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FATİH HEZENCİ

HÜSEYİN BUDAK

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Novel results on trapezoid-type inequalities for conformable fractional integrals

Fatih HEZENCİ*, **Hüseyin BUDAK**
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

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Abstract: This paper establishes an identity for the case of differentiable $s-$convex functions with respect to the conformable fractional integrals. By using this identity, sundry trapezoid-type inequalities are proven by $s-$convex functions with the help of the conformable fractional integrals. Several important inequalities are acquired with taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Moreover, an example using graph is given in order to show that our main results are correct. By using the special choices of the obtained results, we present several new results connected with trapezoid-type inequalities.

Key words: Trapezoid-type inequality, fractional conformable integrals, fractional conformable derivatives, fractional calculus, convex function

1. Introduction & preliminaries

The theory of inequalities has a significant place in the literature. One of the most well-known inequalities for the case of convex functions is the Hermite–Hadamard-type inequality. Thus, a large number of mathematicians has studied Hermite–Hadamard-type inequalities and related inequalities such as trapezoid, midpoint, and Simpson's inequality.

Hermite–Hadamard-type inequalities were first introduced by C. Hermite and J. Hadamard for the case of convex functions. Let $\mathcal{F}: I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $\sigma, \delta \in I$ with $\sigma < \delta$. Then, the double inequality

$$\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \leq \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathcal{F}(x)dx \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} \tag{1.1}$$

is valid. If $\mathcal{F}$ is concave, then both inequalities in (1.1) hold in the reverse direction.

The theory of convexity stages a central and attractive role in numerous fields of research. This theory derives us with a powerful tool in order to solve a large class of problems that arise in applied and pure mathematics. Moreover, the concept of convexity has been extended and improved in many directions.

Definition 1.1 *[25] Let I be an interval of real numbers. Then, a function $\mathcal{F}: I \to \mathbb{R}$ is said to be convex, if*

$$\mathcal{F}(\mu x + (1-\mu) y) \leq \mu f(x) + (1-\mu)\mathcal{F}(y)$$

*Correspondence: fatihezenci@gmail.com
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is valid $\forall x, y \in I$ *and* $\forall \mu \in [0,1]$.

Definition 1.2 *A function* $\mathcal{F} : [0,\infty) \to [0,\infty)$ *will be called* $s-$ *convex in the second sense if the following inequality*

$$\mathcal{F}(\mu x + (1-\mu) y) \leq \mu^s \mathcal{F}(x) + (1-\mu)^s \mathcal{F}(y)$$

is valid for all $x, y \in [0,\infty)$ *with* $\mu \in [0,1]$ *and for some fixed* $s \in (0,1]$.

The s-convex function was introduced in [7] and several properties and connections with s-convexity in the first sense were proven in [14]. Furthermore, it can be easily seen that for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Dragomir and Fitzpatrick [11] investigated a variant of Hermite–Hadamard-type inequalities which are valid for the case of the s-convex functions.

Theorem 1.3 *[11] Suppose* $\mathcal{F} : [0,\infty) \to [0,\infty)$ *is a* s*-convex function in the second sense, where* $s \in [0,1)$ *and suppose also* $\sigma, \delta \in [0,\infty), \sigma < \delta$. *If* $\mathcal{F} \in L_1[0,1]$, *then the following inequalities hold:*

$$2^{s-1}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \leq \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathcal{F}(x)\,dx \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{s+1}. \tag{1.2}$$

For recent results and generalizations associated with s-convex functions see [3, 11, 21].

If we choose $s = 1$ in inequalities (1.2), then inequalities (1.2) reduces to the classical Hermite–Hadamard-type inequalities (1.1).

Fractional calculus, which is calculus of integrals and derivatives of any arbitrary real or complex order, has grown in popularity over the last three decades. It gives some potentially valuable tools in order to solve differential and integral equations. In recent decades, it has piqued the curiosity of mathematicians, physicists, and engineers [4, 28]. Fractional derivatives are also used to model a wide range of mathematical biology, as well as chemical processes and engineering problems [5, 12].

For establishment our main results, we introduce the basic definitions of Riemann-Liouville integrals and conformable integrals, which are well known in the literature.

Definition 1.4 *The gamma function and beta function are defined*

$$\Gamma(x) := \int_{0}^{\infty} \mu^{x-1} e^{-\mu} d\mu$$

and

$$\mathscr{B}(x,y) := \int_{0}^{1} \mu^{x-1}(1-\mu)^{y-1}\,d\mu,$$

respectively for $0 < x, y < \infty$.

In paper [20], Kilbas et al. described fractional integrals, also called *Riemann-Liouville integrals* as follows:

Definition 1.5 *[20] The Riemann-Liouville integrals $J_{\sigma+}^{\beta}\mathcal{F}(x)$ and $J_{\delta-}^{\beta}\mathcal{F}(x)$ of order $\beta > 0$ are given by*

$$J_{\sigma+}^{\beta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_{\sigma}^{x} (x-\mu)^{\beta-1} \mathcal{F}(\mu) d\mu, \quad x > \sigma \tag{1.3}$$

and

$$J_{\delta-}^{\beta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\delta} (\mu - x)^{\beta-1} \mathcal{F}(\mu) d\mu, \quad x < \delta, \tag{1.4}$$

respectively for $\mathcal{F} \in L_1[\sigma, \delta]$. The Riemann-Liouville integrals coincides with the classical integrals for the case of $\beta = 1$.

Dragomir and Agarwal first proved trapezoid inequalities for the case of convex functions in [10], while Kirmacı first established inequalities of midpoint-type for the case of convex functions in [22]. Iqbal et al. and Sarikaya et al. presented several fractional midpoint and trapezoid-type inequalities for the case of convex mappings in papers [16] and [26], respectively. Moreover, several Hermite–Hadamard-type inequalities for the fractional integrals were presented in [16]. In addition to these, twice differentiable functions have been considered in order to obtain important inequalities. For example, Barani et al. [6] established inequalities for the case of twice differentiable convex mappings which are connected with Hermite–Hadamard-type inequalities. Several generalized fractional integral inequalities of midpoint-type and trapezoid-type inequalities for the case of twice differentiable convex functions were proven in [24]. For more information connected with fractional integral inequalities, see [8, 9, 13] and the references cited therein.

By using the derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is improved in paper [23]. Several significant requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. On the other hand, in [2], the author proved that the conformable approach in [23] cannot yield good results if compared to the Caputo definition for specific functions. This flaw in the conformable definition is avoided by several extensions of the conformable approach [15, 29]. Upon this, Jarad et al. [19] introduced the fractional conformable integral operators. They also derived particular characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 1.6 *[19] The fractional conformable integral operator ${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(x)$ and ${}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(x)$ of order $\beta \in \mathbb{C}$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are given by*

$${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_{\sigma}^{x} \left(\frac{(x-\sigma)^{\alpha} - (\mu-\sigma)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(\mu)}{(\mu-\sigma)^{1-\alpha}} d\mu, \qquad \mu > \sigma \tag{1.5}$$

and

$${}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\delta} \left(\frac{(\delta-x)^{\alpha} - (\delta-\mu)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(\mu)}{(\delta-\mu)^{1-\alpha}} d\mu, \qquad \mu < \delta, \tag{1.6}$$

respectively for $\mathcal{F} \in L_1[\sigma, \delta]$.

If we choose $\alpha = 1$, then the integral in (1.5) and (1.6) equals the Riemann-Liouville integral in (1.3) and (1.4), respectively. See [1, 18, 27] and the references cited therein for several recent results regarding fractional integral inequalities.

With the help of the continuing studies and mentioned papers above, we will prove several trapezoid-type inequalities for the case of differentiable $s-$convex function including conformable fractional integrals. The entire form of the study takes the form of four sections including introduction and preliminaries. In Section 2, an equality will be established for the case of differentiable $s-$convex functions with respect to the conformable fractional integrals. By using this equality, several trapezoid-type inequalities will be given by $s-$convex functions with the help of the conformable fractional integrals. Moreover, example will be given by using graph in order to show that our main results are correct. In Section 3, we will present the special cases of the main results, and we prove that our results generalize several well-known trapezoid-type inequalities. Namely, sundry corollaries and remarks will be presented. Finally, summary and concluding remarks are given in Section 4.

2. Main results

Lemma 2.1 *Consider that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable function on (σ,δ) such that $\mathcal{F}'\in L_1[\sigma,\delta]$. Then, the following equality holds:*

$$\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]=\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left\{A_1-A_2\right\}. \tag{2.1}$$

Here,

$$\begin{cases} A_1=\int\limits_0^{\frac{1}{2}}\left[\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right]\mathcal{F}'(\mu b+(1-\mu)\sigma)\,d\mu, \\ A_2=\int\limits_0^{\frac{1}{2}}\left[\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right]\mathcal{F}'(\mu a+(1-\mu)\delta)\,d\mu. \end{cases}$$

Proof From the fact of the integrating by parts, it yields

$$A_1=\int\limits_0^{\frac{1}{2}}\left[\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right]\mathcal{F}'(\mu b+(1-\mu)\sigma)\,d\mu \tag{2.2}$$

$$=\frac{1}{\delta-\sigma}\left[\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right]\mathcal{F}(\mu b+(1-\mu)\sigma)\Bigg|_0^{\frac{1}{2}}$$

$$-\frac{2\beta}{\delta-\sigma}\int\limits_0^{\frac{1}{2}}\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-2\mu)^{\alpha-1}\mathcal{F}(\mu b+(1-\mu)\sigma)\,d\mu$$

$$= \frac{1}{(\delta-\sigma)\,\alpha^{\beta}}\mathcal{F}(\sigma) - \frac{2\beta}{\delta-\sigma}\int\limits_{0}^{\frac{1}{2}}\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-2\mu)^{\alpha-1}\,\mathcal{F}(\mu b+(1-\mu)\,\sigma)\,d\mu.$$

Similar to foregoing process, we get

$$A_2 = -\frac{1}{(\delta-\sigma)\,\alpha^{\beta}}\mathcal{F}(\delta) + \frac{2\beta}{\delta-\sigma}\int\limits_{0}^{\frac{1}{2}}\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-2\mu)^{\alpha-1}\,\mathcal{F}(\mu a+(1-\mu)\,\delta)\,d\mu. \tag{2.3}$$

If we subtract from (2.2) to (2.3), then we have

$$A_1 - A_2 = \frac{1}{(\delta-\sigma)\,\alpha^{\beta}}(\mathcal{F}(\sigma)+\mathcal{F}(\delta)) \tag{2.4}$$

$$-\frac{2\beta}{\delta-\sigma}\int\limits_{0}^{\frac{1}{2}}\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-2\mu)^{\alpha-1}\,\mathcal{F}(\mu b+(1-\mu)\,\sigma)\,d\mu$$

$$-\frac{2\beta}{\delta-\sigma}\int\limits_{0}^{\frac{1}{2}}\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-2\mu)^{\alpha-1}\,\mathcal{F}(\mu a+(1-\mu)\,\delta)\,d\mu.$$

If we use change of variables in (2.4), then equality (2.4) is converted as follows:

$$A_1 - A_2 = \frac{1}{(\delta-\sigma)\,\alpha^{\beta}}(\mathcal{F}(\sigma)+\mathcal{F}(\delta)) \tag{2.5}$$

$$-\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta}\frac{\Gamma(\beta+1)}{\delta-\sigma}\frac{1}{\Gamma(\beta)}\int\limits_{\sigma}^{\frac{\sigma+\delta}{2}}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\sigma+\delta}{2}-x\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(\frac{\sigma+\delta}{2}-x\right)^{1-\alpha}}\mathcal{F}(x)\,dx$$

$$-\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta}\frac{\Gamma(\beta+1)}{\delta-\sigma}\frac{1}{\Gamma(\beta)}\int\limits_{\frac{\sigma+\delta}{2}}^{\delta}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(x-\frac{\sigma+\delta}{2}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(x-\frac{\sigma+\delta}{2}\right)^{1-\alpha}}\mathcal{F}(x)\,dx$$

$$= \frac{1}{(\delta-\sigma)\,\alpha^{\beta}}(\mathcal{F}(\sigma)+\mathcal{F}(\delta)) - \left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta}\frac{\Gamma(\beta+1)}{\delta-\sigma}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right].$$

If the equality (2.5) is multiplied by $\frac{(\delta-\sigma)\alpha^{\beta}}{2}$, then the proof of Lemma 2.1 is finished simultaneously. □

Theorem 2.2 *Note that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable function on (σ,δ) so that $\mathcal{F}'\in L_1[\sigma,\delta]$ and $|\mathcal{F}'|$ is $s-$ convex on $[\sigma,\delta]$. Then, the following inequality*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \tag{2.6}$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\varphi_1(\alpha,\beta,s)+\varphi_1(\alpha,\beta,s)\right)\left[|\mathcal{F}'(\sigma)|+|\mathcal{F}'(\delta)|\right]$$

is valid. Here,

$$\begin{cases}\varphi_1(\alpha,\beta,s)=\int\limits_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\mu^s d\mu,\\[2ex] \varphi_2(\alpha,\beta,s)=\int\limits_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|(1-\mu)^s d\mu.\end{cases} \tag{2.7}$$

Proof Let us consider the modulus of both sides of (2.1). Then, we get

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \tag{2.8}$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left\{\int\limits_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu b+(1-\mu)\sigma)\right|d\mu\right.$$

$$\left.+\int\limits_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\mathcal{F}'(\mu a+(1-\mu)\delta)\right|d\mu\right\}.$$

Since $|\mathcal{F}'|$ is $s-$convex on $[\sigma,\delta]$, it follows

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left\{\int\limits_0^{\frac{1}{2}}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu^s|\mathcal{F}'(\delta)|+(1-\mu)^s|\mathcal{F}'(\sigma)|\right)d\mu\right.$$

$$\left.+\int\limits_0^{\frac{1}{2}}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu^s|\mathcal{F}'(\sigma)|+(1-\mu)^s|\mathcal{F}'(\delta)|\right)d\mu\right\}$$

$$= \frac{(\delta-\sigma)\,\alpha^{\beta}}{2}\left(\varphi_1(\alpha,\beta,s)+\varphi_1(\alpha,\beta,s)\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

□

Theorem 2.3 *If $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable function on (σ,δ) such that $\mathcal{F}'\in L_1[\sigma,\delta]$ and $|\mathcal{F}'|^q$ is $s-$ convex on $[\sigma,\delta]$ with $q>1$, then the following inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[\,{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+\,{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \tag{2.9}$$

$$\leq \frac{(\delta-\sigma)\,\alpha^{\beta}}{2}\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}'(\delta)|^q+\left(2^{s+1}-1\right)|\mathcal{F}'(\sigma)|^q}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^q+\left(2^{s+1}-1\right)|\mathcal{F}'(\delta)|^q}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}\right].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\psi_1^{\alpha,\beta}(p)=\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\mu.$$

Proof Consider Hölder inequality in (2.8). Then, we obtain

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[\,{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+\,{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{(\delta-\sigma)\,\alpha^{\beta}}{2}\left\{\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{F}'(\mu b+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{F}'(\mu a+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right\}.$$

From the fact that $|\mathcal{F}'|^q$ is $s-$convex on $[\sigma,\delta]$, then it yields

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[\,{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+\,{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{(\delta-\sigma)\,\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]^{p}d\mu\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left(\mu^s\left|\mathcal{F}'(\delta)\right|^q+(1-\mu)^s\left|\mathcal{F}'(\sigma)\right|^q\right)d\mu\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\left(\mu^s\left|\mathcal{F}'(\sigma)\right|^q+(1-\mu)^s\left|\mathcal{F}'(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right\}$$

$$=\frac{(\delta-\sigma)\alpha^\beta}{2}\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\left[\left(\frac{\left|\mathcal{F}'(\delta)\right|^q+\left(2^{s+1}-1\right)\left|\mathcal{F}'(\sigma)\right|^q}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left(2^{s+1}-1\right)\left|\mathcal{F}'(\delta)\right|^q}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\right].$$

□

Theorem 2.4 *Assume that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable function on (σ,δ) such that $\mathcal{F}'\in L_1[\sigma,\delta]$ and $|\mathcal{F}'|^q$ is $s-$convex on $[\sigma,\delta]$ with $q\geq 1$. Then, the following inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^\beta\mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \tag{2.10}$$

$$\leq\frac{(\delta-\sigma)\alpha^\beta}{2}\left(\varphi_1(\alpha,\beta,s)\right)^{1-\frac{1}{q}}\left\{\left(\varphi_1(\alpha,\beta,s)\left|\mathcal{F}'(\delta)\right|^q+\varphi_2(\alpha,\beta,s)\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\varphi_1(\alpha,\beta,s)\left|\mathcal{F}'(\sigma)\right|^q+\varphi_2(\alpha,\beta,s)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Here, $\varphi_1(\alpha,\beta,s)$ and $\varphi_2(\alpha,\beta,s)$ are defined in (2.7).

Proof Let us apply power-mean inequality in (2.8). Then, we obtain

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^\beta\mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^\beta}{2}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{\alpha^\beta}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{\alpha^\beta}\right|\left|\mathcal{F}'(\mu b+(1-\mu)\sigma)\right|^q d\mu\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\mu a+(1-\mu)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\Bigg\}.$$

It is known that $|\mathcal{F}'|^q$ is $s-$convex on $[\sigma,\delta]$. Then, we get

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu^s\left|\mathcal{F}'(\delta)\right|^q+(1-\mu)^s\left|\mathcal{F}'(\sigma)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-2\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu^s\left|\mathcal{F}'(\sigma)\right|^q+(1-\mu)^s\left|\mathcal{F}'(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right\},$$

which completes the proof of Theorem 2.4. □

3. Special cases & an example

Corollary 3.1 *If we choose $s=1$ in Theorem 2.2, then Theorem 2.2 becomes*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \tag{3.1}$$

$$\leq\frac{(\delta-\sigma)}{4}\left[1-\frac{1}{\alpha}\mathscr{B}\left(\frac{1}{\alpha},\beta+1\right)\right]\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

where $\mathscr{B}$ denotes the beta function.

Example 3.2 *Let $\mathcal{F}:[0,1]\to\mathbb{R}$ be defined by $\mathcal{F}(x)=x^2$. If we apply Corollary 3.1 with $\beta\in(0,10)$, $\alpha\in(0,1)$, and then the left hand side of (3.1) reduces to*

$$\left|\frac{\mathcal{F}(0)+\mathcal{F}(1)}{2}-2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{1}{2}-}(0)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{1}{2}+}(1)\right]\right|$$

$$= \left| \frac{1}{2} - \frac{\beta}{4} \left[\frac{1}{\beta} + \mathscr{B}\left(\frac{2}{\alpha} + 1, \beta \right) \right] \right|.$$

The right hand side of (3.1) *becomes*

$$\frac{1}{2} \left[1 - \frac{1}{\alpha} \mathscr{B}\left(\frac{1}{\alpha}, \beta + 1 \right) \right].$$

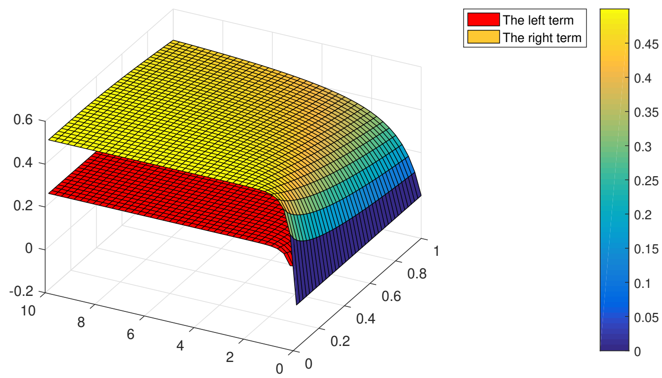

Figure. An example to the inequality (3.1) depending on α and β, computed and plotted with MATLAB.

Remark 3.3 *If we set* $\alpha = 1$ *and* $s = 1$ *in Theorem 2.2, then we have*

$$\left| \frac{\mathcal{F}(\sigma) + \mathcal{F}(\delta)}{2} - \frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}} \left[J^{\beta}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma) + J^{\beta}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta) \right] \right|$$

$$\leq \frac{(\delta-\sigma)}{2} \frac{\beta}{2(\beta+1)} \left[|\mathcal{F}'(\sigma)| + |\mathcal{F}'(\delta)| \right].$$

which is given by Budak et al. in [8, Corollary 5.4].

Remark 3.4 *Let us consider* $\alpha = \beta = 1$ *and* $s = 1$ *in Theorem 2.2. Then, Theorem 2.2 becomes*

$$\left| \frac{\mathcal{F}(\sigma) + \mathcal{F}(\delta)}{2} - \frac{1}{\delta-\sigma} \int_{\sigma}^{\delta} \mathcal{F}(x)\,dx \right| \leq \frac{\delta-\sigma}{8} \left[|\mathcal{F}'(\sigma)| + |\mathcal{F}'(\delta)| \right].$$

which is proven by Dragomir and Agarwal in [10, Theorem 2.2].

Corollary 3.5 *If we choose* $\alpha = 1$ *and* $s = 1$ *in Theorem 2.3, then we derive*

$$\left| \frac{\mathcal{F}(\sigma) + \mathcal{F}(\delta)}{2} - \frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}} \left[J^{\beta}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma) + J^{\beta}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta) \right] \right|$$

$$\leq \frac{\delta-\sigma}{2^{1+\frac{1}{q}}}\left(\frac{1}{2\beta}\mathscr{B}\left(\frac{1}{\beta},p+1\right)\right)^{\frac{1}{p}}\left[\left(\frac{\left|\mathcal{F}'\left(\delta\right)\right|^{q}+3\left|\mathcal{F}'\left(\sigma\right)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\sigma\right)\right|^{q}+3\left|\mathcal{F}'\left(\delta\right)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\delta-\sigma}{2^{1+\frac{1}{q}}}\left(\frac{2}{\beta}\mathscr{B}\left(\frac{1}{\beta},p+1\right)\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'\left(\sigma\right)\right|+\left|\mathcal{F}'\left(\delta\right)\right|\right].$$

Proof The last part of the Corollary 3.5 can be easily seen by setting $\eta_1 = 3\left|\mathcal{F}'\left(\sigma\right)\right|^q$, $\eta_2 = \left|\mathcal{F}'\left(\delta\right)\right|^q$, $\varphi_1 = 3\left|\mathcal{F}'\left(\delta\right)\right|^q$, $\varphi_2 = \left|\mathcal{F}'\left(\sigma\right)\right|^q$ and applying the well-known inequality

$$\sum_{k=1}^{n}\left(\eta_k+\varphi_k\right)^s \leq \sum_{k=1}^{n}\eta_k^s+\sum_{k=1}^{n}\varphi_k^s$$

with $0 \leq s < 1$. This completes the proof of Corollary 3.5. □

Remark 3.6 *Let us consider $\beta = 1$ in Corollary 3.5. Then, we have*

$$\left|\frac{\mathcal{F}\left(\sigma\right)+\mathcal{F}\left(\delta\right)}{2}-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}\left(x\right)dx\right|$$

$$\leq \frac{\delta-\sigma}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|\mathcal{F}'\left(\delta\right)\right|^{q}+3\left|\mathcal{F}'\left(\sigma\right)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\sigma\right)\right|^{q}+3\left|\mathcal{F}'\left(\delta\right)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\delta-\sigma}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'\left(\sigma\right)\right|+\left|\mathcal{F}'\left(\delta\right)\right|\right],$$

which is presented by Budak et al. in [8, Remark 5.5].

Remark 3.7 *If we assign $\alpha = 1$ and $s = 1$ in Theorem 2.4, then we acquire*

$$\left|\frac{\mathcal{F}\left(\sigma\right)+\mathcal{F}\left(\delta\right)}{2}-\frac{2^{\beta-1}\Gamma\left(\beta+1\right)}{\left(\delta-\sigma\right)^{\beta}}\left[J_{\frac{\sigma+\delta}{2}-}^{\beta}\mathcal{F}\left(\sigma\right)+J_{\frac{\sigma+\delta}{2}+}^{\beta}\mathcal{F}\left(\delta\right)\right]\right|$$

$$\leq \frac{\left(\delta-\sigma\right)\beta}{4\left(\beta+1\right)}\left\{\left(\frac{\left(\beta+1\right)\left|\mathcal{F}'\left(\delta\right)\right|^{q}+\left(3\beta+7\right)\left|\mathcal{F}'\left(\sigma\right)\right|^{q}}{4\left(\beta+2\right)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left(\beta+1\right)\left|\mathcal{F}'\left(\sigma\right)\right|^{q}+\left(3\beta+7\right)\left|\mathcal{F}'\left(\delta\right)\right|^{q}}{4\left(\beta+2\right)}\right)^{\frac{1}{q}}\right\},$$

which is given in [8, Corollary 5.9].

Remark 3.8 *Let us take $\alpha = \beta = 1$ and $s = 1$ in Theorem 2.4. Then, we have*

$$\left| \frac{\mathcal{F}(\sigma) + \mathcal{F}(\delta)}{2} - \frac{1}{\delta - \sigma} \int_{\sigma}^{\delta} \mathcal{F}(x)\, dx \right|$$

$$\leq \frac{\delta - \sigma}{8} \left[\left(\frac{|\mathcal{F}'(\delta)|^q + 5 |\mathcal{F}'(\sigma)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{|\mathcal{F}'(\sigma)|^q + 5 |\mathcal{F}'(\delta)|^q}{6} \right)^{\frac{1}{q}} \right],$$

which is established by Budak et al. in [8, Remark 5.6].

4. Summary & concluding remarks

In this paper, we prove an equality for the case of $s-$convex differentiable functions. By using this identity, we established trapezoid-type inequalities for the case of conformable fractional integrals. In addition, an example is given by using graphs in order to show that our main results are correct. Furthermore, we prove that our results generalize several well-known trapezoid-type inequalities with the help of the special cases of the obtained main results.

In future studies, the ideas and strategies for our results about trapezoid-type inequalities by conformable fractional integrals may open new avenues for further research in this area. Moreover, one can obtain likewise inequalities of trapezoid-type by conformable fractional integrals for the case of convex functions by using quantum calculus. Furthermore, one can apply these resulting inequalities to different types of fractional integrals.

Availability of data and material

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Author contributions

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