^4}]_{(x_0)} \\ &= \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[\partial_{x_n} [c(\xi')] c(\xi) \tau c(\xi)]_{(x_0)} + \frac{h'(0)(4i - \xi_n)}{2(\xi_n - i)^4 |\xi|^4} \text{tr}[c(\xi') c(\xi) \tau c(\xi)]_{(x_0)} \\ & \quad - \frac{3h'(0)}{2(\xi_n - i)^4 |\xi|^4} \text{tr}[c(dx_n) c(\xi) \tau c(\xi)]_{(x_0)} \\ &= \left\{ \frac{4\xi_n h'(0)}{(\xi_n - i)^3 |\xi|^4} + \frac{(16\xi_n i - 10\xi_n^2 + 6)h'(0)}{(\xi_n - i)^4 |\xi|^4} \right\} \text{tr}[\sigma_n^H - \Phi^*(\widetilde{e_n})]. \end{aligned} \tag{3.110}$$

So we have:

$$\begin{aligned} \tilde{\Psi}_7 &= \frac{39}{32} (h'(0))^2 \pi \Omega_3 dx' \\ & \quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}^2 \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \frac{c(\xi)\tau c(\xi)}{|\xi|^4}]_{(x_0)} d\xi_n \sigma(\xi') dx' \\ &= \frac{39}{32} (h'(0))^2 \pi \Omega_3 dx' + \frac{(5i - 2)h'}{8} \text{tr}[\sigma_n^H - \Phi^*(\widetilde{e_n})] \pi \Omega_3 dx'. \end{aligned} \tag{3.111}$$

case (8) $r = -1, l = -2, k = 0, j = 0, |\alpha| = 1$.

From 2.15 in [12] and the Leibniz rule, we get:

$$\begin{aligned} \tilde{\Psi}_8 &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-2}((\tilde{D}_H^*)^{-1})]_{(x_0)} d\xi_n \sigma(\xi') dx' \\ &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \sigma_{-2}((\tilde{D}_H^*)^{-1})]_{(x_0)} d\xi_n \sigma(\xi') dx' \\ &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \sigma_{-2}(D^{-1})]_{(x_0)} d\xi_n \sigma(\xi') dx' \\ & \quad + \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \frac{c(\xi)\tau c(\xi)}{|\xi|^4}]_{(x_0)} d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.112}$$

By Lemmas 3.2 and 3.3, computations show:

$$\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1})_{|\xi'|=1} = \frac{-1}{2(\xi_n - i)^2} c(dx_i) - \xi_i \frac{3i - \xi_n}{2(\xi_n - i)^3} c(\xi') + \xi_i \frac{1}{(\xi_n - i)^3} c(dx_n). \tag{3.113}$$

From Lemma 3.2, we have

$$\partial_{x'}^\alpha \frac{c(\xi)\tau c(\xi)}{|\xi|^4} |_{|\xi'|=1} = \frac{c(\xi)}{|\xi|^4} \partial_{x_i}(\tau) c(\xi). \tag{3.114}$$

By the relation of the Clifford action and $trAB = trBA$, then we have the equalities at a fixed point x_0 :

$$\begin{aligned} \operatorname{tr}[c(dx_i)c(\xi)\partial_{x_i}(\tau)c(\xi)](x_0) &= -4(1 - \xi_n^2 + 2\xi_i\xi_j)\operatorname{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\}; \\ \xi_i\operatorname{tr}[c(\xi')c(\xi)\partial_{x_i}(\tau)c(\xi)](x_0) &= 4(1 - \xi_n^2)\xi_i\xi_j\operatorname{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\}; \\ \xi_i\operatorname{tr}[c(dx_n)c(\xi)\partial_{x_i}(\tau)c(\xi)](x_0) &= 8\xi_n\xi_i\xi_j\operatorname{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\}; \end{aligned} \tag{3.115}$$

where $\operatorname{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\}(x_0) = \sum_{j=1}^{n-1} \operatorname{tr}[\nabla_{\partial_{x_j}}^H(\sigma_j^H - \Phi^*(\tilde{e}_j))](x_0)$.

Combining (3.113)-(3.115) and direct computations, we obtain:

$$\begin{aligned} &\sum_{|\alpha|=1} \operatorname{tr}[\partial_{\xi_n} \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x'}^\alpha \frac{c(\xi)\tau c(\xi)}{|\xi|^4}](x_0) \\ &= \frac{-1}{2(\xi_n - i)^2|\xi|^4} \operatorname{tr}[c(dx_i)c(\xi)\partial_{x_i}(\tau)c(\xi)] - \xi_i \frac{3i - \xi_n}{2(\xi_n - i)^3|\xi|^4} \operatorname{tr}[c(\xi')c(\xi)\partial_{x_i}(\tau)c(\xi)] \\ &+ \xi_i \frac{1}{(\xi_n - i)^3|\xi|^4} \operatorname{tr}[c(dx_n)c(\xi)\partial_{x_i}(\tau)c(\xi)] \\ &= \left(\frac{2}{(\xi_n - i)^3(\xi_n + i)} - \xi_i\xi_j \frac{4}{(\xi_n - i)^4(\xi_n + i)^2} + 2\xi_i\xi_j \frac{3\xi_n^2i - \xi_n^3 - 3i + 5\xi_n}{(\xi_n - i)^5(\xi_n + i)^2} \right) \operatorname{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\}. \end{aligned} \tag{3.116}$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi') = 0$, and $\int_{S^3} \xi_i\xi_j = \frac{\pi^2}{2}\delta^{ij}$ (see [6]). Therefore,

$$\begin{aligned} \tilde{\Psi}_8 &= \left(\frac{3}{16} + \frac{3i}{32} \right) s_{\partial M} \pi \Omega_3 dx' \\ &+ \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left(\frac{2}{(\xi_n - i)^3(\xi_n + i)} - \xi_i\xi_j \frac{4}{(\xi_n - i)^4(\xi_n + i)^2} + 2\xi_i\xi_j \frac{3\xi_n^2i - \xi_n^3 - 3i + 5\xi_n}{(\xi_n - i)^5(\xi_n + i)^2} \right) \\ &\operatorname{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\}(x_0) d\xi_n \sigma(\xi') dx' \\ &= \left(\frac{3}{16} + \frac{3i}{32} \right) s_{\partial M} \pi \Omega_3 dx' - \left(4 + \frac{\pi^2}{2} \right) \operatorname{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\} \pi \Omega_3 dx'. \end{aligned} \tag{3.117}$$

case (9) $r = -1, l = -2, k = 1, j = |\alpha| = 0$.

From (2.15) and the Leibniz rule, we get:

$$\begin{aligned} \tilde{\Psi}_9 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \operatorname{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-2}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \operatorname{tr}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \sigma_{-2}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \operatorname{tr}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &+ \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \operatorname{tr} \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.118}$$

By (2.2.29) in [12], we have:

$$\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1})|_{|\xi'|=1} = \frac{1}{(\xi_n - i)^3} c(\xi') + \frac{i}{2(\xi_n - i)^3} c(dx_n). \tag{3.119}$$

From Lemma 3.2, a simple computation shows:

$$\partial_{x_n} \left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4} \right) = \frac{\partial_{x_n}[c(\xi')]\tau c(\xi)}{|\xi|^4} - \frac{2h'c(\xi)\tau c(\xi)}{|\xi|^6} + \frac{c(\xi)\partial_{x_n}(\tau)c(\xi)}{|\xi|^4} + \frac{c(\xi)\tau\partial_{x_n}[c(\xi')]}{|\xi|^4}. \tag{3.120}$$

By the relation of the Clifford action and $trAB = trBA$, then we have the equalities:

$$\begin{aligned} tr[c(\xi)\partial_{x_n}(\tau)c(\xi)c(\xi')](x_0) &= 8\xi_n tr[\partial_{x_n}(\sigma_n^H - \Phi^*(\tilde{e}_n))]; \\ tr[c(\xi)\partial_{x_n}(\tau)c(\xi)c(dx_n)](x_0) &= -4(1 - \xi_n^2)tr[\partial_{x_n}(\sigma_n^H - \Phi^*(\tilde{e}_n))]. \end{aligned} \tag{3.121}$$

Combining (3.119)-(3.121) and direct computations, we obtain:

$$\begin{aligned} &\sum_{|\alpha|=1} tr \left[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{x_n} \left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4} \right) \right] \\ &= \frac{1}{(\xi_n - i)^3 |\xi|^4} tr[\partial_{x_n}[c(\xi')]\tau c(\xi)c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} tr[\partial_{x_n}[c(\xi')]\tau c(\xi)c(dx_n)] \\ &\quad - \frac{2h'}{(\xi_n - i)^3 |\xi|^6} tr[c(\xi)\tau c(\xi)c(\xi')] - \frac{ih'}{(\xi_n - i)^3 |\xi|^6} tr[c(\xi)\tau c(\xi)c(dx_n)] \\ &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} tr[c(\xi)\partial_{x_n}(\tau)c(\xi)c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} tr[c(\xi)\partial_{x_n}(\tau)c(\xi)dx_n] \\ &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} tr[c(\xi)\tau\partial_{x_n}[c(\xi')]c(\xi)] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} tr[c(\xi)\tau\partial_{x_n}[c(\xi')]c(dx_n)] \\ &= \left(\frac{(4\xi_n - 2i)h'(0)}{(\xi_n - i)^5(\xi_n + i)^2} + \frac{h'(0)(4i - 16\xi_n - 4i\xi_n^2)}{(\xi_n - i)^6(\xi_n + i)^3} \right) tr[\sigma_n^H - \Phi^*(\tilde{e}_n)] \\ &\quad + \frac{8\xi_n + 2\xi_n^2 i - 2i}{(\xi_n - i)^5(\xi_n + i)^2} tr[\nabla_{\partial_{x_n}}^H (\sigma_n^H - \Phi^*(\tilde{e}_n))]. \end{aligned} \tag{3.122}$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. Therefore,

$$\begin{aligned} \tilde{\Psi}_9 &= \left(\frac{-367}{128} (h'(0))^2 + \frac{103}{64} h''(0) \right) \pi \Omega_3 dx' \\ &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left(\frac{(4\xi_n - 2i)h'(0)}{(\xi_n - i)^5(\xi_n + i)^2} + \frac{h'(0)(4i - 16\xi_n - 4i\xi_n^2)}{(\xi_n - i)^6(\xi_n + i)^3} \right) tr[\sigma_n^H - \Phi^*(\tilde{e}_n)](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{8\xi_n + 2\xi_n^2 i - 2i}{(\xi_n - i)^5(\xi_n + i)^2} tr[\nabla_{\partial_{x_n}}^H (\sigma_n^H - \Phi^*(\tilde{e}_n))](x_0) d\xi_n \sigma(\xi') dx' \\ &= \left(\frac{-367}{128} (h'(0))^2 + \frac{103}{64} h''(0) \right) \pi \Omega_3 dx' - \frac{153h'}{128} tr[\sigma_n^H - \Phi^*(\tilde{e}_n)] \pi \Omega_3 dx' \\ &\quad - \frac{9}{16} tr[\nabla_{\partial_{x_n}}^H (\sigma_n^H - \Phi^*(\tilde{e}_n))] \pi \Omega_3 dx'. \end{aligned} \tag{3.123}$$

case (10) $r = -2, l = -1, k = 0, j = 1, |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned} \tilde{\Psi}_{10} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.124}$$

By the Leibniz rule, trace property and " + + " and " - - " vanishing after the integration over ξ_n in [3], then,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] d\xi_n \\ &= \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] d\xi_n - \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] d\xi_n. \end{aligned} \tag{3.125}$$

By Lemma 3.2, a simple computation shows:

$$\partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0)|_{|\xi'|=1} = \frac{6\xi_n^2 - 2}{(1 + \xi_n^2)^3} ic(\xi') + \frac{2\xi_n^3 - 6\xi_n}{(1 + \xi_n^2)^3} ic(dx_n). \tag{3.126}$$

From Lemma 3.2, a simple computation shows:

$$\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) = \frac{\partial_{x_n}[c(\xi')] \gamma c(\xi)}{|\xi|^4} - \frac{2h' c(\xi)\gamma c(\xi)}{|\xi|^6} + \frac{c(\xi)\partial_{x_n}(\gamma) c(\xi)}{|\xi|^4} + \frac{c(\xi)\gamma \partial_{x_n}[c(\xi')]}{|\xi|^4}. \tag{3.127}$$

Combining (3.126), (3.127) and direct computations, we obtain:

$$\begin{aligned} &\text{tr} \left[\partial_{x_n} \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] \\ &= \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[\partial_{x_n}[c(\xi')] \gamma c(\xi) c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \text{tr}[\partial_{x_n}[c(\xi')] \gamma c(\xi) c(dx_n)] \\ &\quad - \frac{2h'}{(\xi_n - i)^3 |\xi|^6} \text{tr}[c(\xi)\gamma c(\xi) c(\xi')] - \frac{ih'}{(\xi_n - i)^3 |\xi|^6} \text{tr}[c(\xi)\gamma c(\xi) c(dx_n)] \\ &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\partial_{x_n}(\gamma) c(\xi) c(\xi')] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\partial_{x_n}(\gamma) c(\xi) c(dx_n)] \\ &\quad + \frac{1}{(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\gamma \partial_{x_n}[c(\xi')] c(\xi)] + \frac{i}{2(\xi_n - i)^3 |\xi|^4} \text{tr}[c(\xi)\gamma \partial_{x_n}[c(\xi')] c(dx_n)] \\ &= \left(\frac{32i\partial_{x_n}(\sqrt{h})(x_0)(\xi_n^3 + \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^5} + \frac{-16h'i(2\xi_n^2 + \xi_n + \xi_n^5)}{(\xi_n - i)^5 (\xi_n + i)^5} \right) \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))] \\ &\quad + \frac{8i(2\xi_n^3 + \xi_n^5 + \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^5} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))]. \end{aligned} \tag{3.128}$$

Considering for $i < n$, $\int_{|\xi'|=1} \text{tr} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. Therefore,

$$\begin{aligned} \tilde{\Psi}_{10} &= \left(-\frac{367}{128} h'^2 + \frac{103}{64} h'' \right) \pi \Omega_3 dx' \\ &\quad - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \left(\frac{c(\xi) \gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\ &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\partial_{x_n} \left(\frac{c(\xi) \gamma c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\ &= \left(-\frac{367}{128} h'^2 + \frac{103}{64} h'' \right) \pi \Omega_3 dx' - \frac{129h'}{128} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' \\ &\quad - \frac{9}{16} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx'. \end{aligned} \tag{3.129}$$

case (11) $r = -2, l = -1, k = j = |\alpha| = 0$.

From (2.15), we have:

$$\tilde{\Psi}_{11} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi'}^\alpha \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.130}$$

By Lemmas 3.2 and 3.3, for $i < n$, we have:

$$\partial_{x_i} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) = \partial_{x_i} \left(\frac{ic(\xi)}{|\xi|^2} \right) (x_0) = \frac{i \partial_{x_i} [c(\xi)](x_0)}{|\xi|^2} - \frac{ic(\xi) \partial_{x_i} (|\xi|)^2(x_0)}{|\xi|^4} = 0. \tag{3.131}$$

So $\tilde{\Psi}_{11} = 0$.

case (12) $r = -2, l = -1, k = 1, j = |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned} \tilde{\Psi}_{12} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &= \Psi_7 + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} [\sigma_{-2}(D^{-1}) \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &\quad + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[\frac{c(\xi) \gamma c(\xi)}{|\xi|^4} \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \tag{3.132}$$

From Lemmas 3.2 and 3.4 and direct computations, we obtain:

$$\partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0)|_{|\xi'|=1} = \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^3} \partial_{x_n} [c(\xi')](x_0) + ih'(0) \left[\frac{4(1 - 5\xi_n^2)}{(1 + \xi_n^2)^4} c(\xi') - \frac{12\xi_n(\xi_n - 1)}{(1 + \xi_n^2)^4} c(dx_n) \right]. \tag{3.133}$$

Combining (3.133) and direct computations, we obtain:

$$\begin{aligned}
 & \operatorname{tr} \left[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right] \\
 &= \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^5} \operatorname{tr}[c(\xi)\gamma c(\xi)\partial_{x_n}[c(\xi')]] + ih'(0) \frac{4(1 - 5\xi_n^2)}{(1 + \xi_n^2)^6} \operatorname{tr}[c(\xi)\gamma c(\xi)c(\xi')] - \frac{12\xi_n(\xi_n^2 - 1)}{(1 + \xi_n^2)^6} \operatorname{tr}[c(\xi)\gamma c(\xi)c(dx_n)] \\
 &= \left(\frac{4\xi_n i(6\xi_n^2 - 2)h'(0)}{(1 + \xi_n^2)^5} + \frac{-16i(\xi_n + 4\xi_n^3 + 3\xi_n^5)h'(0)}{(1 + \xi_n^2)^6} \right) \operatorname{tr}[\sigma_n^H + \Phi(\tilde{e}_n)]. \tag{3.134}
 \end{aligned}$$

Combining these assertions, we get:

$$\begin{aligned}
 \tilde{\Psi}_{12} &= \Psi_7 + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr} \left[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \times \partial_{\xi_n}^2 \partial_{x_n} \sigma_{-1}((\tilde{D}_H^*)^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx' \\
 &= \frac{39}{32} (h'(0)^2) \pi \Omega_3 dx' + \frac{(5i + 6)h'}{8} \operatorname{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx'. \tag{3.135}
 \end{aligned}$$

case (13) $r = -2, l = -2, k = j = |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned}
 \tilde{\Psi}_{13} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-2}((\tilde{D}_H)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &+ i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4} \right)](x_0) d\xi_n \sigma(\xi') dx' \\
 &+ i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &+ i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \operatorname{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) \times \left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4} \right)](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.136}
 \end{aligned}$$

By [9], we obtain:

$$\pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) = \frac{(-i\xi_n - 2)c(\xi')\gamma c(\xi') - i[c(dx_n)\gamma c(\xi') + c(\xi')\gamma c(dx_n)] - i\xi_n c(dx_n)\gamma c(dx_n)}{4(\xi_n - i)^2}. \tag{3.137}$$

$$\sigma_{-2}(D^{-1}) = -h' \frac{\xi_n^2 + 2}{(1 + \xi_n^2)^3} c(\xi)c(dx_n)c(\xi) + \frac{1}{(1 + \xi_n^2)^3} c(\xi)c(dx_n)\partial_{x_n}[c(\xi')]. \tag{3.138}$$

Then we have:

$$\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4} \right) = \frac{(\xi_n i + 3)c(\xi')\gamma c(\xi')}{4(\xi_n - i)^3} + \frac{(\xi_n i - 1)c(dx_n)\gamma c(\xi')}{4(\xi_n - i)^3} + \frac{ic(dx_n)\gamma c(\xi')}{2(\xi_n - i)^3} \frac{ic(\xi')\gamma c(dx_n)}{2(\xi_n - i)^3}. \tag{3.139}$$

$$\begin{aligned}
 \partial_{\xi_n} \pi_{\xi_n}^+ (\sigma_{-2}(D^{-1}))(x_0)|_{|\xi'|=1} &= h'(0) \frac{-i\xi_n - 3}{4(\xi_n - i)^3} c(\xi') c(dx_n) c(\xi') + h'(0) \frac{i}{(\xi_n - i)^3} c(\xi') \\
 &+ h'(0) \frac{i\xi_n - 1}{4(\xi_n - i)^3} c(dx_n) + \frac{-i}{2(\xi_n - i)^3} \partial_{x_n} [c(\xi')](x_0) \\
 &+ \frac{i\xi_n + 3}{4(\xi_n - i)^3} c(\xi') c(dx_n) \partial_{x_n} [c(\xi')](x_0) \\
 &+ h'(0) \frac{-2i\xi_n - 8}{8(\xi_n - i)^4} c(\xi') + h'(0) \frac{i\xi_n^2 + 4\xi_n - 9i}{8(\xi_n - i)^4} c(dx_n).
 \end{aligned} \tag{3.140}$$

So

$$\begin{aligned}
 &i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \left(\frac{c(\xi) \tau c(\xi)}{|\xi|^4} \right)](x_0) d\xi_n \sigma(\xi') dx' \\
 &= -\frac{13h'}{16} \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)] \pi \Omega_3 dx'.
 \end{aligned} \tag{3.141}$$

Similarly, we obtain:

$$\begin{aligned}
 &i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi) \gamma c(\xi)}{|\xi|^4} \right) \times \sigma_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &= \frac{73h'}{32} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx'.
 \end{aligned} \tag{3.142}$$

And,

$$\begin{aligned}
 &i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \left(\frac{c(\xi) \gamma c(\xi)}{|\xi|^4} \right) \times \left(\frac{c(\xi) \tau c(\xi)}{|\xi|^4} \right)](x_0) d\xi_n \sigma(\xi') dx' \\
 &= -\frac{1}{16} \sum_{i=1}^n \text{tr}[(\sigma_i^H + \Phi(\tilde{e}_i))(\sigma_i^H - \Phi^*(\tilde{e}_i))] \pi \Omega_3 dx'.
 \end{aligned} \tag{3.143}$$

Combining these assertions, we see:

$$\begin{aligned}
 \tilde{\Psi}_{13} &= -\frac{821}{256} \pi \Omega_3 dx' - \frac{13h'}{16} \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)] \pi \Omega_3 dx' + \frac{73h'}{32} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' \\
 &- \frac{1}{16} \sum_{i=1}^n \text{tr}[(\sigma_i^H + \Phi(\tilde{e}_i))(\sigma_i^H - \Phi^*(\tilde{e}_i))] \pi \Omega_3 dx'.
 \end{aligned} \tag{3.144}$$

case (14) $r = -1, l = -3, k = j = |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned}
 \tilde{\Psi}_{14} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-3}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-3}((\tilde{D}_H^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times \sigma_{-3}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
 &\quad + i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times R_{-3}|_{|\xi'|=1}](x_0) d\xi_n \sigma(\xi') dx'. \tag{3.145}
 \end{aligned}$$

By (2.2.29) in [12], we have:

$$\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}. \tag{3.146}$$

In the orthonormal frame filed, we have:

$$\begin{aligned}
 R_{-3}|_{|\xi'|=1} &= \frac{ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^6} + \frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^6} + \frac{-ic(\xi)\tau c(\xi)\tau c(\xi)}{|\xi|^6} \\
 &\quad + \frac{-ih'c(\xi)\tau c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6} + \frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^8} + \frac{-ic(\xi)}{|\xi|^6} \sum_{j=1}^{n-1} c(\tilde{e}_j)c(\xi)\partial_{x_i}(\tau)c(\xi) \\
 &\quad + \frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\tau c(\xi)}{|\xi|^6} + \frac{2ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^8} + \frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\tau)c(\xi)}{|\xi|^6} \\
 &\quad + \frac{-ic(\xi)c(dx_n)c(\xi)\tau\partial_{x_n}[c(\xi)]}{|\xi|^6}. \tag{3.147}
 \end{aligned}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\begin{aligned}
 \text{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\tau c(\xi)] &= (\xi_n^3 - 3\xi_n)\text{tr}[c(dx_n)\tau] = -4(\xi_n^3 - 3\xi_n)\text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)]; \\
 \text{tr}[c(dx_n)c(\xi)\tau c(\xi)\tau c(\xi)] &= 4(3\xi_n - \xi_n^3)\{\text{tr}[(\sigma_n^H - \Phi^*(\tilde{e}_n))^2] - 4(3\xi_n - \xi_n^3) \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H - \Phi^*(\tilde{e}_j))^2]\}; \\
 \text{tr}[c(dx_n)c(\xi) \sum_{j=1}^{n-1} c(\tilde{e}_j)c(\xi)\partial_{x_i}(\tau)c(\xi)] &= 4(\xi_n^3 - 3\xi_n)\text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\}; \\
 \text{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\partial_{x_n}(\tau)c(\xi)] &= 4(3\xi_n - \xi_n^3)\text{tr}[\nabla_{\partial_{x_n}}^H(\sigma_n^H - \Phi^*(\tilde{e}_n))]. \tag{3.148}
 \end{aligned}$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. From Lemma 3.4, combining (3.146)-(3.148) and direct

computations, we obtain:

$$\begin{aligned}
 & i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_H^{-1}) \times R_{-3|_{|\xi'|=1}}(x_0) d\xi_n \sigma(\xi') dx' \\
 &= \frac{-51h'}{16} \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)] \pi \Omega_3 dx' + \frac{3}{4} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H - \Phi^*(\tilde{e}_n))] \pi \Omega_3 dx' - \frac{3}{4} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\} \pi \Omega_3 dx' \\
 &- \frac{3}{4} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H - \Phi^*(\tilde{e}_j))^2] + \frac{3}{4} \text{tr}[(\sigma_n^H - \Phi^*(\tilde{e}_n))^2]. \tag{3.149}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \tilde{\Psi}_{14} &= \left(\frac{239}{64} (h'(0))^2 - \frac{27}{16} h''(0) - \frac{11}{192} s_{\partial M} \right) \pi \Omega_3 dx' \\
 &- \frac{-51h'}{16} \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)] \pi \Omega_3 dx' + \frac{3}{4} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H - \Phi^*(\tilde{e}_n))] \pi \Omega_3 dx' \\
 &- \frac{3}{4} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\} \pi \Omega_3 dx' - \frac{3}{4} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H - \Phi^*(\tilde{e}_j))^2] \pi \Omega_3 dx' \\
 &+ \frac{3}{4} \text{tr}[(\sigma_n^H - \Phi^*(\tilde{e}_n))^2] \pi \Omega_3 dx'. \tag{3.150}
 \end{aligned}$$

case (15) $r = -3, l = -1, k = j = |\alpha| = 0$.

From (2.15), we have:

$$\begin{aligned}
 \tilde{\Psi}_{15} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-3}(\tilde{D}_H^{-1}) \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) d\xi_n \sigma(\xi') dx' \\
 &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-3}(D^{-1}) \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) d\xi_n \sigma(\xi') dx' \\
 &- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ Q_{-3|_{|\xi'|=1}} \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) d\xi_n \sigma(\xi') dx'. \tag{3.151}
 \end{aligned}$$

By the Leibniz rule, trace property and "++" and "--" vanishing after the integration over ξ_n in [3], then

$$\begin{aligned}
 & \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ Q_{-3|_{|\xi'|=1}} \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) d\xi_n \\
 &= \int_{-\infty}^{+\infty} \text{tr}[Q_{-3|_{|\xi'|=1}} \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) d\xi_n - \int_{-\infty}^{+\infty} \text{tr}[Q_{-3|_{|\xi'|=1}} \times \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) d\xi_n. \tag{3.152}
 \end{aligned}$$

By Lemma 3.2, a simple computation shows

$$\partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1}) = \frac{-2i\xi_n c(\xi') + i(1 - \xi_n^2) c(dx_n)}{(\xi_n^2 + 1)^2}. \tag{3.153}$$

Considering for $i < n, \int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2q+1}} \sigma(\xi') = 0$. From Lemma 3.4, combining (3.153) and direct

computations, we obtain:

$$\begin{aligned}
 & -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[Q_{-3}|_{|\xi'|=1} \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}_H^*)^{-1})(x_0) d\xi_n \sigma(\xi') dx'] \\
 & = \frac{17h'}{48} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' - \frac{11}{24} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx' \\
 & + \frac{11}{24} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \pi \Omega_3 dx' - \frac{11}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \pi \Omega_3 dx' + \frac{11}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \pi \Omega_3 dx'. \quad (3.154)
 \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
 \tilde{\Psi}_{15} & = \left(\frac{239}{64} (h'(0))^2 - \frac{27}{16} h''(0) - \frac{11}{192} s_{\partial M} \right) \pi \Omega_3 dx' \\
 & - \frac{17h'}{6} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' + \frac{7}{24} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx' \\
 & - \frac{7}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \pi \Omega_3 dx' + \frac{7}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \pi \Omega_3 dx' \\
 & - \frac{7}{24} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \pi \Omega_3 dx'. \quad (3.155)
 \end{aligned}$$

Now $\tilde{\Psi}$ is the sum of the $\tilde{\Psi}_{(1,2,\dots,15)}$, then we obtain:

$$\begin{aligned}
 \sum_{i=1}^{15} \tilde{\Psi}_i & = \left(\frac{399}{256} (h'(0))^2 - \frac{29}{32} h''(0) + \left(\frac{71}{96} + \frac{3i}{32} \right) s_{\partial M} \right) \pi \Omega_3 dx' \\
 & + \frac{(40i + 17)h'}{64} \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)] \pi \Omega_3 dx' - \left(\frac{19}{4} + \frac{\pi^2}{2} \right) \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\} \pi \Omega_3 dx' \\
 & - \frac{9}{16} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H - \Phi^*(\tilde{e}_n))] \pi \Omega_3 dx' + \frac{(240i - 323)h'}{384} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \pi \Omega_3 dx' \\
 & - \frac{13}{48} \text{tr}[\nabla_{\partial_{x_n}}^H (\sigma_n^H + \Phi(\tilde{e}_n))] \pi \Omega_3 dx' - \frac{1}{16} \sum_{i=1}^n \text{tr}[(\sigma_i^H + \Phi(\tilde{e}_i))(\sigma_i^H - \Phi^*(\tilde{e}_i))] \pi \Omega_3 dx' \\
 & - \frac{3}{4} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H - \Phi^*(\tilde{e}_j))^2] \pi \Omega_3 dx' + \frac{3}{4} \text{tr}[(\sigma_n^H - \Phi^*(\tilde{e}_n))^2] \pi \Omega_3 dx' \\
 & - \frac{7}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \pi \Omega_3 dx' + \frac{7}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \pi \Omega_3 dx' \\
 & - \frac{7}{24} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \pi \Omega_3 dx'. \quad (3.156)
 \end{aligned}$$

Hence,

Theorem 3.8 *Let M be a 5-dimensional compact spin manifold with the boundary ∂M , and twisted Dirac*

operator $\tilde{D}_H^* = \sum_{j=1}^n c(\tilde{e}_j) \nabla_{\tilde{e}_j}^{S(TM)} \otimes^H - \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi^*(\tilde{e}_j)$, then

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+(\tilde{D}_H^{-1}) \circ \pi^+(\tilde{D}_H^*)^{-1}] &= \int_{\partial M} \left[\left(\frac{399}{256} (h'(0))^2 - \frac{29}{32} h''(0) + \left(\frac{71}{96} + \frac{3i}{32} \right) s_{\partial M} \right) \right. \\ &\quad + \frac{(40i + 17)h'}{64} \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)] + \frac{(240i - 323)h'}{384} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \\ &\quad - \left(\frac{19}{4} + \frac{\pi^2}{2} \right) \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\} - \frac{9}{16} \text{tr}[\nabla_{\partial x_n}^H(\sigma_n^H - \Phi^*(\tilde{e}_n))] \\ &\quad - \frac{13}{48} \text{tr}[\nabla_{\partial x_n}^H(\sigma_n^H + \Phi(\tilde{e}_n))] - \frac{1}{16} \sum_{i=1}^n \text{tr}[(\sigma_i^H + \Phi(\tilde{e}_i))(\sigma_i^H - \Phi^*(\tilde{e}_i))] \\ &\quad - \frac{3}{4} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H - \Phi^*(\tilde{e}_j))^2] + \frac{3}{4} \text{tr}[(\sigma_n^H - \Phi^*(\tilde{e}_n))^2] - \frac{7}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] \\ &\quad \left. + \frac{7}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] - \frac{7}{24} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} \right] \pi \Omega_3 \text{dvol}_{\partial M}. \end{aligned} \tag{3.157}$$

Then, we have:

Theorem 3.9 *Let M be a 5-dimensional compact spin manifold with the boundary ∂M , and twisted Dirac operator $\tilde{D}_H^* = \sum_{j=1}^n c(\tilde{e}_j) \nabla_{\tilde{e}_j}^{S(TM)} \otimes^H - \sum_{j=1}^n c(\tilde{e}_j) \otimes \Phi^*(\tilde{e}_j)$. The following identity holds:*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+(\tilde{D}_H^{-1}) \circ \pi^+(\tilde{D}_H^*)^{-1}] &= \int_{\partial M} \left[\left(\frac{225}{64} K^2 + \frac{29}{8} s_M|_{\partial M} + \left(\frac{197}{24} + \frac{3i}{2} \right) s_{\partial M} \right) \right. \\ &\quad + \left(\frac{(17 + 40i)}{128} \text{tr}[\sigma_n^H - \Phi^*(\tilde{e}_n)] + \frac{(323 - 240i)}{768} \text{tr}[\sigma_n^H + \Phi(\tilde{e}_n)] \right) K \\ &\quad - \left(\frac{19}{4} + \frac{\pi^2}{2} \right) \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H - \Phi^*)]\} - \frac{13}{48} \text{tr}[\nabla_{\partial x_n}^H(\sigma_n^H + \Phi(\tilde{e}_n))] \\ &\quad - \frac{1}{16} \sum_{i=1}^n \text{tr}[(\sigma_i^H + \Phi(\tilde{e}_i))(\sigma_i^H - \Phi^*(\tilde{e}_i))] - \frac{3}{4} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H - \Phi^*(\tilde{e}_j))^2] \\ &\quad + \frac{3}{4} \text{tr}[(\sigma_n^H - \Phi^*(\tilde{e}_n))^2] - \frac{7}{24} \sum_{j=1}^{n-1} \text{tr}[(\sigma_j^H + \Phi(\tilde{e}_j))^2] + \frac{7}{24} \text{tr}[(\sigma_n^H + \Phi(\tilde{e}_n))^2] \\ &\quad \left. - \frac{7}{24} \text{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H + \Phi)]\} - \frac{9}{16} \text{tr}[\nabla_{\partial x_n}^H(\sigma_n^H - \Phi^*(\tilde{e}_n))] \right] \pi \Omega_3 \text{dvol}_{\partial M}. \end{aligned} \tag{3.158}$$

where $s_M, s_{\partial M}$ are respectively scalar curvatures on M and ∂M .

4. Conclusion

In this paper, we mainly give the proof of the Kastler-Kalau-Walze type theorem for twisted Dirac operators on 5-dimensional compact manifolds with boundary. Firstly, we consider an n -dimensional oriented Riemannian manifold (M, g^M) equipped with a fixed spin structure, which are twisted Dirac operators. Secondly, we compute the lower dimensional volume for \tilde{D}_H and \tilde{D}_H^* on five dimensional compact spin manifolds with

boundary. In other words, we need to compute $Vol_5^{(1,1)}$ for \tilde{D}_H and \tilde{D}_H^* on 5-dimensional spin manifolds with boundary, where $\widetilde{Wres}[\pi^+(\tilde{D}_H^{-1}) \circ \pi^+(\tilde{D}_H^{-1})] = \int_{\partial M} \Psi$, and $\widetilde{Wres}[\pi^+(\tilde{D}_H^{-1}) \circ \pi^+(\tilde{D}_H^*)^{-1}] = \int_{\partial M} \tilde{\Psi}$. Thirdly, we divide the boundary term $\int_{\partial M} \Psi$ and $\int_{\partial M} \tilde{\Psi}$ into fifteen cases by using the boundary term formula of the manifolds with boundary, then the boundary terms of twisted Dirac operators are further given, the Kastler-Kalau-Walze type theorem for twisted Dirac operators on compact manifolds with boundary is proved.

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