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## A note on "Some properties of second-order weak subdifferentials" [Turkish Journal of Mathematics (2021)45: 955-960]

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**Abstract:** In this note, we provide an example to illustrate that Proposition 2.4 in [Turkish Journal of Mathematics (2021)45: 955-960] is incorrect, and give a modification of the proposition. Two examples are provided to illustrate the modified result. Meanwhile, we establish a convex function, and correct the proof of Theorem 2.3 in [Turkish Journal of Mathematics (2021)45: 955-960] by the function.

**Key words:** Second-order weak subdifferentials, the subadditionability, the convexity

### 1. Introduction

It is well known that the concept of weak subdifferential plays a crucial role in nonsmooth analysis [1–3]. Recently, İnceoğlu [4] introduced the notion of second-order weak subdifferentials and discussed some of its properties. Unfortunately, the subadditivity of the second-order weak subdifferentials, i.e. [4, Proposition 2.4] is incorrect. At the same time, there exists a flaw in the proof process of [4, Theorem 2.3].

In this note, we first provide an example to demonstrate that [4, Proposition 2.4] is incorrect. Secondly, we propose a modified form of [4, Proposition 2.4]. Finally, we point out that there exists a flaw in the proof process of [4, Theorem 2.3], and provide a correct proof for the theorem.

### 2. Preliminaries

Throughout the paper, let  $X$  be a real normed space and let  $X^*$  be the topological dual space of  $X$ . Let  $\mathbb{R}$  and  $\mathbb{R}_+$  be the set of real numbers and the set of nonnegative real numbers, respectively. Let  $A, B \subset \mathbb{R}$ , the sum  $A + B$  and difference  $A - B$  of  $A$  and  $B$  are defined by  $A + B = \{a + b : a \in A, b \in B\}$  and  $A - B = \{a - b : a \in A, b \in B\}$ , respectively.

Let  $F, G : X \rightarrow \mathbb{R}$  be two single-valued functions, the sum  $F + G$  of functions  $F$  and  $G$  is defined by  $(F + G)(x) = F(x) + G(x), \forall x \in X$ .

Now we recall the definition of the second-order weak subdifferential.

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**Definition 2.1** (see [4, Definition 3.1]) Let  $F : X \rightarrow \mathbb{R}$  be a single-valued function and  $\bar{x} \in X$  be given, where  $F(\bar{x})$  is finite. A pair  $(x^*, c) \in X^* \times \mathbb{R}_+$  is called the second-order weak subgradient of  $F$  at  $\bar{x}$  if

$$F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle^2 - c\|x - \bar{x}\|^2, \forall x \in X.$$

The set

$$\partial_w^2 F(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}_+ : F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle^2 - c\|x - \bar{x}\|^2, \forall x \in X\}$$

of all second-order weak subgradients of  $F$  at  $\bar{x}$  is called the second-order weak subdifferential of  $F$  at  $\bar{x}$ . If  $\partial_w^2 F(\bar{x}) \neq \emptyset$ , then  $F$  is called second-order weakly subdifferentiable at  $\bar{x}$ .

**Definition 2.2** (see [5]) Let  $F : X \rightarrow \mathbb{R}$  be a single-valued function.  $F$  is called a convex function on  $X$  if for any  $x, y \in X$  and  $\lambda \in (0, 1)$ , one has

$$F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y).$$

To prove the convexity of the second-order weak subdifferential, we introduce the following lemma.

**Lemma 2.3** Let  $x \in X$  be given and let  $f_x(\cdot)$  be a single-valued function on  $X^*$  defined by

$$f_x(x^*) = \langle x^*, x \rangle^2, \forall x^* \in X^*.$$

Then  $f_x(\cdot)$  is a convex function on  $X^*$ .

*Proof* Let  $x \in X$ ,  $x_1^*, x_2^* \in X^*$  and  $\lambda \in (0, 1)$ . Then

$$\begin{aligned} f_x(\lambda x_1^* + (1 - \lambda)x_2^*) &= \langle \lambda x_1^* + (1 - \lambda)x_2^*, x \rangle^2 \\ &= \lambda^2 \langle x_1^*, x \rangle^2 + (1 - \lambda)^2 \langle x_2^*, x \rangle^2 + 2\lambda(1 - \lambda) \langle x_1^*, x \rangle \langle x_2^*, x \rangle. \end{aligned} \tag{2.1}$$

Set  $a = f_x(x_1^*) = \langle x_1^*, x \rangle^2$  and  $b = f_x(x_2^*) = \langle x_2^*, x \rangle^2$ . Then it follows from (2.1) that

$$\begin{aligned} f_x(\lambda x_1^* + (1 - \lambda)x_2^*) &= \lambda^2 a^2 + (1 - \lambda)^2 b^2 + 2\lambda(1 - \lambda)ab \\ &\leq \lambda^2 a^2 + (1 - \lambda)^2 b^2 + \lambda(1 - \lambda)(a^2 + b^2) \\ &= \lambda a^2 + (1 - \lambda)b^2 = \lambda f_x(x_1^*) + (1 - \lambda)f_x(x_2^*). \end{aligned}$$

Thus  $f_x(\cdot)$  is a convex function on  $X^*$ , this completes the proof. □

### 3. Properties of the second-order weak subdifferential

In this section, we present an important property of the second-order weak subdifferential and provide a correct proof for [4, Theorem 2.3]. We first recall a result in [4].

**Proposition 3.1** (see [4, Proposition 2.4]) Let  $F, G : X \rightarrow \mathbb{R}$  and  $F + G : X \rightarrow \mathbb{R}$  be single-valued functions and second-order weakly subdifferentiable at  $\bar{x}$ . Then

$$\partial_w^2 F(\bar{x}) + \partial_w^2 G(\bar{x}) \subseteq \partial_w^2 (F + G)(\bar{x}).$$

**Remark 3.2** Since the conditions do not imply the conclusion in [4, Proposition 2.4], [4, Proposition 2.4] is incorrect.

Now, we provide an example to illustrate Remark 3.2.

**Example 3.3** Let  $F, G : \mathbb{R} \rightarrow \mathbb{R}$  be two single-valued functions with  $F(x) = 2x^2$  and  $G(x) = 3x^2$ . Take  $\bar{x} = 0$ . Then

$$\partial_w^2 F(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 2\}, \partial_w^2 G(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 3\},$$

and

$$\partial_w^2 (F + G)(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 5\}.$$

Take  $(a, c) = (\sqrt{3}, 1) \in \partial_w^2 F(\bar{x})$  and  $(b, d) = (\sqrt{3}, 0) \in \partial_w^2 G(\bar{x})$ . Then

$$(a, c) + (b, d) = (2\sqrt{3}, 1) \notin \partial_w^2 (F + G)(\bar{x}).$$

It is obvious that [4, Proposition 2.4] does not hold here.

Next, we give an appropriate modification to rectify Proposition 2.4 in [4].

**Proposition 3.4** Let  $F, G : X \rightarrow \mathbb{R}$  and  $F + G : X \rightarrow \mathbb{R}$  be single-valued functions and second-order weakly subdifferentiable at  $\bar{x} \in X$  and let  $(a, c) \in \partial_w^2 F(\bar{x})$  and  $(b, d) \in \partial_w^2 G(\bar{x})$ . If

$$\langle a, x - \bar{x} \rangle \langle b, x - \bar{x} \rangle \leq 0, \forall x \in X, \tag{3.1}$$

then

$$(a, c) + (b, d) \in \partial_w^2 (F + G)(\bar{x}).$$

*Proof* Since  $(a, c) \in \partial_w^2 F(\bar{x})$  and  $(b, d) \in \partial_w^2 G(\bar{x})$ ,

$$F(x) - F(\bar{x}) \geq \langle a, x - \bar{x} \rangle^2 - c\|x - \bar{x}\|^2, \forall x \in X,$$

and

$$G(x) - G(\bar{x}) \geq \langle b, x - \bar{x} \rangle^2 - d\|x - \bar{x}\|^2, \forall x \in X.$$

Therefore,

$$(F + G)(x) - (F + G)(\bar{x}) \geq \langle a + b, x - \bar{x} \rangle^2 - (c + d)\|x - \bar{x}\|^2 - 2\langle a, x - \bar{x} \rangle \langle b, x - \bar{x} \rangle, \forall x \in X.$$

Then it follows from (3.1) that

$$(F(x) + G(x)) - (F(\bar{x}) + G(\bar{x})) \geq \langle a + b, x - \bar{x} \rangle^2 - (c + d)\|x - \bar{x}\|^2, \forall x \in X.$$

Thus, by virtue of the definition of the second-order weak subgradient, we have

$$(a, c) + (b, d) \in \partial_w^2 (F + G)(\bar{x}),$$

this completes the proof. □

Next, we provide an example to illustrate Proposition 3.4.

**Example 3.5** Consider Example 3.3. Take  $\bar{x} = 0$ . Then

$$\partial_w^2 F(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 2\}, \quad \partial_w^2 G(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 3\},$$

and

$$\partial_w^2 (F + G)(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 5\}.$$

Take  $(a, c) = (-\sqrt{2}, 1) \in \partial_w^2 F(\bar{x})$  and  $(b, d) = (\sqrt{2}, 1) \in \partial_w^2 G(\bar{x})$ . It is obvious that  $\langle a, x - \bar{x} \rangle \langle b, x - \bar{x} \rangle = -2x^2 \leq 0, \forall x \in X$  and

$$(a, c) + (b, d) \in \partial_w^2 (F + G)(\bar{x}).$$

Thus Proposition 3.4 holds here.

**Remark 3.6** The condition (3.1) is essential in Proposition 3.4. The following example explains the case.

**Example 3.7** Consider Example 3.3. Take  $\bar{x} = 0$ . Then

$$\partial_w^2 F(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 2\}, \quad \partial_w^2 G(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 3\},$$

and

$$\partial_w^2 (F + G)(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 5\}.$$

Take  $(a_0, c_0) = (\sqrt{2}, 1) \in \partial_w^2 F(\bar{x})$  and  $(b_0, d_0) = (\sqrt{2}, 0) \in \partial_w^2 G(\bar{x})$ . It is obvious that

$$\langle a_0, x - \bar{x} \rangle \langle b_0, x - \bar{x} \rangle = 2x^2 > 0, \forall x \in X \text{ and } x \neq 0$$

and

$$(a_0, c_0) + (b_0, d_0) \notin \partial_w^2 (F + G)(\bar{x}),$$

that is, the condition (3.1) does not hold here and Proposition 3.4 is not applicable here. Thus the condition (3.1) is essential in Proposition 3.4.

**Remark 3.8** In general, the inequality  $a^2 + b^2 \geq (a + b)^2$  does not hold, where  $a, b \in \mathbb{R}$ , so there is a flaw in the proof process of [4, Theorem 2.3], that is, it follows from the equations (2.8) and (2.9) in [4] that the equation in [4, Page 957, Line -6] may not be obtained.

Next, by virtue of Lemma 2.3, we give an appropriate modification to the proof process of [4, Theorem 2.3].

**Theorem 3.9** Let  $\bar{x} \in X$ ,  $F : X \rightarrow \mathbb{R}$  be a single-valued function and  $\partial_w^2 F(\bar{x}) \neq \emptyset$ . Then the set  $\partial_w^2 F(\bar{x})$  is convex.

*Proof* Let  $(a, c) \in \partial_w^2 F(\bar{x})$ ,  $(b, d) \in \partial_w^2 F(\bar{x})$  and  $\lambda \in (0, 1)$ . Then

$$F(x) - F(\bar{x}) \geq \langle a, x - \bar{x} \rangle^2 - c\|x - \bar{x}\|^2, \forall x \in X,$$

and

$$F(x) - F(\bar{x}) \geq \langle b, x - \bar{x} \rangle^2 - d\|x - \bar{x}\|^2, \forall x \in X.$$

Therefore,

$$F(x) - F(\bar{x}) \geq \lambda \langle a, x - \bar{x} \rangle^2 + (1 - \lambda) \langle b, x - \bar{x} \rangle^2 - (\lambda c + (1 - \lambda)d) \|x - \bar{x}\|^2, \forall x \in X.$$

Then it follows from Lemma 2.3 that

$$F(x) - F(\bar{x}) \geq \langle \lambda a + (1 - \lambda)b, x - \bar{x} \rangle^2 - (\lambda c + (1 - \lambda)d) \|x - \bar{x}\|^2, \forall x \in X.$$

Thus, by virtue of the definition of the second-order weak subgradient, we have

$$\lambda(a, c) + (1 - \lambda)(b, d) \in \partial_w^2 F(\bar{x}),$$

that is, the set  $\partial_w^2 F(\bar{x})$  is convex. So this completes the proof.  $\square$

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