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The class of demi KB-operators on Banach lattices

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Abstract: In this paper, we introduce and study the new concept of demi KB-operators. Let E be a Banach lattice. An operator $T : E \longrightarrow E$ is said to be a demi KB-operator if, for every positive increasing sequence $\{x_n\}$ in the closed unit ball $\mathcal{B}_E$ of E such that $\{x_n - Tx_n\}$ is norm convergent to $x \in E$, there is a norm convergent subsequence of $\{x_n\}$. If the latter sequence has a weakly convergent subsequence then T is called a weak demi KB-operator. We also investigate the relationship of these classes of operators with classical notions of operators, such as b-weakly demicompact operators and demi Dunford-Pettis operators.

Key words: Demi KB-operator, weak demi KB-operator, b-weakly demicompact operator, Demi Dunford-Pettis operator, Banach lattice, KB-space

1. Introduction

Throughout this paper X and Y will denote real Banach spaces, E and F will denote real Banach lattices. The set of all bounded linear operators on X is denoted by $\mathcal{L}(X)$. $\mathcal{B}_E$ is the closed unit ball of E. The positive cone of E will be denoted by $E_+ = \{x \in E; \ \ 0 \leq x\}$.

To state our results, we need to fix some notations and recall some definitions. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A Banach lattice E is said to be a KB-space whenever each increasing norm bounded sequence of E_+ is norm convergent. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf\{x, y\} = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$.

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \in E_+$. We write $S \leq T$ if $(T - S)x \geq 0$ for every $x \in E_+$. We say that S is dominated by T.

We refer the reader to the monographs [2, 14] for ambiguous terminology from Banach lattices and positive operators theory.

An operator T from a Banach lattice E to a Banach space X is said to be b-weakly compact, if the image of every b-order bounded subset of E (that is, order bounded in the topological bidual E'' of E) under

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T is relatively weakly compact. The authors in [6] proved that an operator T from a Banach lattice E into a Banach space X is b-weakly compact if and only if $\{Tx_n\}$ is norm convergent for every positive increasing sequence $\{x_n\}$ of the closed unit ball $\mathcal{B}_E$ of E. The class of b-weakly compact operators was firstly introduced by Alpay, Altin, and Tonyali [3].

In their paper [7], Bahramnezhad and Azar introduced the class of (weak) KB-operators. After that, a series of papers which gave different characterizations of this class were published [4, 16]. Recall that an operator T from a Banach lattice E into a Banach space X is said to be KB-operator (respectively, weak KB-operator) if $\{Tx_n\}$ has a norm (respectively, weak) convergent subsequence in X for every positive increasing sequence $\{x_n\}$ in the closed unit ball $\mathcal{B}_E$ of E. In [4], Altin and Machrafi showed that the three classes of KB, weak KB, and b-weakly compact operators are the same (see [4, Theorem 3.2]).

Let us recall from [15] that an operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$, where $\mathcal{D}(T)$ is a subspace of X, is said to be demicompact if, for every bounded sequence $\{x_n\}$ in the domain $\mathcal{D}(T)$ such that $\{x_n - Tx_n\}$ converges to $x \in X$, there is a convergent subsequence of $\{x_n\}$. Note that each compact operator is demicompact, but the opposite is not always true. In fact, let $Id_X : X \longrightarrow X$ be the identity operator of a Banach space X of infinite dimension. It is clear that $-Id_X$ is demicompact but it is not compact. The concept of demicompactness was introduced by Petryshyn [15] in order to discuss fixed points. Jeribi [12] used the class of demicompact operators to obtain some results on Fredholm and spectral theories.

In [13], Krichen and O'Regan developed some Fredholm and perturbation results including the class of weakly demicompact operators. Moreover, they explored the relationship between this class and measures of weak noncompactness of operators with respect to an axiomatic one. Let us recall that an operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be weakly demicompact if, every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $\{x_n - Tx_n\}$ weakly converges in X, has a weakly convergent subsequence.

Recall that an operator $T : X \longrightarrow X$ is said to be demi Dunford-Pettis, if for every sequence $\{x_n\}$ in X such that $x_n \to 0$ in $\sigma(X, X')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$. Benkhaled et al. [10] introduced the concept of demi Dunford-Pettis operators, a generalization of known classes of operators which are Dunford-Pettis operators.

Recently, a series of papers introduced some operators on Banach lattices involving demi criteria. More precisely, order weakly demicompact operators [8], L-weakly and M-weakly demicompact operators [9] and b-weakly demicompact operators [11]. The purpose of this work is to pursue this analysis in order to define a new class of operators on Banach lattices related to the class of (weak) KB-operators that we call (weak) demi KB-operators.

An outline of this article is as follows. In section 2, we introduce a new class of operators called demi KB-operators (see Definitions 2.1). Note that the class of demi KB-operators involves that of KB-operators (see Proposition 2.3). After that, we illustrate our analysis with some interesting examples (see Examples 2.4, 2.5, and 2.10). Moreover, we characterize Banach lattices on which all operators are demi KB-operators (see Theorem 2.11). In section 3, we introduce the notion of weak demi KB-operators (see Definition 3.1). We also concentrate to solve the following problem "demi KB-operators and their relationships with b-weakly demicompact (see Proposition 3.4) and demi Dunford-Pettis operators (see Propositions 3.8 and 3.11)".

2. Definition and properties

We start this section by the following definition.

Definition 2.1 *Let E be a Banach lattice. An operator $T : E \longrightarrow E$ is said to be a demi KB-operator if, for every positive increasing sequence $\{x_n\}$ in $\mathcal{B}_E$ such that $\{x_n - Tx_n\}$ is norm convergent to $x \in E$, there is a norm convergent subsequence of $\{x_n\}$. The collection of demi KB-operators will be denoted by $\mathcal{L}_{DKB}(E)$.*

Example 2.2 *Let E be a Banach lattice and $\alpha \neq 1$. $\alpha Id_E \in \mathcal{L}_{DKB}(E)$*

Our next result proves that the class of demi KB-operator contains that of KB-operator.

Proposition 2.3 *Let E be a Banach lattice. Every KB-operator $T : E \longrightarrow E$ is a demi KB-operator.*

Proof Let $\{x_n\}$ be a positive increasing sequence in $\mathcal{B}_E$ such that $\{x_n - Tx_n\}$ is norm convergent to $x \in E$. Since $T \in \mathcal{L}_{KB}(E)$, there exists subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ which is norm convergent to $y \in E$. Hence, from the relation

$$x_{n_k} = x_{n_k} - Tx_{n_k} + Tx_{n_k},$$

$\{x_{n_k}\}$ is norm convergent to $x + y$. This means that $\{x_n\}$ has a norm convergent subsequence and so T is a demi KB-operator. □

Note that a demi KB-operator is not necessarily KB-operator. To illustrate this, we give the following example.

Example 2.4 *Let $Id_{L^\infty[0,1]} : L^\infty[0,1] \longrightarrow L^\infty[0,1]$ be the identity operator. It is clear that $-Id_{L^\infty[0,1]}$ is demi KB-operator (see Exemple 2.2). But as $L^\infty[0,1]$ is not KB-space, $-Id_{L^\infty[0,1]}$ is not KB-operator (see [7, Proposition 2.13]).*

It is worth noting that the sum of two demi KB-operators is not necessarily a demi KB-operator. In fact:

Example 2.5 *Let $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ be the identity operator. From Example 2.2, we have $T = -Id_{l^\infty}$ and $S = 2Id_{l^\infty}$ are demi KB-operators. But the sum $T + S = Id_{l^\infty}$ is not. In fact, let $e = (1, 1, 1, ...) \in l^\infty$ and*

$$x_n = (x_m)_n = \begin{cases} 1 & \text{if } m \leq n; \\ 0 & \text{if } m > n. \end{cases}$$

It is clear that $0 \leq x_n \uparrow e$ is satisfied in l^∞. Note that $\{x_n - Id_{l^\infty} x_n\}$ is norm convergent to zero. But $\{x_n\}$ has not any norm convergent subsequence. In fact, assume by way contradiction that there exists a norm convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Hence, it should be cauchy in l^∞. But for each $n_k, n_l \in \mathbb{N}$ with $n_k \geq n_l$ we have

$$\begin{aligned} \|x_{n_k} - x_{n_l}\|_\infty &= \sup\{|x_{n_k}(i) - x_{n_l}(i)|; i \in \mathbb{N}\} \\ &= \sup\{(0, 0, 0, .., \overbrace{0}^{n_l th.}, 1, 1, .., \overbrace{1}^{n_k th.}, 0, 0, .., 0)\} \\ &= 1. \end{aligned}$$

So, $\{x_{n_k}\}$ is not Cauchy sequence. Hence $\{x_n\}$ does not have any norm convergent subsequence.

The following proposition asserts that a KB-operator perturbation of a demi KB-operator is a demi KB-operator.

Proposition 2.6 *Let E be a Banach lattice. Let $T, S : E \longrightarrow E$ be two operators. If T is a demi KB-operator and S is KB-operator, then $T + S$ is a demi KB-operator.*

Proof Let $\{x_n\}$ be a positive increasing sequence in $\mathcal{B}_E$ such that $\{x_n - (T + S)x_n\}$ is norm convergent to $x \in E$. We have to show that $\{x_n\}$ contains a norm convergent subsequence. Since S is KB-operator, we infer that $\{Sx_n\}$ has a norm convergent subsequence $\{Sx_{n_k}\}$ which is norm convergent to y. Since we can write

$$x_{n_k} - Tx_{n_k} = x_{n_k} - (T + S)x_{n_k} + Sx_{n_k},$$

it follows that $x_{n_k} - Tx_{n_k}$ is norm convergent to $x + y$. The fact that T is demi KB-operator, we obtain that $\{x_{n_k}\}$ contains a norm convergent subsequence. Hence, $T + S$ is a demi KB-operator. □

Remark 2.7 *In general, the set of all demi KB-operators is not norm closed in $\mathcal{L}(E)$. In fact, take $E = l^\infty$ and $T_n = \frac{n+1}{n} Id_{l^\infty}$ for each $n \in \mathbb{N}^*$. It is clear that $\{T_n\} \subseteq \mathcal{L}_{DKB}(E)$ (see Example 2.2). On the other hand, from the relation*

$$\|T_n - Id_{l^\infty}\| = \|\frac{n+1}{n} Id_{l^\infty} - Id_{l^\infty}\| = \|\frac{1}{n} Id_{l^\infty}\| = \frac{1}{n},$$

$\lim_{n\to\infty} \|T_n - Id_{l^\infty}\| = 0$ *and so $\{T_n\}$ is norm convergent to Id_{c_0}. But, in view of Example 2.5, Id_{c_0} is not demi KB-operator.*

In what follows, we study the concept of demi KB-operators of the matrix operator $\widetilde{T}$ from E into E expressed by

$$\widetilde{T} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

where $E = E_1 \oplus E_2$ is a direct sum of a family of Banach lattices (E_1, E_2) and $T_1 : E_1 \longrightarrow E_1$, $T_2 : E_2 \longrightarrow E_1$, $T_3 : E_1 \longrightarrow E_2$ and $T_4 : E_2 \longrightarrow E_2$ be four operators.

Proposition 2.8 *Let E_1 and E_2 be two Banach lattices. Let $T_1 : E_1 \longrightarrow E_1$, $T_2 : E_2 \longrightarrow E_1$, $T_3 : E_1 \longrightarrow E_2$, and $T_4 : E_2 \longrightarrow E_2$ be four operators. Consider $E = E_1 \oplus E_2$. Assume that T_1 and T_4 are demi KB-operators. If one of the following statements is valid:*
(i) T_2 is KB-operator.
(ii) T_3 is KB-operator.
Then, the matrix operator $\widetilde{T} : E \longrightarrow E$ determined by

$$\widetilde{T} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

is demi KB-operator.

Proof (i) Assume that T_2 is a KB-operator. Let $\{x_n = (y_n, z_n)\}$, $y_n \in E_1$ and $z_n \in E_2$, be a positive increasing sequence in $\mathcal{B}_E$ such that $\{x_n - \widetilde{T}x_n\}$ is norm convergent to an element $x = (x_1, x_2) \in E$ where $x_1 \in E_1$ and $x_2 \in E_2$. We have to show that $\{x_n\}$ contains a norm convergent subsequence. Since

$x_n = (y_n, z_n)$, it suffices to prove that $\{y_n\}$ and $\{z_n\}$ contain a norm convergent subsequences in E_1 and E_2, respectively. Accordingly, for every n, we get

$$\begin{aligned} x_n - \widetilde{T}x_n &= (y_n, z_n) - \widetilde{T}(y_n, z_n) \\ &= (y_n, z_n) - (T_1 y_n + T_2 z_n, T_3 y_n + T_4 z_n) \\ &= (y_n - T_1 y_n - T_2 z_n, z_n - T_3 y_n - T_4 z_n). \end{aligned}$$

We have $\{x_n - \widetilde{T}x_n\}$ is norm convergent to $x = (x_1, x_2)$, then $\{y_n - T_1 y_n - T_2 z_n\}$ (Resp. $\{z_n - T_3 y_n - T_4 z_n\}$) is norm convergent to x_1 (Resp. x_2). On the other hand, since T_2 is KB-operator, we infer that $\{T_2 z_n\}$ has a norm convergent subsequence $\{T_2 z_{n_k}\}$ which is norm convergent to an element $x_1' \in E_1$. Hence, from the relation

$$y_{n_k} - T_1 y_{n_k} = y_{n_k} - (T_1 y_{n_k} + T_2 z_{n_k}) + T_2 z_{n_k},$$

it follows that $y_{n_k} - T_1 y_{n_k}$ is norm convergent to $x_1 + x_1'$. The fact that T_1 is demi KB-operator, we obtain that $\{y_{n_k}\}$ contains a norm convergent subsequence $\{y_{n_{\varphi(k)}}\}$ which is norm convergent to an element $y \in E_1$. As T_3 is bounded, we have $\{T_3 y_{n_{\varphi(k)}}\}$ is norm convergent to $T_3 y$. Since we can write

$$z_{n_{\varphi(k)}} - T_4 z_{n_{\varphi(k)}} = z_{n_{\varphi(k)}} - (T_4 z_{n_{\varphi(k)}} + T_3 y_{n_{\varphi(k)}}) + T_3 y_{n_{\varphi(k)}},$$

then $z_{n_{\varphi(k)}} - T_4 z_{n_{\varphi(k)}}$ is norm convergent to $x_2 + T_3 y$. The use of T_4 as demi KB-operator implies that $\{z_{n_{\varphi(k)}}\}$ contains a norm convergent subsequence.
(ii) We establish the same reasoning as (i). □

Remark 2.9 *If one of the operators T_1 or T_4 of Proposition 2.8 is not a demi KB-operator then $\widetilde{T}$ is not a demi KB-operator. In fact, let T be an operator from l^1 into l^∞. Consider $E = l^1 \oplus l^\infty$ and let $\widetilde{T}$ be an operator defined via the matrix:*

$$\widetilde{T} = \begin{pmatrix} 0 & 0 \\ T & Id_{l^\infty} \end{pmatrix}.$$

The operator $\widetilde{T}$ is not a demi KB-operator. In fact, consider the sequence $\{x_n\}$ defined by:

$$x_n = (x_m)_n = \begin{cases} 1 & \text{if } m \le n; \\ 0 & \text{if } m > n. \end{cases}$$

Now, put $\widetilde{x}_n = (0, x_n)$ for every n. So that $\{\widetilde{x}_n\}$ is a positive increasing sequence in $\mathcal{B}_E$. Moreover, $\{\widetilde{x}_n - \widetilde{T}\widetilde{x}_n\}$ is norm convergent to 0. Since $\{x_n\}$ has no norm convergent subsequence (see Example 2.5), then $\{\widetilde{x}_n\}$ does not have any norm convergent subsequence.

The following example illustrates that the class of demi KB-operators lacks the vector space structure not only with the sum but also with the external product.

Example 2.10 *Let T be an operator from l^1 into l^∞. Consider $\widetilde{E} = l^1 \oplus l^\infty$ and let $\widetilde{S}$ be an operator defined via the matrix:*

$$\widetilde{S} = \begin{pmatrix} 0 & 0 \\ T & -Id_{l^\infty} \end{pmatrix}.$$

From Proposition 2.8, it follows that $\widetilde{S}$ *is demi KB-operator, but the operator* $-\widetilde{S}$ *given by*

$$-\widetilde{S} = \begin{pmatrix} 0 & 0 \\ -T & Id_{l^\infty} \end{pmatrix}$$

is not (see Remark 2.9).

Next, we characterize Banach lattices on which all operators are demi KB-operators. It is a generalization of Proposition 2.13 obtained by Bahramnezhad-Azar [7] and Corollary 4.6 of Altin-Machrafi [4].

Theorem 2.11 *Let* E *be a Banach lattice. Then the following statements are equivalent:*
(1) E *is a KB-space.*
(2) *Every operator* $T : E \longrightarrow E$ *is KB-operator.*
(3) *Every operator* $T : E \longrightarrow E$ *is demi KB-operator.*
(4) *The identity operator of* E *is demi KB-operator.*

Proof (1) $\Longrightarrow$ (2) Follows from [7, Proposition 2.13].
(2) $\Longrightarrow$ (3) Follows from Proposition 2.3.
(3) $\Longrightarrow$ (4) Obvious.
(4) $\Longrightarrow$ (1) We have to show that E is a KB-space. It suffices to establish that $\{x_n\}$ is norm convergent for every increasing norm bounded sequence $\{x_n\}$ in E_+. Let $\{x_n\}$ be such a sequence. It is clear that $\{x_n - x_n\}$ is norm convergent. Since, Id_E is demi KB-operator, $\{x_n\}$ has a norm convergent subsequence. On the other hand, as $\{x_n\}$ is an increasing sequence, $\{x_n\}$ is norm convergent. Therefore, E is a KB-space. □

Remark 2.12 *There exists an operator from a Banach lattice* E *into* E *that is a demi KB-operator however* E *is not KB-space. In fact, if we take* $X = c_0$ *the Banach lattice of all null convergent sequences, then* $-Id_{c_0} : c_0 \longrightarrow c_0$ *is demi KB-operator (see Example 2.2) but* c_0 *is not KB-space.*

As the class of KB-operators [7], the set of demi KB-operators does not satisfy the duality property.
More precisely, there is a demi KB-operator T from E into E whose dual T' from E' into E' is not demi KB-operator and conversely, there is an operator T from E into E which is not demi KB-operator while its dual T' from E' into E' is one.
In fact, the identity operator Id_{l^1} is a demi KB-operator but its dual which is the identity operator Id_{l^∞} is not a demi KB-operator.
Conversely, the identity operator Id_{c_0} is not demi KB-operator, but its dual which is the identity operator Id_{l^1} is a demi KB-operator.

3. Connection between demi KB-operators and some other operators

In this section, we first introduce the class of weak demi KB-operators.

Definition 3.1 *Let* E *be a Banach lattice. An operator* $T : E \longrightarrow E$ *is said to be a weak demi KB-operator if, for every positive increasing sequence* $\{x_n\}$ *in* $\mathcal{B}_E$ *such that* $\{x_n - Tx_n\}$ *is weakly convergent to* $x \in E$, *there is a weakly convergent subsequence of* $\{x_n\}$. *The collection of weak demi KB-operators will be denoted by* $\mathcal{W}_{DKB}(E)$.

Remark 3.2 (i) *Let* E *be a Banach lattice and* $\alpha \neq 1$. $\alpha Id_E \in \mathcal{W}_{DKB}(E)$.
(ii) *Notice that all of the results obtained in Section 2 about demi KB-operators are also true for weak demi KB-operators, if we replace norm convergence by weak convergence.*

As mentioned in the introduction, this section is dedicated to exhibiting some relationships of demi KB-operators with other operators on Banach lattices.

Recall from [11] that an operator $T : E \longrightarrow E$ is said to be b-weakly demicompact if for every b-order bounded sequence $\{x_n\}$ in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$. The collection of b-weakly demicompact operators will be denoted by $\mathcal{WD}_b(E)$. Note that the sum of b-weakly compact and b-weakly demicompact operators is a b-weakly demicompact operator (see [11, Theorem 2.7]).

Question 3.3 *Is it true that three classes of demi KB, weak demi KB, and b-weakly demicompact operators on a Banach Lattice are the same?*

The result below gives a positive answer to our previous question.

Theorem 3.4 *Let* E *be a Banach lattice and* $T : E \longrightarrow E$ *be an operator. Then, the following assertions are equivalent:*
(1) T *is a b-weakly demicompact operator;*
(2) T *is a demi KB-operator;*
(3) T *is a weak demi KB-operator.*

The proof is based on the following well-known lemma from the KB-space of Banach lattice literature.

Lemma 3.5 *[5, Corollary 2.3] Let* E *be a Banach lattice, then the following assertions are equivalent:*
(i) E *is a KB-space.*
(ii) $\|x_n\| \longrightarrow 0$ *as* $n \to \infty$ *for every b-order bounded sequence* $\{x_n\}$ *of* E_+ *satisfying* $x_n \to 0$ *for the topology* $\sigma(E, E')$.

Proof of Theorem 3.4.
(1) $\Longrightarrow$ (2) Let $\{x_n\}$ be a positive increasing sequence in $\mathcal{B}_E$ such that $\{x_n - Tx_n\}$ is norm convergent to an element $x \in E$. We have to show that $\{x_n\}$ contains a norm convergent subsequence. Hence, by Proposition 2.10 in [6], $Id_E - T$ is b-weakly compact operator. The fact that T is a b-weakly demicompact and from [11, Theorem 2.7], we obtain $Id_E = Id_E - T + T$ is b-weakly demicompact. By [11, Theorem 2.9], E is a KB-space. Then $\{x_n\}$ is norm convergent. So, T is a demi KB-operator.
(2) $\Longrightarrow$ (3) Let $\{x_n\}$ be a positive increasing sequence in $\mathcal{B}_E$ such that $\{x_n - Tx_n\}$ is weakly convergent to an element $x \in E$. We have to show that $\{x_n\}$ contains a weakly convergent subsequence. By our hypothesis, $Id_E - T$ is a weak KB-operator. The use of Theorem 3.2 in [4] asserts that $Id_E - T$ is KB-operator. Then, $\{x_n - Tx_n\}$ is norm convergent to x. Since T is demi KB-operator, we infer that $\{x_n\}$ has a norm convergent subsequence.
(3) $\Longrightarrow$ (1) Let T be a weak demi KB-operator from a Banach lattice E into itself. We have to show that T is a b-weakly demicompact. It suffices to prove that $\|x_n\| \to 0$ as $n \to \infty$ for each b-order bounded sequence $\{x_n\}$ of E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Let $\{x_n\}$ be such a sequence. The use

of [5, Theorem 2.2] implies that $Id_E - T$ is b-weakly compact. Thus, by Theorem 3.2 in [4], $Id_E - T$ is weak KB-operator. Since T is a weak demi KB-operator, it follows from Proposition 2.6 that $Id_E = Id_E - T + T$ is a weak demi KB-operator. We can deduce from Theorem 2.11 that E is a KB-space. Since $\{x_n\}$ be a b-order bounded sequence of E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and by using preceding lemma, $\|x_n\| \to 0$ as $n \to \infty$. □

Note that the class of demi KB-operators does not satisfy the domination problem. To illustrate this, we give the following example.

Example 3.6 *Let $E = l^\infty$ and $Id_E : E \longrightarrow E$ be the identity operator. Clearly, $0 \le Id_E \le 2Id_E$. From Example 2.2, $2Id_E$ is demi KB-operator. But in view of Example 2.5, it follows that Id_E is not a demi KB-operator.*

In [7], the authors proved that the class of KB-operators satisfies the domination property (see [7, Proposition 2.9]). Such a property can be stated as well in the setting for central demi KB-operators. Recall from [1] that an operator $T : E \longrightarrow E$ is called central if it is dominated by a multiple of the identity operator.

Proposition 3.7 *Let E be a Banach lattice and $S, T : E \longrightarrow E$ be two operators with $0 \le S \le T \le Id_E$. If T is a demi KB-operator, then S is also a demi KB-operator.*

Proof Consider $S, T : E \longrightarrow E$ be two operators with $0 \le S \le T \le Id_E$ and T be a demi KB-operator. Let $\{x_n\}$ be a positive increasing sequence in $\mathcal{B}_E$ such that $\{x_n - Sx_n\}$ is norm convergent to $x \in E$. We have to show that $\{x_n\}$ contains a norm convergent subsequence. By our hypothesis, $Id_E - S$ is KB-operator. Since $0 \le Id_E - T \le Id_E - S$ and in view of [7, Proposition 2.9], we obtain $Id_E - T$ that is a KB-operator. The fact that T is a demi KB-operator and from Proposition 2.6, we infer that $Id_E = Id_E - T + T$ is a demi KB-operator. It follows from Theorem 2.11 that E is a KB-space. Then $\{x_n\}$ has a norm convergent subsequence and so S is a demi KB-operator. □

In the next result, we prove that the space of demi KB-operators is bigger than the space of demi Dunford-Pettis operators.

Proposition 3.8 *Let E be a Banach lattice. Every demi Dunford-Pettis operator $T : E \longrightarrow E$ is a demi KB-operator.*

Proof Let $T : E \longrightarrow E$ be a demi Dunford-Pettis operator. We have to show that T is a demi KB-operator. By Theorem 3.4, it suffices to prove that T is b-weakly demicompact. Consider $\{x_n\}$ be a b-order bounded of E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. We claim that $\|x_n\| \to 0$ as $n \to \infty$. In view of T is demi Dunford-Pettis, we can deduce the result. □

Remark 3.9 *There exists a demi KB-operator from a Banach lattice E into E that is not demi Dunford-Pettis. In fact, since $L^1[0,1]$ is a KB-space, it follows from Theorem 2.11 that the identity operator $Id_{L^1[0,1]}$ is a demi KB-operator. On the other side, $L^1[0,1]$ does not have the Schur property, it follows from [10, Theorem 2.4] that $Id_{L^1[0,1]}$ is not demi Dunford-Pettis.*

To give conditions under which a demi KB-operator is demi Dunford-Pettis, we will need the following theorem:

Theorem 3.10 *[17, Theorem 1] Let E be a Banach lattice. Then the following statements are equivalent:*

(i) *E is a discrete KB-space with Dunford-Pettis property.*

(ii) *E has the Schur property.*

Proposition 3.11 *Let E be discrete with Dunford-Pettis property. Then for an operator $T : E \longrightarrow E$ the following assertions are equivalent:*

1. *T is demi Dunford-Pettis.*
2. *T is demi KB-operator.*

Proof (1) $\Longrightarrow$ (2) It follows from Proposition 3.8.
(2) $\Longrightarrow$ (1) Assume that T is a KB-operator. Let $\{x_n\}$ be a sequence of E such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. We have to show that $\|x_n\| \to 0$ as $n \to \infty$. By our hypothesis, $Id_E - T$ is Dunford-Pettis. The use of [7, Proposition 2.21] implies that $Id_E - T$ is KB-operator. The fact that T is a demi KB-operator, it follows from Proposition 2.6 that $Id_E = Id_E - T + T$ is a demi KB-operator. Thus, Theorem 2.11, implies that E is a KB-space. Since E is discrete with Dunford-Pettis property and in view of Theorem 3.10, then E has the Schur property. Note that $x_n \to 0$ in $\sigma(E, E')$ as $n \to \infty$, it follows that $\|x_n\| \to 0$ as $n \to \infty$. □

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