Turkish Journal of Mathematics

Volume 47 | Number 1 Article 20

1-1-2023

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RÜYA ŞEN

NURETTİN CENK TURGAY

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 296 – 316
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doi:10.55730/1300-0098.3361

Biharmonic PNMCV submanifolds in Euclidean 5-space

Rüya ŞEN[1,*], **Nurettin Cenk TURGAY**[2]
[1]Department of Mathematics, Faculty of Engineering and Natural Sciences,
İstanbul Medeniyet University, İstanbul, Turkey
[2]Department of Mathematics,Faculty of Science and Letters, İstanbul Technical University, İstanbul, Turkey

Received: 13.04.2022 • **Accepted/Published Online:** 06.12.2022 • **Final Version:** 13.01.2023

Abstract: In this article, we study 3-dimensional biconservative and biharmonic submanifolds of $\mathbb{E}^5$ with parallel normalized mean curvature vector (PNMCV). First, we prove that the principal curvartures and principal directions of biconservative PNMCV isometric immersions into $\mathbb{E}^5$ can be determined intrinsically. Then, we complete the proof of Chen's biharmonic conjecture for PNMCV submanifolds of $\mathbb{E}^5$.

Key words: Biharmonic submanifolds, biconservative submanifolds, parallel normalized mean curvature vectors

1. Introduction

The study of biharmonic submanifolds was initiated by Chen in the middle of 1980s in his program of understanding finite type submanifolds of Euclidean spaces as well as pseudo-Euclidean spaces [4]. In the mean time, in [12] and [13], Jiang studied biharmonic isometric immersions between Riemannian manifolds by considering the notion of k-harmonic maps proposed by Eells and Sampson in [9].

Chen and Jiang independently showed that there are no biharmonic surfaces in $\mathbb{E}^3$ except the minimal ones. Later, this result was generalized by Dimitric in [8]. In 1991, based on these initial results, Chen claimed that all biharmonic submanifolds of Euclidean spaces are minimal [5]. Although this claim, named as Chen's biharmonic conjecture, was proved to be true in a lot of partial cases (see, for example, [2, 6, 10, 11, 16]), Chen's original problem is still open.

On the other hand, in order to understand the geometrical properties of biharmonic submanifolds, some geometers have shown attention to investigate biconservative submanifolds, [2, 14–16]. For example, the general notion of biconservative submanifolds was introduced in [2]. Also, the complete classification of biconservative hypersurfaces in Euclidean spaces with three distinct principal curvatures is obtained by the second named author in [15].

In [16], authors studied geometrical properties of PNMCV surfaces of $\mathbb{E}^4$ and we also proved that a biharmonic PNMCV surface in $\mathbb{E}^4$ is minimal. Recently, Chen generalized this result into the Euclidean spaces of arbitrary dimension, [6]. In this paper, we study PNMCV isometric immersions from a 3-dimensional Riemannian manifolds into $\mathbb{E}^5$. In Section 2, we give a brief summary of the basic definitions and basic facts of theory of submanifolds. Section 3 is devoted to study some of geometrical properties of biconservative PNMCV

*Correspondence: ruya.yegin@medeniyet.edu.tr
2010 *AMS Mathematics Subject Classification:* 53C42

submanifolds. We obtain our main result in Section 4.

The manifolds that we are dealing with are smooth and connected unless otherwise is stated.

2. Preliminaries

In this section, we would like to give some basic definitions and formulas that we will use in the remaining part of the paper. Moreover, we recall some theorems related with our study.

2.1. Isometric immersions into $\mathbb{E}^5$

Let $\mathbb{E}^n = (\mathbb{R}^n, \tilde{g})$ denote the Euclidean n-space with the metric tensor $\tilde{g}$ given by

$$\tilde{g} = \langle \, .,. \, \rangle = \sum_{i=1}^{n} dx_i^2,$$

where $(x_1, x_2, \ldots, x_n)$ is a Cartesian coordinate system of $\mathbb{E}^n$.

Let $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be an isometric immersion of a 3-dimensional Riemannian manifold (Ω, g) into a Euclidean 5-space $\mathbb{E}^5$. Denote the Levi-Civita connections of Ω and $\mathbb{E}^5$ by ∇ and $\tilde{\nabla}$, respectively. Then the Gauss and Weingarten formulas are given by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X,Y), & (2.1)\\ \widetilde{\nabla}_X \xi &= -A_\xi(X) + \nabla^\perp_X \xi, & (2.2) \end{aligned}$$

respectively, for any vector fields X, Y tangent to Ω and ξ normal to Ω, where h and A_ξ are the second fundamental form and the shape operator of ψ along the normal direction ξ, respectively and $\nabla^\perp$ is the normal connection. Note that h and A_ξ satisfy

$$g(A_\xi(X), Y) = \tilde{g}(h(X,Y), \xi). \tag{2.3}$$

A normal vector field η is called parallel if $\nabla^\perp_X \eta = 0$ whenever X is tangent to Ω. On the other hand, the Ricci tensor Ric and the scalar curvature S of (Ω, g) are defined by

$$\mathrm{Ric}\,(X) = \mathrm{tr}\,\left(R(\cdot, X)\cdot\right) \qquad \text{and} \qquad S = \mathrm{tr}\,(\mathrm{Ric}\,).$$

The mean curvature vector field H of ψ is defined by $H = \frac{1}{3}\mathrm{tr}\, h$ and the mean curvature of ψ is given by $f = \langle H, H\rangle^{1/2}$. ψ is called minimal if f vanishes identically. The covariant derivative $\bar{\nabla} h$ of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any tangent vector fields X, Y and Z on Ω. If R and $\tilde{R}$ stand for the curvature tensor of Ω and $\mathbb{E}^5$, respectively, then, the Gauss equation $\left(\tilde{R}(X,Y)Z\right)^T = 0$ and the Codazzi equation $\left(\tilde{R}(X,Y)Z\right)^\perp = 0$ become

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.4}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z). \tag{2.5}$$

Now, assume that ψ has parallel normalized mean curvature vector e_4. In this case, the Ricci equation $\left(\tilde{R}(X,Y)\xi\right)^T = 0$ yields that all the shape operators of ψ can be diagonalized simultaneously (see [3, Proposition 1.2]). Therefore, by abusing the terminology, we are going to call X as a principal direction of ψ if $A_{e_4}X = kX$, where the smooth function k is going to be called as the corresponding principal curvature. Note that there exists an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that

$$A_{e_4} = \text{diag}\ (k_1, k_2, k_3), \quad A_{e_5} = \text{diag}\ (l_1, l_2, l_3) \tag{2.6}$$

for some smooth functions k_i, l_j satisfying $l_1 + l_2 + l_3 = 0$ and $k_1 + k_2 + k_3 = 3f$.

2.2. Biconservative and biharmonic immersions

In this subsection, we present a summary about biconservative and biharmonic immersions.

A biharmonic map $\psi : (\Omega, g) \to (N, \tilde{g})$ between two Riemannian manifolds is a critical point of the bienergy functional defined by

$$E_2(\psi) = \frac{1}{2}\int_\Omega |\tau(\psi)|^2 v_g \quad ,$$

where ψ is a smooth map, v_g is the volume element of Ω and $\tau(\psi) = \text{tr}\,\nabla d\psi$ is the tension field of ψ. In [13], Jiang obtained the first and second variational formulas for E_2 and proved that ψ is biharmonic if and only if it satisfies the Euler-Lagrange equation associated with bienergy functional given by

$$\tau_2(\psi) = 0, \tag{2.7}$$

where τ_2 is the bitension field of ψ defined by

$$\tau_2(\psi) = \Delta\tau(\psi) - \text{tr}\,\tilde{R}(d\psi, \tau(\psi))d\psi,$$

where Δ is the Rough-Laplacian. On the other hand, a mapping $\psi : (\Omega, g) \to (N, \tilde{g})$ satisfying the condition

$$\langle \tau_2(\psi), d\psi \rangle = 0, \tag{2.8}$$

that is weaker than (2.7) is said to be biconservative. When ψ is an isometric immersion, Equation (2.8) turns into

$$\tau_2(\psi)^T = 0,$$

where $\tau_2(\psi)^T$ denotes the tangential part of $\tau_2(\psi)$. In this case, Ω is said to be a biconservative submanifold of N.

By considering tangential and normal components of $\tau_2(\psi)$ from (2.7), one can obtain the following proposition (see, for example, [14]).

Proposition 2.1 *[14] Let $\psi : (M, g) \hookrightarrow N$ be an isometric immersion between two Riemannian manifolds. Then, ψ is biharmonic if and only if the equations*

$$m\text{grad}\ \|H\|^2 + 4\text{tr}\ A_{\nabla^\perp_\cdot H}(\cdot) + 4\text{tr}\ (\tilde{R}(\cdot, H)\cdot)^T = 0 \tag{2.9}$$

and

$$-\Delta^\perp H + \text{tr}\, h(A_H(\cdot), \cdot) + \text{tr}\, (\tilde{R}(\cdot, H)\cdot)^\perp = 0 \tag{2.10}$$

are satisfied, where m is the dimension of M and $\Delta^\perp$ is the Laplacian associated with $\nabla^\perp$.

By considering Proposition 2.1, one can conclude the following well-known proposition.

Proposition 2.2 *[14] Let $\psi : (M,g) \hookrightarrow N$ be an isometric immersion between two Riemannian manifolds. Then, ψ is biconservative if and only if Equation (2.9) is satisfied.*

The following theorem will be used later.

Theorem 2.3 *[1] Let $\psi : (M,g) \to N$ be a biharmonic map. If ψ is harmonic on an open subset, then it is harmonic everywhere.*

3. Biconservative submanifolds

In this section, we consider biconservative PNMCV isometric immersions into $\mathbb{E}^5$. Let $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be a biconservative PNMCV isometric immersion.

Remark 3.1 *Since the study on biconservative hypersurfaces in $\mathbb{E}^4$ completed in [11], we are going to assume that $\psi(\Omega)$ does not contain any open part lying on a hyperplane of $\mathbb{E}^5$.*

Since the curvature tensor $\tilde{R}$ of $\mathbb{E}^5$ vanishes identically, by using (2.9) one can obtain that ψ is biconservative if and only if

$$A_{e_4}(\operatorname{grad} f) = \frac{-3f}{2}(\operatorname{grad} f), \tag{3.1}$$

where A_{e_4} is the shape operator of Ω along the normalized mean curvature vector e_4 of ψ.

Remark 3.2 *If the mean curvature of ψ is parallel, then (2.9) is satisfied trivally. Furthermore, because of Theorem 2.3 and Equation (2.10), a biharmonic PNMCV immersion must be necessarily harmonic if $\|\operatorname{grad} f\|$ vanishes on an open, nonempty subset of Ω. Therefore, we are going to call a biconservative PNMCV immersion as proper if $\|\operatorname{grad} f\|$ does not vanish.*

Now, assume that ψ is a proper biconservative PNMCV immersion. Then, we have

$$\nabla^{\perp}_{X} e_4 = \nabla^{\perp}_{X} e_5 = 0, \tag{3.2}$$

where e_5 is a unit normal vector field orthogonal to e_4. On the other hand, if e_1 is chosen to be proportional to $\operatorname{grad} f$, then (3.1) implies

$$e_1(f) \neq 0,\ e_2(f) = e_3(f) = 0 \tag{3.3}$$

and $k_1 = -\frac{3f}{2}$. Consequently, the matrix representations of the shape operators of ψ with respect to a suitable frame field $\{e_1, e_2, e_3\}$ takes the form

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \tag{3.4}$$

for some smooth functions k_2, k_3, l_1, l_2, l_3 satisfying

$$k_2 + k_3 = \frac{9f}{2} \text{ and } l_1 + l_2 + l_3 = 0. \tag{3.5}$$

3.1. Two distinct principal curvatures

In this subsection, we focus biconservative PNMCV isometric immersion into $\mathbb{E}^5$ with two distinct principal curvatures. Note that if $k_A = \frac{-3f}{2}$ for $A = 2$ or $A = 3$ on an open subset, then the Codazzi equation (2.5) with $X = Z = e_A$, $Y = e_1$ give $e_1(f) = 0$ which is a contradiction because of (3.3). Therefore, we are going to consider the case $k_1 \neq k_2 = k_3$. First, we consider the shape operators of PNMCV biconservative isometric immersions into $\mathbb{E}^5$.

Lemma 3.3 *Let $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be an isometric immersion with two distinct principal curvatures, where (Ω, g) is a 3-dimensional Riemannian manifold. ψ is proper biconservative PNMCV if and only if there exists an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that*

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & \frac{9f}{4} & 0 \\ 0 & 0 & \frac{9f}{4} \end{pmatrix}, A_{e_5} = \begin{pmatrix} 2c_1 f^{9/5} & 0 & 0 \\ 0 & -c_1 f^{9/5} + f_2 f^{3/5} & 0 \\ 0 & 0 & -c_1 f^{9/5} - f_2 f^{3/5} \end{pmatrix} \tag{3.6}$$

and $\nabla^\perp e_4 = 0$ for some smooth functions f, f_2 and a constant c_1 such that $e_2(f) = e_3(f) = e_1(f_2) = 0$, where f does not vanish.

Proof Let $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be an isometric immersion with two distinct principal curvatures, i.e. k_2, k_3 satisfy

$$k_2 = k_3 = \frac{9f}{4}. \tag{3.7}$$

In order to prove the necessary condition we assume that ψ is proper biconservative and PNMCV. Then, the Codazzi equations $\left(\tilde{R}(e_1, e_A)e_1\right)^\perp = 0$ and $\left(\tilde{R}(e_1, e_A)e_A\right)^\perp = 0$ give

$$\omega_{1A}(e_1) = 0, \quad e_A(l_1) = 0, \tag{3.8a}$$

$$\omega_{1A}(e_2) = \frac{-3}{5}\frac{e_1(f)}{f} \quad \text{and} \tag{3.8b}$$

$$e_1(l_A) = \frac{3e_1(f)}{5f}(l_A - l_1), A = 2, 3, \tag{3.8c}$$

respectively. By considering (3.5), we obtain

$$l_1 = 2c_1 f^{9/5} \tag{3.9}$$

from (3.8c), for a smooth function c_1 satisfying $e_1(c_1) = 0$. By taking into account (3.8a), we get $e_2(c_1) = e_3(c_1) = 0$ which yields that c_1 is a constant. Moreover, from (3.8c) and (3.9) we obtain

$$l_2 = -c_1 f^{9/5} + f_2 f^{9/5} \tag{3.10}$$

for a smooth function f_2 satisfying $e_1(f_2) = 0$. Consequently, (3.5) implies

$$l_3 = -c_1 f^{9/5} - f_2 f^{9/5}. \tag{3.11}$$

By combining (3.4) with (3.7) and (3.9)-(3.11), we obtain (3.6). This completes the proof of the necessary condition. The converse of the lemma is trivial.

Remark 3.4 *One can observe that a frame field $\{e_1, e_2, e_3; e_4, e_5\}$ satisfying the conditions in Lemma 3.3 can be globally constructed if $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ is a proper biconservative PNMCV isometric immersion with two distinct principal curvatures because*

$$e_1 = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}, \qquad e_4 = \frac{H}{f}$$

and e_2, e_3 can be constructed to be an eigenvalue of $A_{e_5}|_D$ at every point of Ω, where

$$D = (\operatorname{span}\{e_1\})^T.$$

Before we continue, we would like to obtain the following result of Lemma 3.3.

Lemma 3.5 *Let $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be a proper biconservative PNMCV isometric immersion with two distinct principal curvatures and put $e_1 = \frac{gradf}{\|gradf\|}$, where f is the mean curvature of ψ. Then,*

(a) An integral curve of e_1 lies on a 3-plane of $\mathbb{E}^5$.

(b) The curvature κ and torsion τ of an integral curve of e_1 satisfy

$$\kappa = f\sqrt{\frac{9}{4} + 4c_1^2 f^{8/5}}, \tag{3.12a}$$

$$\tau = \frac{12}{5}\left(\frac{c_1\|\operatorname{grad} f\| f^{-1/5}}{\frac{9}{4} + 4c_1^2 f^{8/5}}\right). \tag{3.12b}$$

Proof Let $\{e_1, e_2, e_3; e_4, e_5\}$ be an orthonormal frame field on Ω satisfying the properties given in Lemma 3.3 and we suppose that γ is an integral curve of e_1 and it is parametrized by $\gamma(s) = x(s, t_0)$. Consider the Frenet frame $\{T(s), N(s), B_1(s), B_2(s), B_3(s)\}$ at a point $\gamma(s)$, where we put $T(s) = \gamma'(s)$. Note that we have

$$\begin{aligned}
\frac{DT}{ds} &= \kappa_1 N(s),\\
\frac{DN}{ds} &= -\kappa_1 T(s) + \kappa_2(s) B_1(s),\\
\frac{DB_1}{ds} &= -\kappa_2 N(s) + \kappa_3 B_2(s),\\
\frac{DB_2}{ds} &= -\kappa_3 B_1(s) + \kappa_4 B_3(s),\\
\frac{DB_3}{ds} &= -\kappa_4 B_2(s),
\end{aligned}$$

where $\frac{D}{ds}$ denotes the covariant derivative on γ and $\kappa_i(s)$, $i = 1, 2, 3$ is the i-th curvature of γ.

By considering (3.6) with the Gauss formula, we obtain

$$\frac{DT}{ds} = -\frac{3f(s)}{2} e_4(s) + 2c_1 f(s)^{9/5} e_5(s), \tag{3.13}$$

where $e_4(s), e_5(s)$ are restrictions of e_4, e_5 to γ from which we get

$$\kappa_1 = f(s)\sqrt{\frac{9}{4} + 4c_1^2 f(s)^{8/5}} \tag{3.14}$$

and

$$N(s) = \frac{1}{\sqrt{\frac{9}{4} + 4c_1^2 f(s)^{8/5}}} \left(\frac{-3}{2} e_4(s) + 2c_1 f(s)^{4/5} e_5(s) \right). \tag{3.15}$$

By a further computation using (3.3) and (3.15), we get

$$\begin{aligned} \frac{DN}{ds} =& \frac{1}{\left(\frac{9}{4} + 4c_1^2 f(s)^{8/5}\right)^{3/2}} \left(\frac{24}{5} c_1^2 f(s)^{3/5} \|\operatorname{grad} f\| e_4(s) + \frac{18}{5} c_1 f(s)^{-1/5} \|\operatorname{grad} f\| e_5(s) \right) \\ & - f(s)\sqrt{\frac{9}{4} + 4c_1^2 f(s)^{8/5}} e_1(s). \end{aligned}$$

Therefore, we have

$$\kappa_2 = \frac{12}{5} \left(\frac{c_1 \|\operatorname{grad} f\| f(s)^{-1/5}}{\frac{9}{4} + 4c_1^2 f(s)^{8/5}} \right) \tag{3.16}$$

and

$$B(s) = \frac{1}{\sqrt{\frac{9}{4} + 4c_1^2 f(s)^{8/5}}} \left(2c_1 f(s)^{4/5} e_4(s) + \frac{3}{2} e_5(s) \right). \tag{3.17}$$

Next, we compute $\frac{DB}{ds}$ and get $\kappa_3 = 0$ which yields that γ lies on a 3-plane of $\mathbb{E}^5$. Moreover, $\kappa = \kappa_1$ is the curvature and $\tau = \kappa_2$ is the torsion of γ.

Next, by using the Lemma 3.3, we obtain the following characterization of proper biconservative PNMCV immersions.

Proposition 3.6 *Let Ω be a 3-dimensional submanifold of $\mathbb{E}^5$ and $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be an isometric immersion with two distinct principal curvatures. Then, ψ is proper biconservative PNMCV if and only if it is one of the following two classes of isometric immersions.*

Case I. An isometric immersion ψ_1 which has an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & \frac{9f}{4} & 0 \\ 0 & 0 & \frac{9f}{4} \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_2 f^{3/5} & 0 \\ 0 & 0 & -f_2 f^{3/5} \end{pmatrix}, \nabla^\perp e_4 = 0 \tag{3.18}$$

and

$$\begin{aligned} &\omega_{12}(e_1) = \omega_{13}(e_1) = \omega_{12}(e_3) = \omega_{13}(e_2) = \omega_{23}(e_1) = 0, \\ &\omega_{12}(e_2) = \omega_{13}(e_3) = -\frac{3}{5}\frac{e_1(f)}{f}, \\ &\omega_{23}(e_2) = \frac{1}{2}\frac{e_3(f_2)}{f_2}, \quad \omega_{23}(e_3) = -\frac{1}{2}\frac{e_2(f_2)}{f_2} \end{aligned} \tag{3.19}$$

for some smooth functions f, f_2 satisfying $e_2(f) = e_3(f) = e_1(f_2) = 0$, where f does not vanish.

Case II. An isometric immersion ψ_2 *which has an orthonormal* $\{e_1, e_2, e_3; e_4, e_5\}$ *such that*

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & \frac{9f}{4} & 0 \\ 0 & 0 & \frac{9f}{4} \end{pmatrix}, A_{e_5} = \begin{pmatrix} 2c_1 f^{9/5} & 0 & 0 \\ 0 & -c_1 f^{9/5} & 0 \\ 0 & 0 & -c_1 f^{9/5} \end{pmatrix}, \nabla^\perp e_4 = 0 \tag{3.20}$$

and

$$\begin{aligned} &\omega_{12}(e_1) = \omega_{13}(e_1) = \omega_{12}(e_3) = \omega_{13}(e_2) = \omega_{23}(e_2) = \omega_{23}(e_3) = 0, \\ &\omega_{12}(e_2) = \omega_{13}(e_3) = -\frac{3e_1(f)}{5f} \end{aligned} \tag{3.21}$$

for a smooth nonvanishing function f *satisfying* $e_2(f) = e_3(f) = 0$*.*

Proof Assume that ψ is proper biconservative PNMCV. Because of Lemma 3.3, the shape operators of ψ satisfies (3.6) for a constant c_1 and some smooth functions f, f_2 such that $e_2(f) = e_3(f) = e_1(f_2) = 0$. Note that the Codazzi equations $\left(\tilde{R}(e_1, e_2)e_3\right)^\perp = \left(\tilde{R}(e_2, e_3)e_1\right)^\perp = 0$ imply

$$\omega_{13}(e_2) = \omega_{12}(e_3) = 0$$

and

$$f_2 \omega_{23}(e_1) = 0. \tag{3.22}$$

First, we are going to prove the following claim.

Claim 3.7 *If* grad $f_2 = 0$ *on an open, nonempty set* $\mathcal{O}$*, then* $f_2 = 0$ *on* $\mathcal{O}$ *and* $c_1 \neq 0$*.*

Proof of Claim 3.7. Assume that $f_2 = c_2$ on $\mathcal{O}$ for a constant c_2 and toward contradiction assume that $c_2 \neq 0$. Then, on $\mathcal{O}$ we have $\omega_{23}(e_1) = 0$ which implies

$$R(e_1, e_2, e_2, e_1) = R(e_1, e_3, e_3, e_1) = -e_1(\alpha) - \alpha^2 \tag{3.23}$$

because of (3.22), where $\alpha = \omega_{12}(e_2) = \omega_{13}(e_3)$. By combining the Gauss equation (2.4) and (3.23), we get

$$\langle h(e_1, e_1), h(e_2, e_2) - h(e_3, e_3) \rangle = 0$$

which implies $c_1 c_2 f^{12/5} = 0$ because of (2.3) and (3.6). Therefore, we have $c_1 = 0$. In view of the equation of Gauss for $X = Z = e_2$, $Y = e_3$, we obtain $R(e_3, e_2, e_2, e_3) = \frac{81}{16} f^2 - c_2^2 f^{6/5}$ which gives

$$\alpha^2 = c_2^2 f^{6/5} - \frac{81}{16} f^2.$$

By applying e_1 to this equation, we obtain

$$e_1(\alpha) = \frac{1}{2\alpha} \left(c_2^2 \frac{6}{5} f^{1/5} - \frac{81}{8} f \right) e_1(f). \tag{3.24}$$

By combining (3.23) and (3.24), we get

$$\left(c_2^2 \frac{6}{5} f^{1/5} - \frac{81}{8} f \right) e_1(f) = 2\alpha \left(\frac{27}{16} f^2 - c_2^2 f^{6/5} \right). \tag{3.25}$$

Using (3.8b), Equation (3.25) reduces to $f = 0$ which yields a contradiction. Therefore, we have $c_2 = 0$. Since $f(\Omega)$ does not contain any open part lying on a hyperplane, we have $c_1 \neq 0$.

Hence, the proof of the Claim 3.7 is completed. ■

Now, we are going to consider the cases $\operatorname{grad} f_2 = 0$ and $\operatorname{grad} f_2 \neq 0$, separately.

Case I. $\operatorname{grad} f_2 = 0$ on Ω. In this case, Claim 3.7 directly implies $f_2 = 0$ on Ω. Consequently, (3.6) turns into (3.20) on $\mathcal{O}$. A further consideration of Codazzi equations imply (3.21). Hence, we have the Case II of the proposition.

Case II. $\operatorname{grad} f_2 \neq 0$ at a point of Ω. In this case the open subset

$$\mathcal{O} = \{q \in \Omega | (\operatorname{grad} f_2)(q) \neq 0\}$$

of Ω is not empty and we have either $e_2(f_2) \neq 0$ or $e_3(f_2) \neq 0$. Assume that $e_2(f_2) \neq 0$. In this case, the open set $\mathcal{O}_2 = \{q \in \mathcal{O} | f_2(q) \neq 0\}$ is not empty and (3.22) implies $\omega_{23}(e_1) = 0$ on $\mathcal{O}_2$. By considering (3.6) and (3.8a), we see that the Gauss equation $\left(\tilde{R}(e_1, e_2, e_1, e_2)\right)^T = 0$ gives

$$e_1(\omega_{12}(e_2)) - \omega_{12}([e_1, e_2]) = \frac{27f^2}{8} - 2c_1 f^{9/5}(-c_1 f^{9/5} + f_2 f^{3/5}). \tag{3.26}$$

Now, by taking the derivative of Equation (3.26) with respect to e_2 we obtain

$$-2c_1 f^{12/5} e_2(f_2) = 0$$

which implies $c_1 = 0$ because of the assumptions. Consequently, (3.6) turns into (3.18). Hence, we have $\psi = \psi_1$ on $\mathcal{O}$, where ψ_1 is the isometric immersion described in Case I of the proposition.

On the other hand, since $c_1 = 0$, Claim 3.7 implies that $\Omega - \mathcal{O}$ has empty interior because of Remark 3.4. By the continuity of ψ, we have $\psi = \psi_1$ on Ω which yields the Case I of the proposition.

The proof of the converse follows from Lemma 3.3.

Next, we obtain that the mean curvature of a proper biconservative PNMCV immersion can be computed intrinsically as well as the other quantities appearing in the shape operators given by (3.18) and (3.20).

Theorem 3.8 *Let Ω be a 3-dimensional submanifold of $\mathbb{E}^5$ and $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be a proper biconservative PNMCV isometric immersion with two distinct principal curvatures. Then, the vector field e_1 and the quantities f^2, c_1^2, f_2^2 appearing in Proposition 3.6 can be computed intrinsically.*

Proof First, we assume that (Ω, g) admits the biconservative PNMCV isometric immersion ψ_1 described in Case I of Proposition 3.6 for some smooth functions f, f_2. Then, by combining (3.18) with (2.3) we obtain

$$h(e_1, e_1) = -\frac{3f}{2} e_4, \quad h(e_2, e_2) = \frac{9f}{4} e_4 + f_2 f^{3/5} e_5,$$
$$h(e_3, e_3) = \frac{9f}{4} e_4 - f_2 f^{3/5} e_5.$$

After a direct computation by considering the Gauss equation (2.4), we get

$$\begin{aligned} R(e_1, e_2, e_2, e_1) &= R(e_3, e_2, e_2, e_3) = -\frac{27f^2}{8}, \\ R(e_2, e_3, e_3, e_2) &= \frac{81f^2}{16} - f_2^2 f^{6/5}. \end{aligned} \tag{3.27}$$

Consequently, the Ricci tensor Ric of (Ω, g) satisfies $\mathrm{Ric}\,(e_i) = \lambda_i e_i$ for the functions

$$\lambda_1 = -\frac{27f^2}{4}, \lambda_2 = \lambda_3 = \frac{27f^2}{16} - f_2^2 f^{6/5}.$$

Hence, f^2 and f_2^2 can be computed in terms of eigenvalues of Ric and

$$e_1 = \frac{\nabla \lambda_1}{\|\nabla \lambda_1\|}.$$

On the other hand, if (Ω, g) admits the biconservative PNMCV isometric immersion ψ_2 described in Case II of Proposition 3.6, then by a similar way we obtain the eigenvalues of Ric by

$$\lambda_1 = -\left(\frac{27f^2}{4} + 4c_1^2 f^{18/5}\right), \lambda_2 = \lambda_3 = \frac{27f^2}{16} - c_1^2 f^{18/5}$$

and the scalar curvature of (Ω, g) is

$$S = -\left(\frac{27f^2}{16} + 3c_1^2 f^{18/5}\right).$$

Therefore, f^2 and c_1^2 can be computed in terms of λ_1, λ_2. Moreover, we have either $e_1|_p = \frac{(\nabla\lambda_1)_p}{\|(\nabla\lambda_1)_p\|}$ or $e_1|_p = \frac{(\nabla S)_p}{\|(\nabla S)_p\|}$ at a point $p \in \Omega$, because a direct computation yields that

$$e_1(\lambda_1)^2 + e_1(S)^2 \neq 0.$$

By considering the proof of Theorem 3.8 we have the following result.

Corollary 3.9 *Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of the Ricci tensor of a 3-dimensional Riemannian manifold (Ω, g) which admits a proper biconservative PNMCV isometric immersion ψ into $\mathbb{E}^5$ with two distinct principal curvatures. If*

$$\dim\left(\mathrm{span}\left\{\nabla\lambda_1, \nabla\lambda_2, \nabla\lambda_3\right\}\right) = 1,$$

then $\psi = \psi_2$ and otherwise $\psi = \psi_1$, where ψ_1, ψ_2 are the isometric immersions described in Proposition 3.6.

3.2. Biconservative immersions with three distinct principal curvatures

We first want to focus on the PNMCV biconservative isometric immersions with three distinct eigenvalues. Therefore, we assume that

$$k_2 \neq \frac{9f}{4}, \qquad k_3 \neq \frac{9f}{4}. \tag{3.28}$$

By considering Codazzi equations and $[e_2, e_3](k_1) = 0$ similar to the computations in [11], we see that the Levi-Civita connection of Ω satisfies

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\\
&\nabla_{e_2} e_1 = \omega e_2, \nabla_{e_2} e_2 = -\omega e_1 + \omega_{23}(e_2) e_3, \nabla_{e_2} e_3 = -\omega_{23}(e_2) e_2,\\
&\nabla_{e_3} e_1 = \gamma e_3, \nabla_{e_3} e_2 = \omega_{23}(e_3) e_3, \nabla_{e_3} e_3 = -\gamma e_1 - \omega_{23}(e_3) e_2,
\end{aligned}$$

where we put $\omega = \omega_{12}(e_2)$ and $\gamma = \omega_{13}(e_3)$. Moreover, we have

$$e_1(k_2) = \omega(k_1 - k_2), \qquad e_1(k_3) = \gamma(k_1 - k_3), \tag{3.29a}$$

$$e_1(l_2) = \omega(l_1 - l_2), \qquad e_1(l_3) = \gamma(l_1 - l_3), \tag{3.29b}$$

$$e_1(\omega) = -\omega^2 + \frac{3f}{2}k_2 - l_1 l_2, \tag{3.29c}$$

$$e_1(\gamma) = -\gamma^2 + \frac{3f}{2}k_3 - l_1 l_3. \tag{3.29d}$$

Now, we are ready to prove the following result.

Proposition 3.10 *Let $\psi : (\Omega, g) \hookrightarrow \mathbb{E}^5$ be a proper biconservative immersion satisfying $k_2 \neq k_3$ and assume that $\psi(\Omega)$ does not contain any open part lying on a hyperplane of $\mathbb{E}^5$. Then, k_i, l_i, ω and γ satisfies*

$$X(k_i) = X(l_i) = X(\omega) = X(\gamma) = 0 \qquad \textit{whenever } g(X, e_1) = 0, \quad i = 2, 3. \tag{3.30}$$

Proof Let ψ be a proper biconservative immersion, X be a tangent vector field such that $g(X, e_1) = 0$.

First, we apply e_1 on equations in (3.5) and combine the obtained equations with (3.29) and (3.5), we have

$$\gamma\,(-3f - 2k_3) + \omega\,(-3f - 2k_2) \;=\; 9e_1(f), \tag{3.31}$$

$$(l_1 - l_3)\,\gamma + (l_1 - l_2)\,\omega \;=\; -e_1(l_1). \tag{3.32}$$

We apply e_1 on these equations and consider (3.29), we obtain

$$\begin{aligned} &-6\gamma e_1(f) + \gamma^2\,(12f + 8k_3) + \omega^2\,(12f + 8k_2) - 9f^2\,(k_2 + k_3) - 6\omega e_1(f) \\ &\qquad -6f{k_2}^2 - 6f{k_3}^2 + 6fl_1\,(l_2 + l_3) + 4l_1\,(k_2 l_2 + k_3 l_3) = 18e_1^2(f), \end{aligned} \tag{3.33}$$

$$\begin{aligned} &3fk_2 l_1 - 3fk_2 l_2 - l_3\left(3fk_3 + 2{l_1}^2\right) + 2\gamma e_1(l_1) + 4\,(l_3 - l_1)\,\gamma^2 + 3fk_3 l_1 \\ &\qquad +2\omega e_1(l_1) + 4\,(l_2 - l_1)\,\omega^2 + 2l_1{l_2}^2 + 2l_1{l_3}^2 - 2{l_1}^2 l_2 = -\,2e_1^2(l_1) \end{aligned} \tag{3.34}$$

and a further computation give the equations

$$\begin{aligned} &+36\omega^2 e_1(f) - \gamma^3\,(72f + 48k_3) - 12{k_2}^2 e_1(f) - 12{k_3}^2 e_1(f) - 54fk_2 e_1(f) \\ &\gamma\left(-12e_1^2(f) + 126f^2 k_3 + 72f{k_3}^2 + 12f{l_1}^2 - 72fl_1 l_3 + 27f^3 + 8k_3{l_1}^2 - 48k_3 l_1 l_3\right) \\ &+\omega\left(-12e_1^2(f) + 126f^2 k_2 + 72f{k_2}^2 + 12f{l_1}^2 - 72fl_1 l_2 + 27f^3 + 8k_2{l_1}^2 - 48k_2 l_1 l_2\right) \\ &-54fk_3 e_1(f) + 24l_1 l_2 e_1(f) + 24l_1 l_3 e_1(f) + 36\gamma^2 e_1(f) + \omega^3\,(-72f - 48k_2) \\ &\qquad +12fl_2 e_1(l_1) + 12fl_3 e_1(l_1) + 8k_2 l_2 e_1(l_1) + 8k_3 l_3 e_1(l_1) = 36e_1^3(f), \end{aligned} \tag{3.35}$$

$$
\begin{aligned}
&+\left(24l_1-24l_3\right)\gamma^3+\left(24l_1-24l_2\right)\omega^3-12\gamma^2e_1(l_1)-12\omega^2e_1(l_1)\\
&+\gamma\left(-36fk_3l_1+36fk_3l_3-9f^2l_1+9f^2l_3+4e_1^2(l_1)-4{l_1}^3+28l_3{l_1}^2-24{l_3}^2l_1\right)\\
&+\omega\left(-36fk_2l_1+36fk_2l_2-9f^2l_1+9f^2l_2+4e_1^2(l_1)-4{l_1}^3+28l_2{l_1}^2-24{l_2}^2l_1\right)\\
&+12fk_2e_1(l_1)+12fk_3e_1(l_1)+4{l_2}^2e_1(l_1)+4{l_3}^2e_1(l_1)-6k_2l_2e_1(f)-6k_3l_3e_1(f)\\
&+6k_2l_1e_1(f)+6k_3l_1e_1(f)-12l_1l_2e_1(l_1)-12l_1l_3e_1(l_1)=-e_1^3(l_1),
\end{aligned}
\tag{3.36}
$$

where we use the notation $e_1^2(\psi)=e_1e_1(\psi)$ and $e_1^3(\psi)=e_1e_1e_1(\psi)$ for a $\psi\in C^\infty(\Omega)$.

Note that by combining (3.5) with (3.31) and (3.32), we get

$$
B\begin{pmatrix}\omega\\ \gamma\end{pmatrix}=\begin{pmatrix}-\frac{9e_1(f)}{2}\\ -e_1(l_1)\end{pmatrix},\quad B=\begin{pmatrix}\frac{3f}{2}+k_2 & 6f-k_2\\ l_1-l_2 & 2l_1+l_2\end{pmatrix}. \tag{3.37}
$$

Therefore, we have two cases: $\det B=0$ on Ω and $\det B\neq 0$ on an open subset of Ω.

Case I. $\det B=0$ on Ω. In this case, we have

$$
-2fl_1+5l_2f+2k_2l_1=0. \tag{3.38}
$$

By applying e_1 to (3.38), we get

$$
2k_2e_1(l_1)+5l_2e_1(f)=2e_1(fl_1). \tag{3.39}
$$

By combining (3.38) and (3.39) we get

$$
C\begin{pmatrix}k_2\\ l_2\end{pmatrix}=\begin{pmatrix}2fl_1\\ e_1(2fl_1)\end{pmatrix}, \tag{3.40}
$$

where we put $C=\begin{pmatrix}2l_1 & 5f\\ 2e_1(l_1) & 5e_1(f)\end{pmatrix}$. If $\det C\neq 0$, then (3.40) implies

$$
\begin{aligned}
k_2&=\eta_1(f,l_1,e_1(f),e_1(l_1)),\\
l_2&=\eta_2(f,l_1,e_1(f),e_1(l_1))
\end{aligned}
$$

for some smooth functions η_1,η_2. In this case, we have $e_A(k_2)=e_A(l_2)=0$ for $A=2,3$ which completes the proof for this subcase.

Now, we consider the case $\det C=0$ which is equivalent to

$$
l_1=cf \tag{3.41}
$$

for a constant c. Substituting this equation in (3.39), we get

$$
2cf=5l_2+2ck_2. \tag{3.42}
$$

Note that if $c=0$, then (3.5), (3.41) and (3.42) imply $l_1=l_2=l_3=0$ which gives $A_{e_5}=0$. In this case, we have $\widetilde{\nabla}e_5=0$ which yields that $\psi(\Omega)$ lies on a hyperplane of $\mathbb{E}^5$ which is not possible. Therefore, we have $c\neq 0$. However, by combining (3.41) and (3.42), we get

$$
\frac{4c}{5}e_1(f)=0 \tag{3.43}
$$

which is a contradiction.

Case II. $\det B \neq 0$ on an open subset $\mathcal{O}$ of Ω. In this case, from (3.37) we get

$$\begin{aligned}
\omega &= \frac{9\,(2l_1+l_2)\,e_1(f)+2\,(k_2-6f)\,e_1(l_1)}{6fl_1-15fl_2-6k_2l_1}, && \text{(3.44a)}\\
\gamma &= \frac{9\,(l_2-l_1)\,e_1(f)+(3f+2k_2)\,e_1(l_1)}{6fl_1-15fl_2-6k_2l_1} && \text{(3.44b)}
\end{aligned}$$

on $\mathcal{O}$.

By considering (3.5) and (3.44) we see that (3.33) and (3.34) turn into

$$\begin{aligned}
&75f^2l_1\,(4k_2-9f)\,l_2^3-15f\Big(f\left(45e_1^2(f)+42k_2l_1^2\right)-8\left(9e_1(f)^2+2k_2^2l_1^2\right)\\
&+6f^2\left(5k_2^2+4l_1^2\right)-135f^3k_2+405f^4\Big)l_2^2+3\Big(4k_2l_1\left(63e_1(f)^2+4k_2^2l_1^2\right)\\
&-3f^2\big(4l_1\left(-15e_1^2(f)+16k_2l_1^2+10k_2^3\right)+15e_1(f)e_1(l_1)\big)-f\big(207l_1e_1(f)^2\\
&-20k_2\left(e_1(f)e_1(l_1)-9l_1e_1^2(f)\right)+28k_2^2l_1^3\big)-2160f^4k_2l_1+1620f^5l_1\\
&+12f^3\left(55k_2^2l_1+17l_1^3\right)\Big)l_2+6f^2\big(-9l_1\left(2l_1e_1^2(f)+23e_1(f)e_1(l_1)\right)+78k_2^3l_1^2 \qquad \text{(3.45)}\\
&+k_2\left(30e_1(l_1)^2+52l_1^4\right)\big)-2f\Big(+3k_2l_1\left(121e_1(f)e_1(l_1)-36l_1e_1^2(f)\right)\\
&+2268f^4k_2l_1^2-972f^5l_1^2+4k_2^2\left(5e_1(l_1)^2+24l_1^4\right)-621l_1^2e_1(f)^2+36k_2^4l_1^2\Big)\\
&-144f^3l_1^4+6k_2l_1\left(63l_1e_1(f)^2+2k_2\left(14e_1(f)e_1(l_1)-9l_1e_1^2(f)\right)+4k_2^2l_1^3\right)\\
&-36f^3\left(47k_2^2l_1^2-10e_1(l_1)^2\right)=0
\end{aligned}$$

and

$$\begin{aligned}
&-72fl_1^2\,(l_1+2l_2)\,k_2^3+4l_1\Big(9f^2\left(22l_1^2+7l_2l_1-20l_2^2\right)-16e_1(l_1)^2\\
&+4\left(3l_1\left(e_1^2(l_1)+2l_1\left(l_1^2+l_2l_1+l_2^2\right)\right)\right)\Big)k_2^2-6\Big(12l_1\,(l_1+2l_2)\,e_1(f)e_1(l_1)\\
&+4f\big(-l_1\left(7e_1(l_1)^2+10l_2e_1^2(l_1)\right)+10l_2e_1(l_1)^2+4l_1^2\left(e_1^2(l_1)-5l_2^3\right)\\
&+8l_1^5-12l_2l_1^4-12l_2^2l_1^3\big)+3f^3\,(2l_1-5l_2)\left(38l_1^2+17l_2l_1-10l_2^2\right)\Big)k_2 \qquad \text{(3.46)}\\
&+108f\left(14l_1^2-7l_2l_1-10l_2^2\right)e_1(f)e_1(l_1)-648l_1\,(l_1-l_2)\,(2l_1+l_2)\,e_1(f)^2\\
&+12f^2\big(-42l_1e_1(l_1)^2+45l_2e_1(l_1)^2+(2l_1-5l_2)^2\,e_1^2(l_1)+8l_1^5-32l_2l_1^4\\
&+18l_2^2l_1^3+10l_2^3l_1^2+50l_2^4l_1\big)+81f^4\,(2l_1+l_2)\,(2l_1-5l_2)^2=0,
\end{aligned}$$

respectively.

On the other hand, by combining (3.29c) and (3.29d) with (3.44a), (3.44b), we get

$$\begin{aligned}
-54f{l_1}^2k_2^3 &+ \left(6l_1\left(9f^2\left(2l_1-5l_2\right)-2e_1^2(l_1)+6{l_1}^2l_2\right)+16e_1(l_1)^2\right)k_2^2\\
&+\Big(-54l_1\left(2l_1+l_2\right)e_1^2(f)+6\left(31l_1+20l_2\right)e_1(f)e_1(l_1)-\frac{27}{2}f^3\left(2l_1-5l_2\right)^2\\
&+6f\left(-27e_1(l_1)^2+\left(14l_1-5l_2\right)e_1^2(l_1)+6l_2\left(5l_2-2l_1\right){l_1}^2\right)\Big)k_2\\
&+9\Big(44f^2e_1(l_1)^2+3\left(2l_1+l_2\right)\left(13l_1+8l_2\right)e_1(f)^2-4f^2e_1^2(l_1)\\
&+3f\left(\left(2l_1-5l_2\right)\left(2l_1+l_2\right)e_1^2(f)-\left(38l_1+25l_2\right)e_1(f)e_1(l_1)\right)\\
&+f^2l_1l_2\left(2l_1-5l_2\right)\left(2l_1-5l_2\right)\Big)=0
\end{aligned} \tag{3.47a}$$

and

$$\begin{aligned}
54f{l_1}^2k_2^3 &+ \Big(16e_1(l_1)^2-3l_1\big(9f^2\left(13l_1-10l_2\right)+4e_1^2(l_1)+12\left(l_1+l_2\right){l_1}^2\big)\Big)k_2^2\\
&+\Big(54l_1\left(l_1-l_2\right)e_1^2(f)+6\left(20l_2-11l_1\right)e_1(f)e_1(l_1)-6f\left(l_1+5l_2\right)e_1^2(l_1)\\
&+18fe_1(l_1)^2+36f\left(l_1+l_2\right)\left(2l_1-5l_2\right){l_1}^2+\frac{135}{2}f^3\left(4l_1-l_2\right)\left(2l_1-5l_2\right)\Big)k_2\\
&+27\left(l_1-l_2\right)\left(5l_1-8l_2\right)e_1(f)^2-27f\left(2l_1-5l_2\right)\left(\left(l_1-l_2\right)e_1^2(f)+e_1(f)e_1(l_1)\right)\\
&-9f^2\left(e_1(l_1)^2+\left(2l_1-5l_2\right)\left(l_1\left(2l_1-5l_2\right)\left(l_1+l_2\right)-e_1^2(l_1)\right)\right)\\
&-\frac{243}{4}f^4\left(2l_1-5l_2\right)^2=0.
\end{aligned} \tag{3.47b}$$

First, we are going to prove the following claims.

Claim 3.11 *The interior of the subset $E=\{p\in\mathcal{O}|l_1(p)=0\}$ is empty.*

Proof of Claim 3.11. Assume that $l_1=0$ on an open subset $\mathcal{O}_2$ of $\mathcal{O}$. Then, (3.46) turns into

$$f^3{l_2}^3\left(9f-4k_2\right)=0$$

from which we get $l_2=0$. Therefore, we have $l_1=l_2=l_3=0$ on $\mathcal{O}_2$ which yields that $\psi(\mathcal{O}_2)$ is contained on a hyperplane of $\mathbb{E}^5$ which is a contradiction if $\mathcal{O}$ is not empty.

Hence, the proof of the Claim 3.11 is completed. ■

Next, we prove the following claim.

Claim 3.12 *$X(k_2)=0$ on $\mathcal{O}$ if and only if $X(l_2)=0$ on $\mathcal{O}$.*

Proof of Claim 3.12. Assume that $X(k_2)=0$ and $X(l_2)\neq 0$ at a point $p\in\mathcal{O}$. Then, the third degree polynomial of l_2 appearing in left hand-side of (3.45) is a trivial polynomial. Therefore, we have $f^2l_1\left(4k_2-9f\right)=0$ which is not possible because of (3.28) and Claim 3.11.

Conversely, if $X(l_2)=0$ and $X(k_2)\neq 0$ at a point $q\in\mathcal{O}$, then we have $fl_1^2=0$ from the coefficient of k_2^4 in the left hand-side of (3.45). However, this is a contradiction.

Hence, the proof of the Claim 3.12 is completed. ■

Now, towards contradiction assume that $X(k_2) \neq 0$ on an open subset of $\mathcal{O}$ which yields $X(l_2) \neq 0$ because of Claim 3.12. By considering (3.5) and (3.44) we see that (3.35) and (3.36) turn into,

$$C_4 l_2^4 + C_3 l_2^3 + C_2 l_2^2 + C_1 l_2 + C_0 = 0 \tag{3.48}$$

and

$$D_4 k_2^4 + D_3 k_2^3 + D_2 k_2^2 + D_1 k_2 + D_0 = 0, \tag{3.49}$$

respectively, where C_i and D_i are some smooth functions satisfying $X(C_i) = X(D_i) = 0$. Next, we want to prove the following claim.

Claim 3.13 *The interior of the subsets $E_1 = \{p \in \mathcal{O} | 9l_1(p)e_{1p}(f) - 5f(p)e_{1p}(l_1) = 0\}$ is empty.*

Proof of Claim 3.13. Assume that the interior of E_1 is not empty, i.e.

$$l_1 e_1(f) - 5f e_1(l_1) = 0$$

on a nonempty open subset $\mathcal{O}_3$ of $\mathcal{O}$. Then, on $\mathcal{O}_3$ we have $l_1 = cf^{9/5}$ for a constant c. We have $c \neq 0$ because of Claim 3.13. Consequently, (3.47a) turns into

$$\left(50cf^{19/5} l_2 - 30fe_1^2(f) + 48e_1(f)^2 - 75f^3 k_2\right) = 0. \tag{3.50}$$

By applying e_1 to (3.50) and using (3.29a), (3.29b), we get

$$15f^3\left(4c^2 f^{8/5} + 9\right) e_1(f) - 440cf^{14/5} l_2 e_1(f) + 60e_1^3(f)f + 540f^2 k_2 e_1(f) - 132e_1(f)e_1^2(f) = 0. \tag{3.51}$$

From (3.50) and (3.51) we get

$$l_2 = \frac{3\left(100c^2 f^{28/5} e_1(f) + 100e_1^3(f)f^2 + 225f^4 e_1(f) + 576e_1(f)^3 - 580fe_1(f)e_1^2(f)\right)}{400cf^{19/5} e_1(f)}$$

which implies $X(l_2) = 0$ which is a contradiction. Hence, the interior of E_1 is empty.

On the other hand, if we assume that the interior of E_2 is not empty, then we have $l_1 = cf$ for a nonzero constant on a nonempty open subset $\mathcal{O}_4$ of $\mathcal{O}$.

Hence, the proof of the Claim 3.13 is completed. ■

Next, we combine (3.45) with (3.48) and (3.46) with (3.49) to get

$$\begin{aligned}
(9l_1 e_1(f) - 5fe_1(l_1))^6 \left(P_0 + P_1 k_2 + P_2 k_2^2 + \cdots + P_{13} k_2^{13}\right) &= 0, && (3.52)\\
(9l_1 e_1(f) - 5fe_1(l_1))^6 \left(Q_0 + Q_1 l_2 + Q_2 l_2^2 + \cdots + Q_{13} l_2^{13}\right) &= 0, && (3.53)
\end{aligned}$$

for some P_i, Q_i satisfying $X(P_i) = X(Q_i) = 0$, where we have

$$P_{13} = Af^4 \left(l_1 e_1(f) - fe_1(l_1)\right)\left(3\left(35f^2 + 16l_1{}^2\right) e_1(f) - 20fl_1 e_1(l_1)\right)^2,$$

$$Q_{13} = Bl_1^4 \left(l_1 e_1(f) - f e_1(l_1)\right) \left(15 f l_1 e_1(f) - 2\left(15 f^2 + 2{l_1}^2\right) e_1(l_1)\right)^2,$$

for some $A, B \in \mathbb{R}$. Because of Claim 3.13, we have $P_i = Q_i = 0$ for $i = 0, 1, \ldots, 13$. Note that if $l_1 e_1(f) - f e_1(l_1) \neq 0$ at a point, then $P_{13} = Q_{13} = 0$ implies

$$\begin{aligned} 3\left(35 f^2 + 16{l_1}^2\right) e_1(f) - 20 f l_1 e_1(l_1) &= 0, \\ 15 f l_1 e_1(f) - 2\left(15 f^2 + 2{l_1}^2\right) e_1(l_1) &= 0, \end{aligned}$$

from which we get $e_1(f) = 0$ which is a contradiction. Therefore, we have (3.41) on $\mathcal{O}$, where c is a nonzero constant. By a direct computation, we see that P_{11}, Q_{11} and P_9 turn into

$$\begin{aligned} P_{11} =& 49 c f^3 e_1(f)^2 \Big(108 a_1^2 e_1^3(f) f^2 + e_1(f)\big(- 12 a_1 \left(116c^2 + 675\right) f e_1^2(f) \\ & + 448\left(8c^4 + 90c^2 + 243\right) e_1(f)^2 + 27 a_1 \left(48c^4 + 352c^2 + 405\right) f^4\big)\Big), \end{aligned}$$

$$\begin{aligned} Q_{11} =& c^3 f^3 e_1(f)^2 \Big(12 a_1 e_1^3(f) f^2 + e_1(f)\big(- 4 a_2 \left(44c^2 + 261\right) f e_1^2(f) \\ & + 9 a_2 \left(16c^2\left(c^2 + 6\right) - 9\right) f^4 + 64\left(8c^4 + 90c^2 + 243\right) e_1(f)^2\big)\Big) \end{aligned}$$

and

$$\begin{aligned} P_9 = 4cf\Big(& 405000 a_2 \left(2c^2 + 9\right)^2 f^{10} e_1^3(f) + 526848\left(a_2 + 12\right) e_1(f)^7 \\ & - 90000\left(c^2 + 6\right)\left(2c^2 + 9\right)\left(11 a_2 + 240\right) f^9 e_1(f) e_1^2(f) \\ & - 1568\left(1007 a_2 + 9492\right) f e_1(f)^5 e_1^2(f) \\ & - 20000\left(116c^2 + 675\right) f^3 e_1(f) e_1^2(f)^3 \\ & - 7560\left(132c^2 + 295\right) f^3 e_1(f)^2 e_1^2(f) e_1^3(f) \\ & + 560\left(10876c^2 + 66825\right) f^2 e_1(f)^3 e_1^2(f)^2 \\ & + 4704\left(176c^2 + 135\right) f^2 e_1(f)^4 e_1^3(f) \\ & + 101250\left(2c^2 + 9\right)^2\left(48c^4 + 352c^2 + 405\right) f^{12} e_1(f) \\ & + 540000\left(8c^4 + 66c^2 + 135\right) e_1^3(f) f^7 e_1^2(f) + a_3 f^5 e_1(f)^3 e_1^2(f) \\ & - 15000\left(784c^4 + 8520c^2 + 23085\right) f^6 e_1^2(f)^2 e_1(f) \\ & - 315\left(9008c^4 + 85800c^2 + 184275\right) f^6 e_1^3(f) e_1(f)^2 \\ & + 180000 a_2 e_1^3(f) e_1^2(f)^2 + 151200 a_2 e_1^3(f)^2 e_1(f) \\ & - 392\left(1928c^4 + 103626c^2 + 564975\right) f^4 e_1(f)^5\Big) \\ & + 105c\left(221632c^6 + 4561488c^4 + 30995460c^2 + 70038675\right) f^9 e_1(f)^3, \end{aligned} \tag{3.54}$$

where we put $a_1 = c^2 + 15$, $a_2 = 4c^2 + 15$, $a_3 = 35\left(327568c^4 + 4714920c^2 + 16099965\right)$. By a direct computation considering $P_{11} = Q_{11} = 0$, we get

$$e_1^2(f) = -\frac{-16c^2e_1(f)^2+108c^2f^4-72e_1(f)^2+405f^4}{3a_2f}, \tag{3.55}$$

$$e_1^3(f) = \frac{32\left(2c^2+9\right)(5a_2+24)e_1(f)^3-27a_2(a_2+8)(a_2+30)f^4e_1(f)}{36a_2^2f^2}. \tag{3.56}$$

By taking derivative (3.55) in the direction of e_1 and considering (3.56), we get

$$\left(81-16c^4\right)f^2e_1(f)=0$$

which implies $c=\frac{3\epsilon}{2}$, where $\epsilon=\pm1$. Consequently, the first equation in (3.55) turns into

$$2fe_1^2(f)-3e_1(f)^2+18f^4=0$$

which implies

$$e_1(f)=\delta\sqrt{bf^3-18f^4} \tag{3.57}$$

for some constants b,δ such that $\delta=\pm1$. By combining (3.55)-(3.57) with (3.54), we get

$$\begin{aligned}P_9 =& \frac{243\epsilon}{2}f^{23/2}(b-18f)^{3/2}\Big(77824b^2(\delta-1)\\&+32b(1243091-1243679\delta)f+9(760492257\delta-75972425)f^2\Big)\end{aligned}$$

which implies that P_9 does not vanish outside of a set with empty interior. However, this is a contradiction.

Hence, we have $X(k_2)=X(l_2)=0$. Consequently, (3.31),(3.32) and (3.5) imply $X(k_3)=X(l_3)=X(\omega)=X(\gamma)=0$.

Similar to Theorem 3.8, we obtain the following theorem.

Theorem 3.14 *Let (Ω,g) be a 3-dimensional Riemannian manifold and $\psi:(\Omega,g)\hookrightarrow\mathbb{E}^5$ be a proper biconservative PNMCV isometric immersion with three distinct principal curvatures. Then, the principal directions e_1,e_2,e_3, principal curvatures k_2,k_3 and the functions f,l_1,l_2,l_3 can be determined intrinsically up to their signature.*

Proof By considering the Gauss equation and the shape operators of ψ, we obtain Ric of (Ω,g) satisfies $Ric(e_i)=-\lambda_ie_i$ for some functions λ_1, λ_2 and λ_3. Note that Proposition 3.10 implies $e_A(\lambda_i)=0$ and we have

$$\lambda_1=\frac{27f^2}{4}+l_1^2. \tag{3.58}$$

Consequently, $D=\operatorname{span}\{\nabla\lambda_1,\nabla\lambda_2,\nabla\lambda_3\}=\operatorname{span}\{e_1\}$ which yields that e_1 and λ_1 can be determined intrinsically. Furthermore, e_2,e_3 and ω,γ are unit eigenvectors and eigenvalues of the linear transformation

$$L:D^\perp\to D^\perp, L(X)=\nabla_Xe_i.$$

Therefore, they also can be determined intrinsically. Define τ_{ij} by

$$\tau_{ij}=R(e_i,e_j,e_i,e_j),\qquad 1\le i<j\le3. \tag{3.59}$$

By a direct computation combining Codazzi equations (3.29a) and (3.29b) with Gauss equations, we obtain

$$(\omega+\gamma)\left(\frac{9f^2}{4}+l_1^2\right)=e_1\left(\tau_{12}+\tau_{13}+\frac{1}{2}\lambda_1\right)+\omega\tau_{12}+\gamma\tau_{13}. \tag{3.60}$$

From (3.58) and (3.60), we see that f and l_1 can be determined intrinsically. A further computation by considering (3.28) and (3.59), one can obtain k_2, k_3, l_2, l_3 in terms of $\tau_{12}, \tau_{13}, \tau_{23}, f$ and l_1.

Consequently, we have the following result which can be obtained using [7, Theorem 1.1 at page 7].

Corollary 3.15 *If a 3-dimensional Riemannian manifold (Ω, g) admits two proper biconservative PNMCV isometric immersions into $\mathbb{E}^4$ with three distinct principal curvatures, then these immersions differ by an isometry of $\mathbb{E}^5$.*

4. Biharmonic submanifolds

In this section we consider biharmonic PNMCV submanifolds of dimension 3 and prove the following theorem.

Theorem 4.1 *Let $\psi:(\Omega,g)\hookrightarrow\mathbb{E}^5$ be a PNMCV isometric immersion, where (Ω,g) is a three-dimensional Riemannian manifold. Then, ψ cannot be biharmonic.*

Proof Suppose that ψ is a biharmonic PNMCV isometric immersion. It was proved in [11] that a biharmonic hypersurface in $\mathbb{E}^4$ is harmonic. Therefore, by considering Theorem 2.3, we assume that $\psi(\Omega)$ does not contain any open part lying on a hyperplane of $\mathbb{E}^5$.

Since ψ is biharmonic, it is biconservative and (2.10) is satisfied. By a direct computation considering (2.10) and (3.4), we get

$$\Delta f = f\left(\frac{9f^2}{4}+k_2^2+k_3^2\right), \tag{4.1a}$$

$$4l_2k_2+4l_3k_3+l_2k_3+l_3k_2=0. \tag{4.1b}$$

Note that because of (3.29a), (3.29b), (3.5) and (4.1b), l_2, k_2, l_3 and k_3 does not vanish outside of a subset Ω with empty interior.

First, towards contradiction, we assume $k_2\neq k_3$ at a point $p\in\Omega$. Then, on a neighborhood $\mathcal{N}_p$ of p, we have (3.30) because of Proposition 3.10. Therefore, the Gauss equation $\left(\tilde{R}(e_2,e_3,e_2,e_3)\right)^T=0$ gives

$$\omega\gamma \quad = \quad k_2k_3+l_2l_3 \qquad \text{if } k_2\neq k_3. \tag{4.2}$$

By applying e_1 to (4.1b), (4.2), then consider (3.29), (3.5) and (4.2), we obtain

$$(4k_2+k_3)(5l_2+2l_3)\,\omega+(k_2+4k_3)(2l_2+5l_3)\,\gamma \quad = \quad 0, \tag{4.3a}$$

$$P_1(k_2,k_3,l_2,l_3)\omega+P_1(k_3,k_2,l_3,l_2)\gamma \quad = \quad 0, \tag{4.3b}$$

where P is the polynomial given by

$$P_1(x_1,x_2,y_1,y_2)=-3x_1y_1-3x_2y_2+2x_1^2+5x_1x_2+6y_1^2+9y_1y_2.$$

By considering Theorem 2.3 and using (3.29a), (3.5), we observe that the interior of the set $K = \{p \in \Omega : \omega(p) = \gamma(p) = 0\}$ is empty. Therefore, (4.1b) and (4.3) imply

$$\begin{aligned} &32k_2^3l_3 - 6k_2^2l_3^2 - 232k_3k_2^2l_3 - 120k_2l_3^3 - 135k_3k_2l_3^2 - 958k_3^2k_2l_3 - 480k_3l_3^3 - 30k_3^2l_2^2 \\ &\quad +36k_3^2l_3^2 - 2040k_3l_2l_3^2 - 30k_3^3l_2 + 8k_3^3l_3 - 960k_3l_2^2l_3 - 135k_3^2l_2l_3 = 0 \end{aligned} \tag{4.4}$$

outside of K. We apply e_1 to (4.4) and use the same procedure to get

$$P_2(k_2,k_3,l_2,l_3)\omega + P_3(k_2,k_3,l_2,l_3)\gamma = 0 \tag{4.5}$$

for some fourth degree polynomials P_2, P_3. By considering (4.3a), (4.5), we obtain

$$\begin{aligned} &-1920l_2^2k_2^4 + 2240l_3^2k_2^4 + 800l_2l_3k_2^4 + 1416l_3^3k_2^3 + 13440k_3l_2^2k_2^3 + 32848k_3l_3^2k_2^3 \\ &+4332l_2l_3^2k_2^3 + 720l_2^2l_3k_2^3 + 89848k_3l_2l_3k_2^3 + 18720l_3^4k_2^2 + 1200k_3l_2^3k_2^2 \\ &+13398k_3l_3^3k_2^2 + 76800l_2l_3^3k_2^2 + 62760k_3^2l_2^2k_2^2 + 58896k_3^2l_3^2k_2^2 + 70080l_2^2l_3^2k_2^2 \\ &+50019k_3l_2l_3^2k_2^2 + 19200l_2^3l_3k_2^2 + 22260k_3l_2^2l_3k_2^2 + 291372k_3^2l_2l_3k_2^2 + 57600k_3l_2^4k_2 \\ &+48960k_3l_3^4k_2 + 12480k_3^2l_2^3k_2 - 15324k_3^2l_3^3k_2 + 420000k_3l_2l_3^3k_2 + 21180k_3^3l_2^2k_2 \\ &-65536k_3^3l_3^2k_2 + 837840k_3l_2^2l_3^2k_2 + 1146k_3^2l_2l_3^2k_2 + 441600k_3l_2^3l_3k_2 \\ &+40380k_3^2l_2^2l_3k_2 + 35312k_3^3l_2l_3k_2 + 14400k_3^2l_2^4 - 103680k_3^2l_3^4 + 345k_3^3l_2^3 \\ &-12240k_3^3l_3^3 - 283200k_3^2l_2l_3^3 + 240k_3^4l_2^2 - 21248k_3^4l_3^2 - 145920k_3^2l_2^2l_3^2 \\ &-24372k_3^3l_2l_3^2 + 22800k_3^2l_2^3l_3 - 5460k_3^3l_2^2l_3 - 12032k_3^4l_2l_3 = 0 \end{aligned} \tag{4.6}$$

outside of K. Finally, by obtaining the resultant of the polynomials appearing on the right hand side of (4.1b),(4.4),(4.6) with respect to l_2 and l_3, we get

$$\begin{aligned} &-43200\,(4k_2+k_3)^2\,(k_2+4k_3)^3\left(4k_2^2 - 13k_3k_2 + 4k_3^2\right)\Big(73381632k_2^{18} \\ &-689651232k_3k_2^{17} + 95630336k_3^2k_2^{16} + 20048552092k_3^3k_2^{15} - 51528775696k_3^4k_2^{14} \\ &-156852284797k_3^5k_2^{13} + 662866585600k_3^6k_2^{12} + 140071434296k_3^7k_2^{11} \\ &-1622719053552k_3^8k_2^{10} + 2974299612642k_3^9k_2^9 - 1622719053552k_3^{10}k_2^8 \\ &+140071434296k_3^{11}k_2^7 + 662866585600k_3^{12}k_2^6 - 156852284797k_3^{13}k_2^5 \\ &-51528775696k_3^{14}k_2^4 + 20048552092k_3^{15}k_2^3 + 95630336k_3^{16}k_2^2 \\ &-689651232k_3^{17}k_2 + 73381632k_3^{18}\Big) = 0 \end{aligned}$$

from which we see

$$k_2 = ck_3 \tag{4.7}$$

for a constant c such that $c \notin \{-1, 1\}$. From (3.5), (4.1b) and (4.7) we get

$$\begin{aligned} k_2 &= \frac{9f}{2(c+1)}, & k_3 &= \frac{9cf}{2(c+1)}, \\ l_2 &= -\frac{(4c+1)l_1}{3(c-1)}, & l_3 &= \frac{(c+4)l_1}{3(c-1)}. \end{aligned} \tag{4.8}$$

Finally, by combining (3.29a) and (3.29b) with (4.8), we obtain

$$(4c+1)fe_1(l_1) - 3l_1e_1(f) = 0,$$
$$(c+4)fe_1(l_1) - 3cl_1e_1(f) = 0$$

which implies $e_1(f) = 0$ unless $c^2 - 1 = 0$. However, this is a contradiction. Hence, we have $k_2 = k_3$ on Ω.

Since we have $k_2 = k_3$, (3.5) and (4.1b) imply $l_1 = 0$. Therefore, ψ is the isometric immersion given in Case I of Proposition 3.6. Consequently, (3.19) and (4.1a) give

$$40fe_1^2(f) - 48e_1(f)^2 + 495f^4 = 0. \quad (4.9)$$

On the other hand, the Gauss equation $\left(\tilde{R}(e_1, e_2, e_1, e_2)\right)^T = 0$ implies

$$40fe_1^2(f) - 64e_1(f)^2 + 225f^4 = 0. \quad (4.10)$$

However, (4.9) and (4.10) imply $8e_1(f)^2 + 135f^4 = 0$ which yields a contradiction.

Combining Theorem 4.1 with [6, Theorem 1] and [10, Theorem 1.1] provides the following partial answer for Chen's biharmonic conjecture.

Theorem 4.2 *There do not exist proper biharmonic submanifolds in $\mathbb{E}^5$ with the parallel normalized mean curvature vector.*

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