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Hahn-Hamiltonian systems

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Abstract: In this paper, we study the basic theory of regular Hahn-Hamiltonian systems. In this context, we establish an existence and uniqueness result. We introduce the corresponding maximal and minimal operators for this system and some properties of these operators are investigated. Moreover, we give a criterion under which these operators are self-adjoint. Finally, an expansion theorem is proved.

Key words: Regular Hahn-Hamiltonian system, maximal and minimal operators, self-adjoint operator, eigenfunction expansion

1. Introduction

The Hahn difference operators were introduced by Hahn [11], [12]. These operators are receiving an increase of interest since their applications in the construction of families of orthogonal polynomials and approximation problems (see [6, 10, 18, 19, 21]). There exist some papers including the Hahn difference operator in the literature (see [5, 8, 13–15, 23]). Recently, in [1, 2, 16], the authors studied Hahn–Dirac systems. In [5], the authors studied matrix-valued Hahn–Sturm–Liouville equations.

In this paper, we discuss the basic properties of the regular Hahn–Hamiltonian system defined as

$$l(\mathcal{Z}) := J\mathcal{Z}^{[\varrho]}(x) - M(x)\mathcal{Z}(x) = \lambda N(x)\mathcal{Z}(x), \ x \in [\omega_0, a],$$

where the matrices

$$M(x) = \begin{pmatrix} M_1(x) & M_2^*(x) \\ M_2(x) & M_3(x) \end{pmatrix}$$

and

$$N(x) = \begin{pmatrix} N_1(x) & 0 \\ 0 & N_2(x) \end{pmatrix}$$

are $2n \times 2n$ complex Hermitian matrix-valued functions defined on $[\omega_0, a]$ and are continuous at ω_0; $\mathcal{Z}(x)$ is $2n \times 1$ vector-valued function and

$$\mathcal{Z}^{[\varrho]}(x) = \begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ \frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Z}_2(x) \end{pmatrix} = \begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ D_{\omega,q}\mathcal{Z}_2\left(h^{-1}(x)\right) \end{pmatrix},$$

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where $\mathcal{Z}_1, \mathcal{Z}_2 : [\omega_0, h^{-1}(a)] \to \mathbb{C}^n$, $h(x) := \omega + qx$, $h^{-1}(x) = q^{-1}(x - \omega)$;

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. In the analysis that follows, we will largely follow the development of the theory in [4, 5, 9, 17, 22, 24].

Our paper is organized as follows. The second section introduces the fundamental concepts of Hahn calculus. In the third section, an existence and uniqueness theorem is proved for the regular Hahn–Hamiltonian system. We also introduce the corresponding maximal and minimal operators for this system and some properties of these operators are investigated. In the fourth section, a criterion under which the Hahn–Hamiltonian operators are self-adjoint is given. Finally, an expansion theorem is proved in the last section.

2. Preliminaries

In this section, we recall some necessary concepts of Hahn calculus. For more details, the reader may want to consult [7, 8, 11, 12].

Throughout the paper, we let $\omega > 0$ and $q \in (0, 1)$. Let I be a real interval containing ω_0, where $\omega_0 := \frac{\omega}{1-q}$.

Definition 2.1 ([11],[12]) *Let $u : I \to \mathbb{R}$ be a function. If u is differentiable at ω_0, then the Hahn difference operator $D_{\omega,q}$ is given by the formula*

$$D_{\omega,q}u(x) = \begin{cases} [\omega + (q-1)x]^{-1}[u(\omega + qx) - u(x)], & x \neq \omega_0, \\ u'(\omega_0), & x = \omega_0. \end{cases}$$

Remark 2.2 *The operator $D_{\omega,q}$ unifies two well-known operators. In fact,*

$$\lim_{q\to 1} D_{\omega,q}u(x) = \Delta_\omega u(x) := [(\omega + x) - x]^{-1}[u(\omega + x) - u(x)], \ x \in \mathbb{R},$$

$$\lim_{\omega\to 0} D_{\omega,q}u(x) = D_q u(x) := [(qx) - x]^{-1}[u(qx) - u(x)], \ x \neq 0,$$

and

$$\lim_{\substack{q\to 1 \\ \omega\to 0}} D_{\omega,q}u(x) = u'(x).$$

Now, we give some properties of the Hahn difference operator $D_{\omega,q}$.

Theorem 2.3 ([7]) *Let $u, v : I \to \mathbb{R}$ be ω, q-differentiable at $x \in I$. Then we have*

i) $D_{\omega,q}(au + bv)(x) = aD_{\omega,q}u(x) + bD_{\omega,q}v(x), \quad a, b \in I,$

ii) $D_{\omega,q}(u \backslash v)(x) = \frac{D_{\omega,q}(u(x))v(x) - u(x)D_{\omega,q}v(x)}{v(x)v(\omega + xq)},$

iii) $D_{\omega,q}(uv)(x) = (D_{\omega,q}u(x))v(x) + u(\omega + xq)D_{\omega,q}v(x),$

iv) $D_{\omega,q}u\left(h^{-1}(x)\right) = \frac{1}{q}D_{-\omega q^{-1},q^{-1}}u(x),$

where $h(x) := \omega + qx$, $h^{-1}(x) = q^{-1}(x - \omega)$, and $x \in I$.

Definition 2.4 ([7]) *Let $u : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. The $\omega, q-$integral of the function u is given by*

$$\int_a^b u(x)\, d_{\omega,q}x := \int_{\omega_0}^b u(x)\, d_{\omega,q}x - \int_{\omega_0}^a u(x)\, d_{\omega,q}x,$$

where

$$\int_{\omega_0}^x u(x)\, d_{\omega,q}x := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n u\left(\omega \frac{1-q^n}{1-q} + xq^n\right),\ x \in I$$

provided that the series converges.

Next, we give the $\omega, q-$integration by parts.

Lemma 2.5 ([7]) *Let $u, v : I \to \mathbb{R}$ be ω, q-integrable on I, $a, b \in I$, and $a < b$. Then the following formula holds:*

$$\int_a^b u(x)\, D_{\omega,q} v(x)\, d_{\omega,q}(x) + \int_a^b v(\omega + qx)\, D_{\omega,q} u(x)\, d_{\omega,q}(x)$$
$$= u(b)\, v(b) - u(a)\, v(a).$$

3. Hahn–Hamiltonian systems

Consider the following Hahn–Hamiltonian system

$$l(\mathcal{Z}) := J\mathcal{Z}^{[\varrho]}(x) = [\lambda N(x) + M(x)]\, \mathcal{Z}(x),\ x \in [\omega_0, a], \tag{3.1}$$

where $I + [\omega + (q-1)x]\, M_2(x)$ is invertible; $N(x)$ is nonnegative definite; and λ is a complex spectral parameter.

Let $\mathcal{H} = L^2_{\omega,q,N}((\omega_0, a); \mathbb{C}^{2n})$ be the Hilbert space $2n$-dimensional vector-valued functions $\mathcal{U}, \mathcal{V}$, with the inner product and

$$(\mathcal{U}, \mathcal{V}) = \int_{\omega_0}^a \mathcal{V}^*(x) N(x) \mathcal{U}(x) d_{\omega,q}x,$$

and norm $\|\mathcal{U}\| = \sqrt{(\mathcal{U}, \mathcal{U})}$.

The following assumption will be needed throughout the paper. For every nontrivial solution $\mathcal{Z}$ of (3.1), we have

$$\int_{\omega_0}^a \mathcal{Z}^*(x) N(x) \mathcal{Z}(x) d_{\omega,q}x > 0.$$

Let $C_{\omega,q}((\omega_0, a); \mathbb{C}^{2n})$ be the space of all vector-valued functions u such that u are continuous at ω_0. It is obvious that $C_{\omega,q}((\omega_0, a); \mathbb{C}^{2n}) \subset \mathcal{H}$.

Theorem 3.1 *Eq. (3.1) with initial condition*

$$\mathcal{Z}(\omega_0, \lambda) = \begin{pmatrix} \mathcal{Z}_1(\omega_0, \lambda) \\ \mathcal{Z}_2(\omega_0, \lambda) \end{pmatrix} = K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \tag{3.2}$$

where $k_1, k_2 \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, has a unique solution in $C_{\omega,q}((\omega_0, a); \mathbb{C}^{2n})$.

Proof From (3.1), we obtain

$$\widehat{\mathcal{Z}}(x,\lambda)=K-q\int_{\omega_0}^{x}J\left[\lambda N\left(h\left(s\right),\lambda\right)+M\left(h\left(s\right),\lambda\right)\right]\widehat{\mathcal{Z}}\left(h\left(s\right),\lambda\right)d_{\omega,q}s, \tag{3.3}$$

where

$$\widehat{\mathcal{Z}}\left(x\right)=\begin{pmatrix}\mathcal{Z}_1\left(x\right)\\ \mathcal{Z}_2\left(h^{-1}\left(x\right)\right)\end{pmatrix},$$

and $x\in(\omega_0,a)$. In fact, one can easily prove that (3.3) satisfies (3.1) by using the Hahn difference operator $D_{\omega,q}$. We solve (3.3) by the method of successive approximations:

$$\widehat{\mathcal{Z}}^{(0)}(\omega_0,\lambda)=K,$$

$$\widehat{\mathcal{Z}}^{(k+1)}(x,\lambda)=K-q\int_{\omega_0}^{x}J\left[\lambda N\left(h\left(s\right),\lambda\right)+M\left(h\left(s\right),\lambda\right)\right]\widehat{\mathcal{Z}}^{(k)}\left(h\left(s\right),\lambda\right)d_{\omega,q}s, \tag{3.4}$$

where $x\in[\omega_0,a]$ and $k=0,1,2,....$

We wish to show that the sequence $\left\{\widehat{\mathcal{Z}}^{(k)}\right\}_{k\in\mathbb{N}}$ $(\mathbb{N}:=\{1,2,3,...\})$ converges uniformly on each compact subset of $[\omega_0,a]$. There exist positive numbers $\tau\left(\lambda\right)$ and $\mu\left(\lambda\right)$ such that

$$\left\|J\left[\lambda N\left(h\left(s\right),\lambda\right)+M\left(h\left(s\right),\lambda\right)\right]\right\|_{\mathbb{C}^{2n}}\leq\tau\left(\lambda\right),$$

$$\left\|\mathcal{Z}^{(1)}\left(x,\lambda\right)\right\|_{\mathbb{C}^{2n}}\leq\mu\left(\lambda\right),$$

where $x\in[\omega_0,a]$. By induction, we obtain

$$\left\|\widehat{\mathcal{Z}}^{(k+1)}(x,\lambda)-\widehat{\mathcal{Z}}^{(k)}(x,\lambda)\right\|_{\mathbb{C}^{2n}}\leq q^{\frac{k(k+1)}{2}}\tau\left(\lambda\right)\frac{\left(\mu\left(\lambda\right)x\left(1-q\right)-\omega\right)^k}{\left(q;q\right)_k},$$

where $(q;q)_k=\prod_{i=0}^{k-1}\left(1-q^{i+1}\right)$ and $k\in\mathbb{N}$. According to Weierstrass M-test, we see that $\left\{\widehat{\mathcal{Z}}^{(k)}\right\}_{k\in\mathbb{N}}$ converges to a function $\widehat{\mathcal{Z}}$ uniformly on each compact subset of $[\omega_0,a]$. One can prove that $\widehat{\mathcal{Z}}$ is continuous at ω_0. $\widehat{\mathcal{Z}}$ satisfies condition (3.2). It remains to be proved that the system (3.1)-(3.2) has a unique solution. Assume $\widehat{\mathcal{Y}}$ is another solution. Then $\widehat{\mathcal{Y}}$ is continuous at ω_0. Therefore, there exists a positive number ζ such that $\left\|\widehat{\mathcal{Z}}-\widehat{\mathcal{Y}}\right\|\leq\zeta$. By induction, we conclude that

$$\left\|\widehat{\mathcal{Z}}\left(x,\lambda\right)-\widehat{\mathcal{Y}}(x,\lambda)\right\|_{\mathbb{C}^{2n}}\leq q^{\frac{k(k+1)}{2}}\zeta\tau\left(\lambda\right)\frac{\left(x\left(1-q\right)-\omega\right)^k}{\left(q;q\right)_k},$$

where $k\in\mathbb{N}$. Since

$$\lim_{k\to\infty}\zeta\mu\left(\lambda\right)q^{\frac{k(k+1)}{2}}\frac{\left(x\left(1-q\right)-\omega\right)^k}{\left(q;q\right)_k}=0,$$

we see that $\widehat{\mathcal{Z}} = \widehat{\mathcal{Y}}$ on $[\omega_0, a]$. □

Consider the sets

$$D_{\max} = \left\{ \mathcal{Z} \in \mathcal{H} : \begin{array}{c} \mathcal{Z} \text{ is continuous at } \omega_0, \\ J\mathcal{Z}^{[\varrho]}(x) - M(x)\mathcal{Z}(x) = N(x)F(x) \\ \text{exists in } [\omega_0, a] \text{ and } F \in \mathcal{H} \end{array} \right\},$$

and

$$D_{\min} = \left\{ \mathcal{Z} \in D_{\max} : \widehat{\mathcal{Z}}(\omega_0) = \widehat{\mathcal{Z}}(a) = 0 \right\}. \tag{3.5}$$

Then the *maximal operator* $L_{\max}$ on $D_{\max}$ is defined by $L_{\max}\mathcal{Z} = l(\mathcal{Z})$. Similarly, we define the *minimal operator* $L_{\min}$ on $D_{\min}$ using the following rule $L_{\min}\mathcal{Z} = l(\mathcal{Z})$.

Let us denote by $W_h(\mathcal{U}, \mathcal{V})$ the *Wronskian* of $\mathcal{U}, \mathcal{V}$ with the rule

$$W_h(\mathcal{U}, \mathcal{V})(x) = \mathcal{V}_2^*\left(h^{-1}(x)\right)\mathcal{U}_1(x) - \mathcal{V}_1^*(x)\mathcal{U}_2\left(h^{-1}(x)\right),$$

where

$$\mathcal{U}(x) = \begin{pmatrix} \mathcal{U}_1(x) \\ \mathcal{U}_2(x) \end{pmatrix}, \ \mathcal{V}(x) = \begin{pmatrix} \mathcal{V}_1(x) \\ \mathcal{V}_2(x) \end{pmatrix}.$$

One may infer that the Wronskian of any solutions of Eq. (3.1) is independent of x.

Now, we can introduce the following Green's formula.

Theorem 3.2 *For two functions $\mathcal{U}, \mathcal{V} \in D_{\max}$ we have the following relation*

$$(L_{\max}\mathcal{U}, \mathcal{V}) - (\mathcal{U}, L_{\max}\mathcal{V}) = [\mathcal{U}, \mathcal{V}](a) - [\mathcal{U}, \mathcal{V}](\omega_0), \tag{3.6}$$

where $[\mathcal{U}, \mathcal{V}](x) := \widehat{\mathcal{V}}^*(x)J\widehat{\mathcal{U}}(x)$ and $x \in [\omega_0, a]$.

Proof For $\mathcal{U}, \mathcal{V} \in D_{\max}$, there exist $F, G \in \mathcal{H}$ such that $L_{\max}\mathcal{U} = F$ and $L_{\max}\mathcal{V} = G$. Then we have

$$(L_{\max}\mathcal{U}, \mathcal{V}) - (\mathcal{U}, L_{\max}\mathcal{V}) = (F, \mathcal{V}) - (\mathcal{U}, G)$$

$$= \int_{\omega_0}^{a} \mathcal{V}^*(x)N(x)F(x)\,d_{\omega,q}x - \int_{\omega_0}^{a} G^*(x)N(x)\mathcal{U}(x)\,d_{\omega,q}x$$

$$= \int_{\omega_0}^{a} \mathcal{V}^*(x)\{l(\mathcal{U})\}\,d_{\omega,q}x - \int_{\omega_0}^{a} \{l(\mathcal{V})\}^*\mathcal{U}(x)\,d_{\omega,q}x$$

$$= \int_{\omega_0}^{a} \mathcal{V}^*(x)\left\{J\mathcal{U}^{[h]}(x) - [\lambda N(x) + M(x)]\,\mathcal{U}(x)\right\} d_{\omega,q}x$$

$$- \int_{\omega_0}^{a} \left\{J\mathcal{V}^{[h]}(x) - [\lambda N(x) + M(x)]\,\mathcal{V}(x)\right\}^* \mathcal{U}(x)\, d_{\omega,q}x$$

$$= \int_{\omega_0}^{a} \mathcal{V}^*(x)\, J\mathcal{U}^{[h]}(x) d_{\omega,q}x - \int_{\omega_0}^{a} \left\{J\mathcal{V}^{[h]}(x)\right\}^* \mathcal{U}(x)\, d_{\omega,q}x$$

$$= \int_{\omega_0}^{a} \left\{-\frac{1}{q}\mathcal{V}_1^*(x)\, D_{-\omega q^{-1},q^{-1}}\mathcal{U}_2(x) + \mathcal{V}_2^*(x)\, D_{\omega,q}\mathcal{U}_1(x)\right\} d_{\omega,q}x$$

$$- \int_{\omega_0}^{a} \left[\left\{-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{V}_2^*(x)\right\}\mathcal{U}_1(x) + D_{\omega,q}\mathcal{V}_1^*(x)\mathcal{U}_2(x)\right] d_{\omega,q}x$$

$$= \int_{\omega_0}^{a} \left\{\mathcal{V}_1^*(x)\left[-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{U}_2(x)\right] - D_{\omega,q}\mathcal{V}_1^*(x)\mathcal{U}_2(x)\right\} d_{\omega,q}x$$

$$+ \int_{\omega_0}^{a} \left[\mathcal{V}_2^*(x)\, D_{\omega,q}\mathcal{U}_1(x) - \left\{-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{V}_2^*(x)\right\}\mathcal{U}_1(x)\right] d_{\omega,q}x.$$

On the other hand

$$D_{\omega,q}\left(\mathcal{V}_1^*(x)\,\mathcal{U}_2\left(h^{-1}(x)\right)\right)$$

$$= \left(\mathcal{V}_1^*(x)\, D_{\omega,q}\mathcal{U}_2\left(h^{-1}(x)\right)\right) + D_{\omega,q}\mathcal{V}_1^*(x)\,\mathcal{U}_2(x)$$

$$= \mathcal{V}_1^*(x)\left[\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{U}_2(x)\right] + \left(D_{\omega,q}\mathcal{V}_1(x)\right)^* \mathcal{U}_2(x)$$

and

$$D_{\omega,q}\left(\mathcal{V}_2^*\left(h^{-1}(x)\right)\mathcal{U}_1(x)\right)$$

$$= D_{\omega,q}\mathcal{V}_2^*\left(h^{-1}(x)\right)\mathcal{U}_1(x) + \mathcal{V}_2^*(x)\left(D_{\omega,q}\mathcal{U}_1(x)\right)$$

$$= \left(\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{V}_2^*(x)\right)\mathcal{U}_1(x) + \mathcal{V}_2^*(x)\left(D_{\omega,q}\mathcal{U}_1(x)\right).$$

Therefore

$$(L_{\max}\mathcal{U},\mathcal{V})-(\mathcal{U},L_{\max}\mathcal{V})$$

$$=\int_{\omega_0}^{a} D_{\omega,q}\left\{\mathcal{V}_2^*\left(h^{-1}(x)\right)\mathcal{U}_1(x)-\mathcal{V}_1^*(x)\mathcal{U}_2\left(h^{-1}(x)\right)\right\}d_{\omega,q}x$$

$$=\widehat{\mathcal{V}}^*(a)J\widehat{\mathcal{U}}(a)-\widehat{\mathcal{V}}^*(\omega_0)J\widehat{\mathcal{U}}(\omega_0)=[\mathcal{U},\mathcal{V}](a)-[\mathcal{U},\mathcal{V}](\omega_0).$$

□

From (3.5) and (3.6), we obtain the following lemmas.

Lemma 3.3 *The operator $L_{\min}$ is Hermitian.*

Lemma 3.4 *The relation*

$$(L_{\min}\mathcal{U},\mathcal{V})=(\mathcal{U},L_{\max}\mathcal{V})$$

holds for all $\mathcal{U}\in D_{\min}$ and all $\mathcal{V}\in D_{\max}$.

Lemma 3.5 *Denote by $\mathcal{N}ul(L)$ and $\mathcal{R}an(L)$ the null space and the range of an operator L, respectively. Then we have the following relation*

$$\mathcal{R}an(L_{\min})=\mathcal{N}ul(L_{\max})^{\perp}.$$

Proof Given any $F\in\mathcal{R}an(L_{\min})$, there exists $\mathcal{U}\in D_{\min}$ such that $L_{\min}\mathcal{U}=F$. For each $\mathcal{V}\in\mathcal{N}ul(L_{\max})$, from Lemma 3.4, we see that $(F,\mathcal{V})=(L_{\min}\mathcal{U},\mathcal{V})=(\mathcal{U},L_{\max}\mathcal{V})=0$. Consequently, $F\in\mathcal{N}ul(L_{\max})^{\perp}$.

Consider the following problem:

$$J\mathcal{Z}^{[\varrho]}(x)-M(x)\mathcal{Z}(x)=N(x)F(x),\ \widehat{\mathcal{Z}}(\omega_0)=0, \tag{3.7}$$

where $x\in[\omega_0,a]$ and $F\in\mathcal{N}ul(L_{\max})^{\perp}$. By Theorem 3.1, Eq. (3.7) has a unique solution on $[\omega_0,a]$. Let $\Psi(x)=(\psi_1,\psi_2,...,\psi_{2n})$ be the fundamental solution of the system

$$J\mathcal{Z}^{[\varrho]}(x)-M(x)\mathcal{Z}(x)=0,\ \widehat{\Psi}(a)=J,$$

where $x\in[\omega_0,a]$. It is clear that $\psi_i\in\mathcal{N}ul(L_{\max})$, where $1\leq i\leq 2n$. It follows from Theorem 3.2 that

$$0=(F,\psi_i)=\int_{\omega_0}^{a}\psi_i^*(x)N(x)F(x)d_{\omega,q}x=\int_{\omega_0}^{a}\psi_i^*(x)l(\mathcal{Z})(x)d_{\omega,q}x$$

$$=\int_{\omega_0}^{a}\psi_i^*(x)l(\mathcal{Z})(x)d_{\omega,q}x-\int_{\omega_0}^{a}l(\psi_i)^*(x)\mathcal{Z}(x)d_{\omega,q}x$$

$$=\widehat{\psi}_i^*(a)J\widehat{\mathcal{Z}}(a)-\widehat{\psi}_i^*(0)J\widehat{\mathcal{Z}}(0)=\widehat{\psi}_i^*(a)J\widehat{\mathcal{Z}}(a),$$

where $1\leq i\leq 2n$. Hence $\widehat{\Psi}^*(a)J\widehat{\mathcal{Z}}(a)=\widehat{\mathcal{Z}}(a)=0$, i.e. $F\in\mathcal{R}an(L_{\min})$. □

Theorem 3.6 *$L_{\min}$ is a densely defined operator and is a symmetric operator. Furthermore $L^*_{\min} = L_{\max}$.*

Proof Suppose that $F \in D^{\perp}_{\min}$. Then, for all $\mathcal{Y} \in D_{\min}$, we have $(F, \mathcal{Y}) = 0$. Let $L_{\min}\mathcal{Y}(x) = H(x)$ and let $\mathcal{Z}(x)$ be any solution of the system

$$J\mathcal{Z}^{[\varrho]}(x) - M(x)\mathcal{Z}(x) = N(x)F(x),$$

where $x \in [\omega_0, a]$. From Theorem 3.2, we obtain

$$\begin{aligned}(\mathcal{Z}, H) - (F, \mathcal{Y}) &= \int_{\omega_0}^{a} H^*(x) N(x) \mathcal{Z}(x) d_{\omega,q}x - \int_{\omega_0}^{a} \mathcal{Y}^*(x) N(x) F(x) d_{\omega,q}x \\ &= \int_{\omega_0}^{a} (l(\mathcal{Y})(x))^* \mathcal{Z}(x) d_{\omega,q}x - \int_{\omega_0}^{a} \mathcal{Y}^*(x) l(\mathcal{Z})(x) d_{\omega,q}x \\ &= -\widehat{\mathcal{Y}}^*(a) J\widehat{\mathcal{Z}}(a) + \widehat{\mathcal{Y}}^*(\omega_0) J\widehat{\mathcal{Z}}(\omega_0) = 0.\end{aligned}$$

This implies that $(\mathcal{Z}, H) = (F, \mathcal{Y}) = 0$. It follows from Lemma 3.5 that $\mathcal{Z} \in \mathcal{R}an(L_{\min})^{\perp} = \mathcal{N}ul(L_{\max})$. Therefore $F = 0$, i.e. $D^{\perp}_{\min} = \{0\}$. According to Lemma 3.3, $L_{\min}$ is a symmetric operator.

For any given $\mathcal{Z} \in D_{\max}$, we get from Lemma 3.4 $(\mathcal{Z}, L_{\min}\mathcal{Y}) = (L_{\max}\mathcal{Z}, \mathcal{Y})$, for all $\mathcal{Y} \in D_{\min}$. Thus the functional $(\mathcal{Z}, L_{\min}(.))$ is continuous on $D_{\min}$ and $\mathcal{Z} \in D^*_{\min}$, i.e. $D_{\max} \subset D^*_{\min}$.

If $\mathcal{Z} \in D^*_{\min}$, then $\mathcal{Z}$ and $H := L^*_{\min}\mathcal{Z}$ are all in $\mathcal{H}$. Assume that $\mathcal{U}$ is a solution of the system

$$J\mathcal{U}^{[h]}(x) - M(x)\mathcal{U}(x) = N(x)H(x). \tag{3.8}$$

It follows from Lemma 3.4 that $(H, \mathcal{Y}) = (L_{\max}\mathcal{U}, \mathcal{Y}) = (\mathcal{U}, L_{\min}\mathcal{Y})$. Hence

$$\begin{aligned}(\mathcal{Z} - \mathcal{U}, L_{\min}\mathcal{Y}) &= (\mathcal{Z}, L_{\min}\mathcal{Y}) - (\mathcal{U}, L_{\min}\mathcal{Y}) \\ &= (L^*_{\min}\mathcal{Z}, \mathcal{Y}) - (H, \mathcal{Y}) = 0,\end{aligned}$$

i.e. $\mathcal{Z} - \mathcal{U} \in \mathcal{R}an(L_{\min})^{\perp}$. According to Lemma 3.5, we get $\mathcal{Z} - \mathcal{U} \in \mathcal{N}ul(L_{\max})$.

From (3.8), we see that

$$J\mathcal{Z}^{[\varrho]}(x) - M(x)\mathcal{Z}(x) = J\mathcal{U}^{[h]}(x) - M(x)\mathcal{U}(x) = N(x)H(x),$$

where $x \in [\omega_0, a]$. Consequently, we conclude that $\mathcal{Z} \in D_{\max}$ and $L_{\max}\mathcal{Z} = H = L^*_{\min}\mathcal{Z}$, due to $\mathcal{Z}, H \in \mathcal{H}$. □

4. Self–adjoint operator

In this section, we introduce self-adjoint Hahn–Hamiltonian problems.

Let $D = \left\{\mathcal{Z} \in D_{\max} : U\widehat{\mathcal{Z}}(\omega_0) + V\widehat{\mathcal{Z}}(a) = 0\right\}$, where U and V are $m \times 2n$ matrices such that $rank(U : V) = m$.

Then we define the *operator* L on D as

$$L\mathcal{Z} = F \Leftrightarrow J\mathcal{Z}^{[\varrho]}(x) - M(x)\,\mathcal{Z}(x) = N(x)\,F(x),$$

where $x \in [\omega_0, a]$.

Assume that

$$\begin{pmatrix} U & V \\ \Phi & \Gamma \end{pmatrix}$$

is a nonsingular matrix, where Φ and Γ are $(4n-m)\times 2n$ matrices such that $rank\,(\Phi : \Gamma) = 4n - m$. Choose $\begin{pmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{\Phi} & \widetilde{\Gamma} \end{pmatrix}$ so that

$$\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} = \begin{pmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{\Phi} & \widetilde{\Gamma} \end{pmatrix}^{*} \begin{pmatrix} U & V \\ \Phi & \Gamma \end{pmatrix}. \tag{4.1}$$

Then we have the following theorem.

Theorem 4.1 *For* $\mathcal{Z}, \mathcal{Y} \in D_{\max}$, *we have*

$$\begin{aligned}(L_{\max}\mathcal{Z}, \mathcal{Y}) - (\mathcal{Z}, L_{\max}\mathcal{Y}) &= \left[\widetilde{U}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{V}\widehat{\mathcal{Y}}(a)\right]^{*}\left[U\widehat{\mathcal{Z}}(\omega_0) + V\widehat{\mathcal{Z}}(a)\right] \\ &+ \left[\widetilde{\Phi}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{\Gamma}\widehat{\mathcal{Y}}(a)\right]^{*}\left[\Phi\widehat{\mathcal{Z}}(\omega_0) + \Gamma\widehat{\mathcal{Z}}(a)\right].\end{aligned}$$

Proof It follows from (3.6) and (4.1) that

$$\begin{aligned}&\widehat{\mathcal{Y}}^{*}(a)J\widehat{\mathcal{Z}}(a) - \widehat{\mathcal{Y}}^{*}(\omega_0)J\widehat{\mathcal{Z}}(\omega_0) \\ &= \left(\widehat{\mathcal{Y}}^{*}(\omega_0), \widehat{\mathcal{Y}}^{*}(a)\right)\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} \widehat{\mathcal{Z}}(\omega_0) \\ \widehat{\mathcal{Z}}(a) \end{pmatrix} \\ &= \left(\widehat{\mathcal{Y}}^{*}(\omega_0), \widehat{\mathcal{Y}}^{*}(a)\right)\begin{pmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{\Phi} & \widetilde{\Gamma} \end{pmatrix}^{*}\begin{pmatrix} U & V \\ \Phi & \Gamma \end{pmatrix}\begin{pmatrix} \widehat{\mathcal{Z}}(\omega_0) \\ \widehat{\mathcal{Z}}(a) \end{pmatrix} \\ &= \left[\begin{pmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{\Phi} & \widetilde{\Gamma} \end{pmatrix}\begin{pmatrix} \widehat{\mathcal{Y}}(\omega_0) \\ \widehat{\mathcal{Y}}(a) \end{pmatrix}\right]^{*}\left[\begin{pmatrix} U & V \\ \Phi & \Gamma \end{pmatrix}\begin{pmatrix} \widehat{\mathcal{Z}}(\omega_0) \\ \widehat{\mathcal{Z}}(a) \end{pmatrix}\right] \\ &= \begin{pmatrix} \widetilde{U}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{V}\widehat{\mathcal{Y}}(a) \\ \widetilde{\Phi}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{\Gamma}\widehat{\mathcal{Y}}(a) \end{pmatrix}^{*}\begin{pmatrix} U\widehat{\mathcal{Z}}(\omega_0) + V\widehat{\mathcal{Z}}(a) \\ \Phi\widehat{\mathcal{Z}}(\omega_0) + \Gamma\widehat{\mathcal{Z}}(a) \end{pmatrix}.\end{aligned}$$

□

Now, we shall study the operator L^{*}.

Theorem 4.2 *Let*

$$D^* = \left\{ \mathcal{Z} \in \mathcal{H} : \begin{array}{c} \mathcal{Z} \text{ is continuous at } \omega_0, \\ J\mathcal{Z}^{[\varrho]}(x) - M(x)\mathcal{Z}(x) = N(x)G(x) \\ \text{exists in } [\omega_0, a],\ G \in \mathcal{H} \\ \text{and } \widetilde{\Phi}\widehat{\mathcal{Z}}(\omega_0) + \widetilde{\Gamma}\widehat{\mathcal{Z}}(a) = 0 \end{array} \right\}.$$

For $\mathcal{Z} \in D^*$,

$$L^*\mathcal{Z} = G \text{ if and only if } J\mathcal{Z}^{[\varrho]}(x) - M(x)\mathcal{Z}(x) = N(x)G(x).$$

Proof It is clear that $L_{\min} \subset L^* \subset L_{\max}$ due to $L_{\min} \subset L \subset L_{\max}$. By Theorem 4.1, we see that

$$(L\mathcal{Z}, \mathcal{Y}) - (\mathcal{Z}, L^*\mathcal{Y}) = \left[\widetilde{U}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{V}\widehat{\mathcal{Y}}(a)\right]^* \left[U\widehat{\mathcal{Z}}(\omega_0) + V\widehat{\mathcal{Z}}(a)\right]$$
$$+ \left[\widetilde{\Phi}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{\Gamma}\widehat{\mathcal{Y}}(a)\right]^* \left[\Phi\widehat{\mathcal{Z}}(\omega_0) + \Gamma\widehat{\mathcal{Z}}(a)\right],$$

where $\mathcal{Z} \in D$ and $\mathcal{Y} \in D^*$. Thus

$$0 = \left[\widetilde{\Phi}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{\Gamma}\widehat{\mathcal{Y}}(a)\right]^* \left[\Phi\widehat{\mathcal{Z}}(\omega_0) + \Gamma\widehat{\mathcal{Z}}(a)\right].$$

This implies that $\widetilde{\Phi}\widehat{\mathcal{Y}}(\omega_0) + \widetilde{\Gamma}\widehat{\mathcal{Y}}(a) = 0$, due to $\Phi\widehat{\mathcal{Z}}(\omega_0) + \Gamma\widehat{\mathcal{Z}}(a)$ is arbitrary.

Conversely, if $\mathcal{Z}$ satisfies the criteria listed above then $\mathcal{Z} \in D^*$. □

Now, we shall give parametric boundary conditions for the sets D and D^*. Using these conditions, we obtain a criterion under which the operator L is self-adjoint.

Note that

$$\begin{pmatrix} U & V \\ \Phi & \Gamma \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{Z}}(\omega_0) \\ \widehat{\mathcal{Z}}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}, \tag{4.2}$$

where F is arbitrary. Multiplying both sides of (4.2) by the following matrices

$$\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{\Phi} & \widetilde{\Gamma} \end{pmatrix}^*,$$

we see that

$$\begin{pmatrix} \widehat{\mathcal{Z}}(\omega_0) \\ \widehat{\mathcal{Z}}(a) \end{pmatrix} = \begin{pmatrix} J\widetilde{\Phi}^*F \\ -J\widetilde{\Gamma}^*F \end{pmatrix}. \tag{4.3}$$

Similarly, we have the following equality

$$\begin{pmatrix} \widehat{\mathcal{Y}}^*(\omega_0) & \widehat{\mathcal{Y}}^*(a) \end{pmatrix} \begin{pmatrix} \widetilde{U} & \widetilde{V} \\ \widetilde{\Phi} & \widetilde{\Gamma} \end{pmatrix}^* = \begin{pmatrix} G^* & 0 \end{pmatrix}, \tag{4.4}$$

where G is arbitrary. Multiplying both sides of (4.4) by

$$\begin{pmatrix} U & V \\ \Phi & \Gamma \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}$$

we deduce that

$$\widehat{\mathcal{Y}}(\omega_0) = -JU^*G,\ \widehat{\mathcal{Y}}(a) = JV^*G. \tag{4.5}$$

Now we have the following.

Theorem 4.3 *L is a self-adjoint operator if and only if*

$$UJU^* = VJV^*$$

and

$$rank\,(U : V) = m = 2n.$$

Proof Let L be a self-adjoint operator. Then $\mathcal{Z}$ satisfies the following boundary condition

$$U\widehat{\mathcal{Z}}\,(\omega_0) + V\widehat{\mathcal{Z}}\,(a) = 0.$$

From (4.5), we obtain

$$U\,(-JU^*G) + V\,(JV^*G) = 0,\ \ [UJU^* - VJV^*]\,G = 0.$$

Hence $UJU^* = VJV^*$, due to G is arbitrary.

Conversely, let $UJU^* = VJV^*$. Then we see that

$$\begin{pmatrix} -UJ & VJ \end{pmatrix}\begin{pmatrix} U^* \\ V^* \end{pmatrix} = 0.$$

It follows from (4.2) and (4.3) that

$$\begin{pmatrix} -UJ & VJ \end{pmatrix}\begin{pmatrix} \widetilde{\Phi}^* \\ \widetilde{\Gamma}^* \end{pmatrix} = 0.$$

Thus, we have

$$\begin{pmatrix} \widetilde{\Phi}^* \\ \widetilde{\Gamma}^* \end{pmatrix}\Theta = \begin{pmatrix} U^* \\ V^* \end{pmatrix},$$

where Θ is a constant, nonsingular matrix. Then, the following boundary conditions are equivalent

$$U\widehat{\mathcal{Y}}\,(\omega_0) + V\widehat{\mathcal{Y}}\,(a) = 0,\ \widetilde{\Phi}\widehat{\mathcal{Y}}\,(\omega_0) + \widetilde{\Gamma}\widehat{\mathcal{Y}}(a) = 0.$$

Since the forms of L and L^* are the same, we reach the assertion of the theorem. □

5. The expansion theorem

In this section, we shall give an expansion theorem.

Assume that $Z\,(x,\lambda)$ is a fundamental matrix for the following system

$$J\mathcal{Z}^{[\varrho]}(x) = [\lambda N\,(x) + M\,(x)]\,\mathcal{Z}\,(x)$$

satisfying $\widehat{Z}\,(\omega_0,\lambda) = I$.

Consider the following nonhomogeneous system

$$J\mathcal{Z}^{[\varrho]}(x) = [\lambda N(x) + M(x)]\, \mathcal{Z}(x) + N(x)\, F(x)\,. \tag{5.1}$$

We can represent the general solution of Eq. (5.1) in the form $\mathcal{Z}(x,\lambda) = Z(x,\lambda)\, C(x,\lambda)\,,$ where $C(x,\lambda)$ is $2n\times 1$ vector-valued function.

By applying the standard method of variation of constants we get

$$\mathcal{Z}(x,\lambda) = Z(x,\lambda)\left[U + V\widehat{Z}(a)\right]^{-1} U \int_{\omega_0}^{x} \left[J\widetilde{Z}(t)\right]^{-1} N(t)\, F(t)\, d_{\omega,q}t$$

$$- Z(x,\lambda)\left[U + V\widehat{Z}(a)\right]^{-1} V\widehat{Z}(a) \int_{x}^{a} \left[J\widetilde{Z}(t)\right]^{-1} N(t)\, F(t)\, d_{\omega,q}t,$$

where

$$\widetilde{Z}(x) = \begin{pmatrix} Z_1(h(x)) \\ Z_2\left(h^{-1}(x)\right) \end{pmatrix}.$$

Putting

$$G(x,t,\lambda)$$

$$= \begin{cases} Z(x,\lambda)\left[U + V\widehat{Z}(a)\right]^{-1} U\left[J\widetilde{Z}(t)\right]^{-1}, & \omega_0 \le t \le x \le a \\ -Z(x,\lambda)\left[U + V\widehat{Z}(a)\right]^{-1} V\widehat{Z}(a)\left[J\widetilde{Z}(t)\right]^{-1}, & \omega_0 \le x \le t \le a, \end{cases}$$

we obtain

$$R(\lambda) = (L - \lambda I)^{-1} F = \mathcal{Z}(x,\lambda) = \int_{\omega_0}^{a} G(x,t,\lambda)\, N(t)\, F(t)\, d_{\omega,q}t. \tag{5.2}$$

Theorem 5.1 *Let $\lambda \notin \mathbb{R}$. Then the operator $R(\lambda)$ defined by (5.2) exists and is a bounded operator. It exists also for all real λ for which* $\det\left[U + V\widehat{\mathcal{Z}}(a)\right] \neq 0$ *as a bounded operator. $\sigma(L)$, the spectrum of L, consists entirely of isolated eigenvalues, zeros of* $\det\left[U + V\widehat{\mathcal{Z}}(a)\right] = 0$.

Proof It is evident that $R(\lambda)$ exists for all real λ except the zeros of $\det\left[U + V\widehat{\mathcal{Z}}(a)\right] = 0$. For all nonreal λ, $R(\lambda)$ exists due to L is a self-adjoint operator. Since $\det\left[U + V\widehat{\mathcal{Z}}(a)\right]$ is analytic in λ and is not identically zero, $\sigma(L)$ consists entirely of isolated eigenvalues, zeros of

$$\det\left[U + V\widehat{\mathcal{Z}}(a)\right] = 0.$$

These zeros can accumulate only at $\pm\infty$.

Let $\mathcal{F}(\eta) = N^{1/2}(\eta)\, F(\eta)$ and $\mathcal{W}(x,\eta,\lambda) = N^{1/2}(\eta)\, G(x,t,\lambda)\, N^{1/2}(x)\,,$ where $N^{1/2}$ is a square root of the matrix N. Then, we see that

$$\left\|(L - \lambda I)^{-1} F\right\|^2 = \|\mathcal{Z}\|^2 = \int_{\omega_0}^{a} \mathcal{Z}^*(x)\, N(x)\, \mathcal{Z}(x)\, d_{\omega,q}x$$

$$= \int_{\omega_0}^{a} \left\{ \begin{array}{c} \left[\int_{\omega_0}^{a} G(x,\eta,\lambda) N(\eta) F(\eta) d_{\omega,q}\eta\right]^{*} \times \\ \times N(x) \left[\int_{\omega_0}^{a} G(x,t,\lambda) N(t) F(t) d_{\omega,q}t\right] \end{array} \right\} d_{\omega,q}x$$

$$= \int_{\omega_0}^{a} \left[\int_{\omega_0}^{a} \mathcal{F}^{*}(\eta) \mathcal{W}^{*}(x,\eta,\lambda) d_{\omega,q}\eta\right] \left[\int_{\omega_0}^{a} \mathcal{W}(x,t,\lambda) \mathcal{F}(t) d_{\omega,q}t\right] d_{\omega,q}x.$$

Using Schwarz's inequality, we conclude that

$$\left\|(L-\lambda I)^{-1} F\right\|^{2} \leq \|\mathcal{W}\|^{2} \|F\|^{2},$$

where

$$\|\mathcal{W}\|^{2} = \int_{\omega_0}^{a} \int_{\omega_0}^{a} \sum_{i=1}^{n} \sum_{j=1}^{n} |\mathcal{W}_{ij}(x,\eta,\lambda)|^{2} d_{\omega,q}\eta d_{\omega,q}x.$$

Theorem is proved. □

There is no loss of generality in assuming that 0 is not an eigenvalue. Then, the solution of the following system

$$J\mathcal{Z}^{[\varrho]}(x) - M(x)\mathcal{Z}(x) = N(x) F(x),$$

$$U\widehat{\mathcal{Z}}(\omega_0) + V\widehat{\mathcal{Z}}(a) = 0,$$

is given by the formula

$$\mathcal{Z}(x) = \int_{\omega_0}^{a} G(x,t) N(t) F(t) d_{\omega,q}t,$$

where $G(x,t) = G(x,t,0)$.

Theorem 5.2 *Let $\mathcal{Z} = \Upsilon F = L^{-1}F$. Then Υ is bounded and*

$$\|\Upsilon\| = \sup\left\{\left|\frac{1}{\lambda_k}\right| : \lambda_k \in \sigma(L)\right\}.$$

Proof Let $L\chi_k = \lambda_k \chi_k$, where $k \in \mathbb{N}$. Then we have

$$\Upsilon\chi_k = \frac{1}{\lambda_k}\chi_k.$$

□

We will order the eigenvalues of Υ,

$$\tau_k = \frac{1}{\lambda_k},$$

such that $|\tau_1| \geq |\tau_2| \geq ... \geq |\tau_k| \geq ...,$ where

$$\lim_{k\to\infty} |\tau_k| = 0. \tag{5.3}$$

Then we have the following.

Theorem 5.3 *Let*

$$\Upsilon_k F = \Upsilon F - \sum_{i=1}^{k-1} \tau_i \chi_i (F, \chi_i).$$

Then we have $\|\Upsilon_k\| = |\tau_k|$, *where* $k \in \mathbb{N}$, *and*

$$\lim_{k\to\infty} \Upsilon_k = 0. \tag{5.4}$$

Proof It is evident that

$$\Upsilon_k \chi_j = \begin{cases} 0, & \text{if } 1 \le j \le k-1 \\ \tau_j \chi_j, & \text{if } k \le j < \infty. \end{cases}$$

Since Υ_k is bounded and self-adjoint, we deduce that

$$\|\Upsilon_k\| = \sup_{\chi\in\mathcal{H}, \|\chi\|=1} |(\Upsilon_k \chi, \chi)| = \sup_{\substack{\chi\in\mathcal{H}, \|\chi\|=1 \\ \chi \neq \chi_1,\dots,\chi_{k-1}}} |(\Upsilon_k \chi, \chi)| = |\tau_k|.$$

From (5.3), we see that

$$\lim_{k\to\infty} \Upsilon_k = 0.$$

□

Theorem 5.4 *For all* $F \in \mathcal{H}$, *we have*

$$F = \sum_{i=1}^{\infty} \chi_i (F, \chi_i),$$

and

$$\Upsilon F = \sum_{i=1}^{\infty} \tau_i \chi_i (F, \chi_i).$$

For all $\mathcal{Z} \in D$, *we have*

$$L\mathcal{Z} = \sum_{i=1}^{\infty} \lambda_i \chi_i (\mathcal{Z}, \chi_i).$$

Proof From (5.4), we obtain

$$\Upsilon F = \sum_{i=1}^{\infty} \tau_i \chi_i (F, \chi_i). \tag{5.5}$$

By (5.5), we see that

$$F = \sum_{i=1}^{\infty} \chi_i (F, \chi_i).$$

Further, $(F, \chi_i) = (L\mathcal{Z}, \chi_i) = (\mathcal{Z}, L\chi_i) = \lambda_i (\mathcal{Z}, \chi_i)$. Therefore, we obtain

$$L\mathcal{Z} = \sum_{i=1}^{\infty} \lambda_i \chi_i (\mathcal{Z}, \chi_i).$$

□

Theorem 5.5 *There exists a collection of projection operators* $\{E(\lambda)\}$ *satisfying*
(a)

$$\lim_{\lambda\to\infty} E(\lambda) = I, \quad \lim_{\lambda\to-\infty} E(\lambda) = 0,$$

(b)

$$E(\lambda_1) \leq E(\lambda_2)$$

when $\lambda_1 \leq \lambda_2$,
(c) $E(\lambda)$ *is continuous from above,*
(d)

$$F = \int_{-\infty}^{\infty} dE(\lambda) F, \ \Upsilon f = \int_{-\infty}^{\infty} \frac{1}{\lambda} dE(\lambda) F,$$

$$L\mathcal{Z} = \int_{-\infty}^{\infty} \lambda dE(\lambda) \mathcal{Z},$$

where $F \in \mathcal{H}$ *and* $\mathcal{Z} \in D$.

Proof Define

$$P_i F = \chi_i (F, \chi_i), \ i \in \mathbb{N},$$

where P_i is a projection operator. If we define

$$E(\lambda) F = \sum_{\lambda_i \leq \lambda} P_i F,$$

then $E(\lambda)$ generates a Stieltjes measure. The integrals in (d) are obtained from this series. □

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