Turkish Journal of Mathematics

Volume 47 | Number 1 Article 15

1-1-2023

Geodesics and isocline distributions in tangent bundles of nonflat Lorentzian-Heisenberg spaces

MURAT ALTUNBAŞ

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 234 – 247
© TÜBİTAK
doi:10.55730/1300-0098.3356

Geodesics and isocline distributions in tangent bundles of nonflat Lorentzian-Heisenberg spaces

Murat ALTUNBAŞ*
Department of Mathematics, Faculty of Arts and Sciences, Erzincan Binali Yıldırım University, Erzincan, Turkey

Received: 24.06.2022 • **Accepted/Published Online:** 18.11.2022 • **Final Version:** 13.01.2023

Abstract: Let (H_3, g_1) and (H_3, g_2) be the Lorentzian-Heisenberg spaces with nonflat metrics g_1 and g_2, and (TH_3, g_1^s), (TH_3, g_2^s) be their tangent bundles with the Sasaki metric, respectively. In the present paper, we find nontotally geodesic distributions in tangent bundles by using lifts of contact forms from the base manifold H_3. We give examples for totally geodesic but not isocline distributions. We study the geodesics of tangent bundles by considering horizontal and natural lifts of geodesics of the base manifold H_3. We also investigate more general classes of geodesics which are not obtained from horizontal and natural lifts of geodesics.

Key words: Geodesic, Lorentzian-Heisenberg space, tangent bundle

1. Introduction

Geometric properties of tangent bundles TM of (pseudo-) Riemannian manifolds (M, g) have been the subject of much attention by researchers since the eminent work of Sasaki [15]. The metric g^s introduced by Sasaki in this paper gives a natural splitting of the tangent bundle TTM into the horizontal distribution HTM and the vertical distribution VTM by means of the Levi-Civita connection ∇ on (M, g). It is known that the vertical distribution VTM is integrable but the horizontal distribution HTM is not integrable unless the base manifold (M, g) is flat. In [5], Druta and Piu showed the totally geodesicity and isoclinity of such distributions.

An important problem on tangent bundles of pseudo-Riemannian manifolds is to find geodesics with respect to the Riemannian metrics (for the background of the geodesics, we refer to [9, 11–13]). Geodesics have been studied intensively with respect to various Riemannian metrics in tangent bundles (for example see [2, 6, 7, 14, 16, 18–20]).

In this paper, we deal with the tangent bundles (TH_3, g_1^s) and (TH_3, g_2^s) of nonflat Lorentzian-Heisenberg spaces (H_3, g_1) and (H_3, g_2). It is known that there exist three nonisometric left-invariant Lorentzian metrics g_1, g_2, g_3 on the 3-dimensional Heisenberg group where g_3 is flat [10]. Using contact forms in these spaces, we find distributions which are not totally geodesic on tangent bundles, and by defining different distributions, show that they are totally geodesic and not isocline. We study the geodesics of the tangent bundles (TH_3, g_1^s) and (TH_3, g_2^s) in a classical way, i.e. by considering horizontal and natural lifts of geodesics from the base manifolds (H_3, g_1) and (H_3, g_2). We also search for some classes of geodesics by studying in a more general context which are not occured from horizontal and natural lifts of geodesics.

*Correspondence: maltunbas@erzincan.edu.tr
2010 *AMS Mathematics Subject Classification:* 53C22, 53C30, 53C50

2. Preliminaries

2.1. Tangent bundle

We consider (M, g) to be an $n-$dimensional, connected, smooth pseudo-Riemannian manifold and its tangent bundle TM has the natural projection $\pi : TM \to M$. To any local chart $(U, \varphi) = (U, x^1, ..., x^n)$ on M correspond a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), x^1, ..., x^n, y^1, ..., y^n)$ on TM. Therefore, TM has a $2n-$dimensional smooth manifold structure.

The Levi Civita connection ∇ of g on M infers the following direct sum decomposition

$$TTM = VTM \oplus HTM \tag{2.1}$$

of the tangent bundle to TM into the vertical distribution $VTM = Ker\pi_*$ and the horizontal distribution given by ∇.

The set of vector fields $\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}$ on $\pi^{-1}(U)$ defines a local frame field for VTM, and for HTM we have the local frame field $\{\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}\}$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{ki} y^k \frac{\partial}{\partial y^h}$$

and Γ^h_{ki} are the Christoffel symbols of g.

The set $\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}\}$ defines a local frame on TM, adapted to the direct sum decomposition (2.1). Remark that

$$(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i},\ (\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} - \Gamma^h_{ki} y^k \frac{\partial}{\partial y^h},$$

where $X^V \in VTM$ and $X^H \in HTM$ denote the vertical and horizontal lift of the vector field X on M.

The Sasaki metric g^s is defined by the following three relations:

$$\begin{cases} g^s(X^H, Y^H) = g(X, Y), \\ g^s(X^H, Y^V) = 0, \\ g^s(X^V, Y^V) = g(X, Y),\ \forall X, Y \in \chi(M). \end{cases}$$

The Levi-Civita connection $\tilde{\nabla}$ of the metric g^s satisfies the following relations:

$$\begin{cases} \tilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H - \frac{1}{2}(R(X, Y)y)^V, \\ \tilde{\nabla}_{X^H} Y^V = (\nabla_X Y)^V - \frac{1}{2}(R(Y, y)X)^H, \\ \tilde{\nabla}_{X^V} Y^H = -\frac{1}{2}(R(X, y)Y)^H, \\ \tilde{\nabla}_{X^V} Y^V = 0,\ \forall X, Y \in \chi(M). \end{cases} \tag{2.2}$$

When the metric g of M has the components g_{ij}, the metric g^s of TM has the components

$$g^s = g_{ij} dx^i dx^j + g_{ij} Dy^i Dy^j,\ \forall i, j = 1, ..., n,$$

where $\{Dy^i, dx^j\}_{i,j=1,...,n}$ is the dual frame of $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}_{i,j=1,...,n}$. The covariant derivative of y^i with respect to ∇ is given by

$$Dy^i = dy^i + \Gamma^i_{hj} y^h dx^j.$$

For further information about tangent bundles see [4] and [19].

2.2. Totally geodesic and isocline distributions on tangent bundles

A distribution F of a pseudo-Riemannian manifold (M, g) is said to be totally geodesic if every geodesic tangent to F at one point remains everywhere tangent to F.

A distribution F is totally geodesic if $\nabla_X Y + \nabla_Y X \in C^\infty(F)$ for all $X, Y \in C^\infty(F)$ [17].

Definition 2.1 *([5]) Consider that F is a totally geodesic distribution and N a unit vector field normal to F. We say that the distribution F is isocline when for every arc-parametrized geodesic curve, the angle between the tangent vector field $\dot{\gamma}(s)$ and the distribution F is constant along the geodesic.*

Proposition 2.2 *([8],[5]) Let (M, g) be a pseudo-Riemannian manifold and ∇ Levi-Civita connection of g. In order that, a totally geodesic distribution F is isocline it is necessary and sufficient for all vector field N normal to F, the vector field $\nabla_N N$ is normal to F.*

Let $\{X_i, N_\alpha\},\ i = 1, ..., p,\ \alpha = 1, ..., q,\ p+q = n,$ be an orthonormal frame of (M, g) for the distribution F ($X_i \in C^\infty(F)$ and $N_\alpha \in C^\infty(F^\perp)$), then F is isocline if and only if

$$g(\nabla_{X_i} X_j + \nabla_{X_j} X_i, N_\alpha) = 0 \ \text{ geodesicity}$$

$$g(\nabla_{N_\alpha} N_\beta + \nabla_{N_\beta} N_\alpha, X_i) = 0,$$

where $i, j = 1, ..., p; \alpha, \beta = 1, ..., q.$

Proposition 2.3 *([5]) Let (M, g) be a Riemannian manifold and (TM, g^s) its tangent bundle with the Sasaki metric. Then the horizontal distribution HTM and the vertical distribution VTM are isocline.*

2.3. Lorentzian-Heisenberg spaces

Each left-invariant Lorentzian metric on the 3-dimensional Heisenberg group H_3 is isometric to one of the following metrics:

$$g_1 = -\frac{1}{\lambda^2}(dx^1)^2 + (dx^2)^2 + (x^1 dx^2 + dx^3)^2, \tag{2.3}$$

$$g_2 = \frac{1}{\lambda^2}(dx^1)^2 + (dx^2)^2 - (x^1 dx^2 + dx^3)^2,\ \lambda > 0, \tag{2.4}$$

$$g_3 = (dx^1)^2 + (x^1 dx^2 + dx^3)^2 - ((1 - x^1)dx^2 - dx^3)^2.$$

Furthermore, the Lorentzian metrics g_1, g_2, g_3 are nonisometric and the Lorentzian metric g_3 is flat (see [1],[3]). We will deal with the metrics g_1 and g_2 (i.e. nonflat cases) and suppose that $\lambda = 1$.

2.3.1. The metric g_1

Let $\{\omega^1, \omega^2, \omega^3\}$ be an orthonormal coframe field defined by

$$\omega^1 = x^1 dx^2 + dx^3,\ \omega^2 = dx^2,\ \omega^3 = dx^1.$$

The dual orthonormal frame field $\{E_1, E_2, E_3\}$ of $\{\omega^1, \omega^2, \omega^3\}$ is given by

$$E_1 = \frac{\partial}{\partial x^3},\ E_2 = \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3},\ E_3 = \frac{\partial}{\partial x^1}.$$

Moreover, the 1-form $\omega = x^1dx^2 + dx^3$ is a contact form on (H_3, g_1) (since $\omega \wedge (d\omega)^n \neq 0$).

Let ∇ and R denote the Levi-Civita connection and Riemannian curvature tensor of (H_3, g_1), respectively. Then we have

$$\begin{gathered}
\nabla_{E_1}E_1 = 0,\ \nabla_{E_1}E_2 = \frac{1}{2}E_3,\ \nabla_{E_1}E_3 = \frac{1}{2}E_2, \\
\nabla_{E_2}E_1 = \frac{1}{2}E_3,\ \nabla_{E_2}E_2 = 0,\ \nabla_{E_2}E_3 = \frac{1}{2}E_1, \\
\nabla_{E_3}E_1 = \frac{1}{2}E_2, \nabla_{E_3}E_2 = -\frac{1}{2}E_1, \nabla_{E_3}E_3 = 0, \\
[E_2, E_3] = E_1,\ [E_1, E_2] = [E_1, E_3] = 0
\end{gathered} \tag{2.5}$$

and

$$\begin{gathered}
R(E_1, E_2)E_1 = \frac{1}{4}E_2,\ R(E_1, E_2)E_2 = -\frac{1}{4}E_1,\ R(E_1, E_3)E_1 = \frac{1}{4}E_3, \\
R(E_1, E_3)E_3 = \frac{1}{4}E_1,\ R(E_2, E_3)E_2 = -\frac{3}{4}E_3,\ R(E_2, E_3)E_3 = -\frac{3}{4}E_2,
\end{gathered}$$

where the other components of R are zero.

2.3.2. The metric g_2

Let $\{\theta^1, \theta^2, \theta^3\}$ be an orthonormal coframe field defined by

$$\theta^1 = dx^2,\ \theta^2 = dx^1,\ \theta^3 = x^1dx^2 + dx^3.$$

The dual orthonormal frame field $\{E_1, E_2, E_3\}$ of $\{\theta^1, \theta^2, \theta^3\}$ is given by

$$E_1 = \frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial x^3},\ E_2 = \frac{\partial}{\partial x^1},\ E_3 = \frac{\partial}{\partial x^3}.$$

Moreover, the 1-form $\omega = x^1dx^2 + dx^3$ is a contact form on (H_3, g_2) (since $\omega \wedge (d\omega)^n \neq 0$).

Let ∇ and R denote the Levi-Civita connection and Riemannian curvature tensor of (H_3, g_2), respectively. Then we have

$$\begin{gathered}
\nabla_{E_1}E_1 = 0,\ \nabla_{E_1}E_2 = \frac{1}{2}E_3,\ \nabla_{E_1}E_3 = \frac{1}{2}E_2, \\
\nabla_{E_2}E_1 = -\frac{1}{2}E_3,\ \nabla_{E_2}E_2 = 0,\ \nabla_{E_2}E_3 = -\frac{1}{2}E_1, \\
\nabla_{E_3}E_1 = \frac{1}{2}E_2, \nabla_{E_3}E_2 = -\frac{1}{2}E_1, \nabla_{E_3}E_3 = 0, \\
[E_1, E_2] = E_3,\ [E_1, E_3] = [E_2, E_3] = 0
\end{gathered} \tag{2.6}$$

and

$$\begin{gathered}
R(E_1, E_2)E_1 = -\frac{3}{4}E_2,\ R(E_1, E_2)E_2 = \frac{3}{4}E_1,\ R(E_1, E_3)E_1 = \frac{1}{4}E_3, \\
R(E_1, E_3)E_3 = \frac{1}{4}E_1,\ R(E_2, E_3)E_2 = \frac{1}{4}E_3,\ R(E_2, E_3)E_3 = \frac{1}{4}E_2,
\end{gathered}$$

where the other components of R are zero.

3. Isocline distributions and geodesics of (TH_3, g_1^s)

3.1. Isoclinity

The following proposition is obtained by using the contact form $\omega = x^1 dx^2 + dx^3$.

Proposition 3.1 *Let ω^H (resp. ω^V) be the horizontal (vertical) lift of the contact form ω of (H_3, g_1). Then, the 1-codimensional distribution $F = Ker(\omega^H)$ (resp. $F = Ker(\omega^V)$) is not totally geodesic in (TH_3, g_1^s).*

Proof Consider the distribution $Ker(\omega^H)$. We can determine a basis for this distribution as $\{E_2^H, E_3^H, E_1^V, E_2^V, E_3^V\}$. Thus, we have

$$\begin{aligned} g_1^s(\tilde{\nabla}_{E_2^H} E_2^V + \tilde{\nabla}_{E_2^V} E_2^H, E_1^H) &= -\frac{1}{2} g_1^s((R(E_2, y)E_2)^H, {E_1}^H) - \frac{1}{2} g_1^s((R(E_2, y)E_2)^H, E_1^H) \\ &= -g_1(R(E_2, y)E_2, E_1) \neq 0, \end{aligned}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection of the Sasaki metric given by (2.2) and y is a tangent vector in TH_3.

Now, consider the distribution $Ker(\omega^V)$. In this case, a basis can be chosen as $\{E_1^H, E_2^H, E_3^H, E_2^V, E_3^V\}$, so we get from (2.5)

$$g_1^s(\tilde{\nabla}_{E_2^H} E_3^V + \tilde{\nabla}_{E_3^V} E_2^H, E_1^V) = g_1^s((\nabla_{E_2} E_3)^V, E_1^V) = g_1(\frac{1}{2} E_1, E_1) = \frac{1}{2} \neq 0.$$

Thus we proved the proposition. □

In the following proposition, we obtain totally geodesic distributions which are not isocline in (TH_3, g_1^s).

Proposition 3.2 *The distributions $F^H = L(E_2^H, E_3^H)$ and $F^V = L(E_2^V, E_3^V)$ are totally geodesic but they are not isocline.*

Proof From (2.2) and (2.5), we easily verify the following relations:

$$\tilde{\nabla}_{E_2^H} E_3^H + \tilde{\nabla}_{E_3^H} E_2^H = 0, \tilde{\nabla}_{E_2^V} E_3^V + \tilde{\nabla}_{E_3^V} E_2^V = 0.$$

So, the distributions F^H and F^V are totally geodesic.

We also have

$$\begin{aligned} g_1^s(\tilde{\nabla}_{E_1^H} E_2^V + \tilde{\nabla}_{E_2^V} E_1^H, E_2^H) &= -\frac{1}{2} g_1^s((R(E_2, y)E_1)^H, {E_2}^H) - \frac{1}{2} g_1^s((R(E_2, y)E_1)^H, {E_2}^H) \\ &= -g_1(R(E_2, y)E_1, E_2) \neq 0. \end{aligned}$$

$$g_1^s(\tilde{\nabla}_{E_1^V} E_2^H + \tilde{\nabla}_{E_2^H} E_1^V, E_3^V) = g_1^s((\nabla_{E_2} E_1)^V, {E_3}^V) = \frac{1}{2} g_1(E_3, E_3) = -\frac{1}{2} \neq 0 \ (E_3 \text{ is timelike}).$$

This shows that these distributions are not isocline. □

3.2. Geodesics

The Sasaki lift metric g_1^s of the metric g_1 in (2.3) is given by

$$g_1^s = -(dx^1)^2 + (dx^2)^2 + (x^1dx^2 + dx^3)^2 - (Dy^1)^2 + (Dy^2)^2 + (x^1Dy^2 + Dy^3)^2.$$

Lagrangian of the metric g_1^s is expressed as

$$L = -\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(x^1\frac{dx^2}{dt} + \frac{dx^3}{dt}\right)^2 - \left(\frac{Dy^1}{dt}\right)^2 + \left(\frac{Dy^2}{dt}\right)^2 + \left(x^1\frac{Dy^2}{dt} + \frac{Dy^3}{dt}\right)^2$$

and the corresponding Euler-Lagrange equations are

$$\frac{d}{dt}\frac{dx^1}{dt} = -\frac{dx^2}{dt}\left(x^1\frac{dx^2}{dt} + \frac{dx^3}{dt}\right),$$

$$\frac{d}{dt}\left(x^1(x^1\frac{dx^2}{dt} + \frac{dx^3}{dt}) + \frac{dx^2}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(x^1\frac{dx^2}{dt} + \frac{dx^3}{dt}\right) = 0,$$

$$\frac{d}{dt}\frac{Dy^1}{dt} = 0,$$

$$\frac{d}{dt}\left(x^1(x^1\frac{Dy^2}{dt} + \frac{Dy^3}{dt}) + \frac{Dy^2}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(x^1\frac{Dy^2}{dt} + \frac{Dy^3}{dt}\right) = 0.$$

Let the curve $\gamma : I \to TH_3,\ \gamma(t) = (x^1(t), x^2(t), x^3(t), y^1(t), y^2(t), y^3(t))$ be a geodesic which satisfies the initial conditions $\gamma(0) = (0,0,0,0,0,0)$ and $\dot{\gamma}(0) = (u, v, w, l, m, n)$. Then the above Euler-Lagrange equations turn in

$$x^1\frac{dx^2}{dt} + \frac{dx^3}{dt} = w, \tag{3.1}$$

$$\frac{d}{dt}\frac{dx^1}{dt} = -\frac{dx^2}{dt}w, \tag{3.2}$$

$$\frac{d}{dt}\left(x^1w + \frac{dx^2}{dt}\right) = 0, \tag{3.3}$$

$$x^1\frac{Dy^2}{dt} + \frac{Dy^3}{dt} = n, \tag{3.4}$$

$$\frac{Dy^1}{dt} = l, \tag{3.5}$$

$$x^1n + \frac{Dy^2}{dt} = m. \tag{3.6}$$

Remark 3.3 *The equations (3.1)-(3.3) are valid if and only if the curve* $\alpha : I \to (H_3, g_1)$ *is a geodesic which satisfies the initial conditions* $\alpha(0) = (0,0,0)$ *and* $\dot{\alpha}(0) = (u,v,w)$.

Since $\dot{\gamma}(0) = (u,v,w,l,m,n)$, from equation (3.3) we get

$$\frac{dx^2}{dt} = -x^1 w + v.$$

By substituting above into (3.2) we obtain

$$\frac{d^2x^1}{dt^2} - x^1 w^2 = 0.$$

Solution of this equation is

$$x^1(t) = \frac{u}{w}\sinh(wt) - \frac{v}{w}\cosh(wt) + \frac{v}{w}. \tag{3.7}$$

Similarly, from (3.2) and (3.3) we have

$$x^2(t) = -\frac{u}{w}\cosh(wt) + \frac{v}{w}\sinh(wt) + \frac{u}{w}. \tag{3.8}$$

Setting the solutions (3.7) and (3.8) in (3.1) we deduce

$$x^3(t) \quad = \quad \frac{(u-v)^2}{4w^2}\sinh(2wt) - \frac{uv}{2w^2}\cosh(2wt) + \frac{v}{2w^2}(u\cosh(wt) - v\sinh w(t)) + (\frac{v^2-u^2}{2w} + w)t - \frac{uv}{2w^2}.$$

If $w = 0$, the solution of the system formed by (3.1)-(3.3) is

$$x^1(t) = ut,\ x^2(t) = vt,\ x^3(t) = 0. \tag{3.9}$$

Now we can express the following theorem.

Theorem 3.4 *The horizontal lift* $\tilde{C}$ *and the natural lift* $\hat{C}$ *of a curve* C *from* H_3 *are geodesics in* (TH_3, g_1^s) *if and only if* C *is a geodesic in* (H_3, g_1), *and* $\tilde{C}$ *and* $\hat{C}$ *pass through the origin and satisfy* $\dot{\tilde{C}}(0) = \dot{\hat{C}}(0) = (u,v,w,0,0,0)$.

Proof Let the curve $\tilde{C}(t) = (C(t), Y(t))$ be the horizontal lift to (TH_3, g_1^s) of the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ from (H_3, g_1). From the definition of the horizontal lift of a curve we have

$$\frac{Dy^1}{dt} = \frac{Dy^2}{dt} = \frac{Dy^3}{dt} = 0,$$

and then equations (3.4)-(3.6) yield $l = m = n = 0$.

Let the curve $\hat{C}(t) = (C(t), Y(t))$ be the natural lift to (TH_3, g_1^s) of the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ from (H_3, g_1). From the definition of the natural lift of a curve we have

$$y^h = \frac{dx^h}{dt}.$$

Therefore, we can write the covariant derivative of Y as follows:

$$\frac{Dy^h}{dt} = \frac{d^2x^h}{dt^2} + \Gamma^h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}, \ \forall i,j,h = 1,2,3. \tag{3.10}$$

By virtue of Remark 3.3, we see that the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ is a geodesic in (H_3, g_1) and the expression of $\frac{Dy^h}{dt}$ in (3.10) reduces zero, and the last three Euler-Lagrange equations (3.4)-(3.6) yield again $l = m = n = 0$. This ends the proof. □

Now, we investigate more general examples for geodesics except for horizontal and natural lifts. To do this, we need Christoffel symbols of the metric g_1. The metric g_1 in (2.3) and its inverse are rewritten in matrix form as follows:

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1+(x^1)^2 & x^1 \\ 0 & x^1 & 1 \end{pmatrix}, \quad g_1^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -x^1 \\ 0 & -x^1 & 1+(x^1)^2 \end{pmatrix}.$$

Then, we easily compute the nonzero coeffiecients of the Christoffel symbols of the metric g_1 as follows:

$$\Gamma^1_{22} = x^1, \ \Gamma^1_{23} = \frac{1}{2}, \ \Gamma^2_{12} = \frac{x^1}{2}, \ \Gamma^2_{13} = \frac{1}{2}, \ \Gamma^3_{12} = \frac{1}{2}(1-(x^1)^2), \ \Gamma^3_{13} = -\frac{x^1}{2}.$$

Therefore, equations (3.4)-(3.6) become

$$\frac{dy^1}{dt} + (x^1y^2 + \frac{y^3}{2})\frac{dx^2}{dt} + \frac{y^2}{2}\frac{dx^3}{dt} = l,$$

$$\frac{dy^2}{dt} + (\frac{x^1y^2}{2} + \frac{y^3}{2})\frac{dx^1}{dt} + \frac{x^1y^1}{2}\frac{dx^2}{dt} + \frac{y^1}{2}\frac{dx^3}{dt} = m - unt,$$

$$\frac{dy^3}{dt} + \frac{(1+(x^1)^2)y^2}{2}\frac{dx^1}{dt} + \frac{(1+(x^1)^2)y^1}{2}\frac{dx^2}{dt} + x^1(m - x^1n) = n.$$

If the geodesic in (H_3, g_1) is given as (3.9), then above equations reduce to

$$\frac{dy^1}{dt} + (uty^2 + \frac{y^3}{2})v = l,$$

$$\frac{dy^2}{dt} + (\frac{uty^2}{2} + \frac{y^3}{2})u + \frac{uvty^1}{2} = m - unt,$$

$$\frac{dy^3}{dt} + \frac{(1+u^2t^2)y^2}{2}u + \frac{(1+u^2t^2)y^1}{2}v + ut(m - unt) = n,$$

in other words, we have the following system

$$(*) : \begin{cases} \frac{dy_1}{dt} + utvy_2 + \frac{v}{2}y_3 = l, \\ \frac{dy_2}{dt} + \frac{ut}{2}y_1 + \frac{u^2t}{2}y_2 + \frac{u}{2}y_3 = m - unt, \\ \frac{dy_3}{dt} + \frac{(1+u^2t^2)v}{2}y_1 + \frac{(1+u^2t^2)u}{2}y_2 + ut(m-unt) = n. \end{cases}$$

Remark 3.5 *When $u = v = 0$, a particular solution for the above system is obtained by*

$$y_1(t) = lt, \ y_2(t) = mt, \ y_3(t) = nt.$$

Having in mind this remark, we can express the theorem below.

Theorem 3.6 *If the curve from* (H_3, g_1) *reduces to the origin point* $(0,0,0)$*, then the geodesics from* (TH_3, g_1^s) *are the curves* $\gamma(t) = (0,0,0,lt,mt,nt)$*.*

We know that if a geodesic being in a fiber of the tangent bundle (TM, g^s) of a Riemannian manifold (M,g) defined by $x^h = c^h$, $\forall h = 1,...,n$, where c^h is a constant, then the geodesic is expressed as $x^h = c^h$, $y^h = a^h t + b^h$, according to the induced coordinates $\{x^h, y^h\}_{h=1,\dots n}$, where a^h, b^h, c^h are constants (see [19]). When we consider (TH_3, g_1^s) this result becomes to Theorem 3.6 , because if x^i are constants, we get $\frac{Dy^i}{dt} = \frac{dy^i}{dt}$, $i = 1,2,3$, and equations (3.4)-(3.6) with the conditions $\frac{dy^1}{dt}(0) = l$, $\frac{dy^2}{dt}(0) = m$, $\frac{dy^3}{dt}(0) = n$ yield $x^i = 0$, $i = 1,2,3$.

Remark 3.7 *When* $m = n = 0$*, a particular solution of the system* $(*)$ *is given by*

$$y_1(t) = lt, \ y_2(t) = 0, \ y_3(t) = 0.$$

Having in mind above remark and equations (3.9), we may express the final theorem of this section.

Theorem 3.8 *One of the geodesics of the tangent bundle* (TH_3, g_1^s) *has the form* $\tilde{\gamma}(t) = (ut, vt, 0, lt, 0, 0)$*.*

4. Isocline distributions and geodesics of (TH_3, g_2^s)

In this final section, we follow same way in the previous section for the metric g_2 which is defined in (2.4).

4.1. Isoclinity

We write the following proposition by considering the contact form $\omega = x^1 dx^2 + dx^3$.

Proposition 4.1 *Let* ω^H (ω^V) *be the horizontal (vertical) lift of the contact form* ω *of* (H_3, g_2)*. Then, the 1-codimensional distribution* $F = Ker(\omega^H)$ $(F = Ker(\omega^V))$ *is not totally geodesic in* (TH_3, g_2^s)*.*

Proof If the distribution $Ker(\omega^H)$ is given, then one can constitute a basis as $\{E_1^H, E_2^H, E_1^V, E_2^V, E_3^V\}$. Thus

$$\begin{aligned} g_2^s(\tilde{\nabla}_{E_1^H} E_1^V + \tilde{\nabla}_{E_1^V} E_1^H, E_3^H) &= -\frac{1}{2} g_2^s((R(E_1,y)E_1)^H, E_3{}^H) - \frac{1}{2} g_2^s((R(E_1,y)E_1)^H, E_3^H) \\ &= -g_2(R(E_1,y)E_1, E_3) \neq 0, \end{aligned}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the Sasaki metric given by (2.2) and y denotes a tangent vector in TH_3.

Now, let the distribution $Ker(\omega^V)$ is given. In this case, a basis can be formed as $\{E_1^H, E_2^H, E_3^H, E_1^V, E_2^V\}$, so from (2.6)

$$g_2^s(\tilde{\nabla}_{E_1^H} E_2^V + \tilde{\nabla}_{E_2^V} E_1^H, E_3^V) = g_2^s((\nabla_{E_1} E_2)^V, E_3^V) = g_2(\frac{1}{2} E_3, E_3) = -\frac{1}{2} \neq 0 \ (E_3 \text{ is timelike}).$$

Thus the proposition is proved. □

In the following proposition, we determine totally geodesic distributions which are not isocline in (TH_3, g_2^s).

Proposition 4.2 *The distributions* $F^H = L(E_1^H, E_2^H)$ *and* $F^V = L(E_1^V, E_2^V)$ *are totally geodesic but they are not isocline.*

Proof Using (2.2) and (2.6), we get the following relations:

$$\tilde{\nabla}_{E_1^H} E_2^H + \tilde{\nabla}_{E_2^H} E_1^H = 0, \tilde{\nabla}_{E_1^V} E_2^V + \tilde{\nabla}_{E_2^V} E_1^V = 0.$$

So, the distributions F^H and F^V are totally geodesic.

We also occur

$$\begin{aligned} g_2^s(\tilde{\nabla}_{E_3^H} E_1^V + \tilde{\nabla}_{E_1^V} E_3^H, E_1^H) &= -\frac{1}{2} g_2^s((R(E_1, y)E_3)^H, E_1{}^H) - \frac{1}{2} g_2^s((R(E_1, y)E_3)^H, E_1{}^H) \\ &= -g_2(R(E_1, y)E_3, E_1) \neq 0, \end{aligned}$$

$$g_2^s(\tilde{\nabla}_{E_3^V} E_2^H + \tilde{\nabla}_{E_2^H} E_3^V, E_1^V) = g_2^s((\nabla_{E_2} E_3)^V, E_1^V) = -\frac{1}{2} g_2(E_1, E_1) = -\frac{1}{2} \neq 0.$$

This shows that F^V and F^H are not isocline. □

4.2. Geodesics

The Sasaki lift metric g_2^s of the metric g_2 in (2.4) has expression

$$g_2^s = (dx^1)^2 + (dx^2)^2 - (x^1 dx^2 + dx^3)^2 + (Dy^1)^2 + (Dy^2)^2 - (x^1 Dy^2 + Dy^3)^2.$$

Lagrangian of this metric is

$$L = \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 - \left(x^1 \frac{dx^2}{dt} + \frac{dx^3}{dt}\right)^2 + \left(\frac{Dy^1}{dt}\right)^2 + \left(\frac{Dy^2}{dt}\right)^2 - \left(x^1 \frac{Dy^2}{dt} + \frac{Dy^3}{dt}\right)^2$$

and the corresponding Euler-Lagrange equations are written as

$$\frac{d}{dt}\frac{dx^1}{dt} = -\frac{dx^2}{dt}\left(x^1 \frac{dx^2}{dt} + \frac{dx^3}{dt}\right),$$

$$\frac{d}{dt}\left(-x^1(x^1 \frac{dx^2}{dt} + \frac{dx^3}{dt}) + \frac{dx^2}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(x^1 \frac{dx^2}{dt} + \frac{dx^3}{dt}\right) = 0,$$

$$\frac{d}{dt}\frac{Dy^1}{dt} = 0,$$

$$\frac{d}{dt}\left(-x^1(x^1 \frac{Dy^2}{dt} + \frac{Dy^3}{dt}) + \frac{Dy^2}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(x^1 \frac{Dy^2}{dt} + \frac{Dy^3}{dt}\right) = 0.$$

Let the curve $\gamma : I \to TH_3,\ \gamma(t) = (x^1(t), x^2(t), x^3(t), y^1(t), y^2(t), y^3(t))$ be a geodesic which satisfies the initial conditions $\gamma(0) = (0,0,0,0,0,0)$ and $\dot{\gamma}(0) = (u,v,w,l,m,n)$. Then the above Euler-Lagrange equations reduce to

$$x^1 \frac{dx^2}{dt} + \frac{dx^3}{dt} = w, \tag{4.1}$$

$$\frac{d}{dt}\frac{dx^1}{dt} = -\frac{dx^2}{dt} w, \tag{4.2}$$

$$\frac{d}{dt}\left(-x^1 w + \frac{dx^2}{dt}\right) = 0, \tag{4.3}$$

$$x^1 \frac{Dy^2}{dt} + \frac{Dy^3}{dt} = n, \tag{4.4}$$

$$\frac{Dy^1}{dt} = l, \tag{4.5}$$

$$-x^1 n + \frac{Dy^2}{dt} = m. \tag{4.6}$$

Remark 4.3 *The equations (4.1)-(4.3) are fulfilled if and only if the curve $\alpha : I \to (H_3, g_2)$ is a geodesic which satisfies the initial conditions $\alpha(0) = (0,0,0)$ and $\dot{\alpha}(0) = (u,v,w)$.*

Since $\dot{\gamma}(0) = (u,v,w,l,m,n)$, from equation (4.3) we obtain

$$\frac{dx^2}{dt} = x^1 w + v.$$

By putting above in (4.2) we obtain

$$\frac{d^2x^1}{dt^2} + x^1 w^2 = 0.$$

The solution of this equation is

$$x^1(t) = \frac{u}{w}\sin(wt) + \frac{v}{w}\cos(wt) + \frac{v}{w}. \tag{4.7}$$

Similarly, from (4.2) and (4.3) we get

$$x^2(t) = -\frac{u}{w}\cos(wt) + \frac{v}{w}\sin(wt) + \frac{u}{w}. \tag{4.8}$$

Putting the solutions (4.7) and (4.8) in (4.1) we deduce

$$x^3(t) \quad = \quad \frac{u^2 - v^2}{4w^2}\sin(2wt) - w(\frac{u^2+v^2}{2w^2} - 1)t + \frac{uv}{2w^2}\cos(2wt).$$

If $w = 0$, the solution of the system formed by (4.1)-(4.3) is

$$x^1(t) = ut,\ x^2(t) = vt,\ x^3(t) = 0. \tag{4.9}$$

Now we can give the following theorem.

Theorem 4.4 *The horizontal lift $\tilde{C}$ and the natural lift $\hat{C}$ of a curve C from H_3 are geodesics in (TH_3, g_2^s) if and only if C is a geodesic in (H_3, g_2), and $\tilde{C}$ and $\hat{C}$ pass through the origin and satisfy $\dot{\tilde{C}}(0) = \dot{\hat{C}}(0) = (u, v, w, 0, 0, 0)$.*

Proof Let the curve $\tilde{C}(t) = (C(t), Y(t))$ be the horizontal lift to (TH_3, g_2^s) of the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ from (H_3, g_2). From the definition of the horizontal lift of a curve we have

$$\frac{Dy^1}{dt} = \frac{Dy^2}{dt} = \frac{Dy^3}{dt} = 0,$$

and then equations (4.4)-(4.6) lead $l = m = n = 0$.

Let the curve $\hat{C}(t) = (C(t), Y(t))$ be the natural lift to (TH_3, g_2^s) of the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ from (H_3, g_2). From the definition of the natural lift of a curve we have

$$y^h = \frac{dx^h}{dt}.$$

Therefore, we can write the covariant derivative of Y as follows:

$$\frac{Dy^h}{dt} = \frac{d^2x^h}{dt^2} + \Gamma_{ij}^h \frac{dx^i}{dt}\frac{dx^j}{dt}, \ \forall i, j, h = 1, 2, 3. \tag{4.10}$$

By virtue of Remark 4.1, we see that the curve $C(t) = (x^1(t), x^2(t), x^3(t))$ is a geodesic in (H_3, g_2) and the expression of $\frac{Dy^h}{dt}$ in (4.10) reduces zero, and the last three Euler-Lagrange equations (4.4)-(4.6) lead again $l = m = n = 0$. This ends the proof. □

Now, we study more general examples for geodesics except for horizontal and natural lifts. For this, we need Christoffel symbols of the metric g_2. The metric g_2 in (2.4) and its inverse are rewritten in matrix form as follows:

$$g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-(x^1)^2 & -x^1 \\ 0 & -x^1 & -1 \end{pmatrix}, \quad g_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -x^1 \\ 0 & -x^1 & (x^1)^2-1 \end{pmatrix}.$$

Then, we compute the nonzero coefficients of the Christoffel symbols of the metric g_2 as follows:

$$\Gamma_{22}^1 = x^1, \ \Gamma_{23}^1 = \frac{1}{2}, \ \Gamma_{12}^2 = -\frac{x^1}{2}, \ \Gamma_{13}^2 = -\frac{1}{2}, \Gamma_{12}^3 = \frac{1}{2}(1+(x^1)^2), \Gamma_{13}^3 = \frac{x^1}{2}.$$

Therefore, equations (4.4)-(4.6) reduce to

$$\frac{dy^1}{dt} + (x^1y^2 + \frac{y^3}{2})\frac{dx^2}{dt} + \frac{y^2}{2}\frac{dx^3}{dt} = l,$$

$$\frac{dy^2}{dt} - (\frac{x^1y^2}{2} + \frac{y^3}{2})\frac{dx^1}{dt} - \frac{x^1y^1}{2}\frac{dx^2}{dt} - \frac{y^1}{2}\frac{dx^3}{dt} = m - unt,$$

$$\frac{dy^3}{dt} + \frac{(1+(x^1)^2)y^2}{2}\frac{dx^1}{dt} + \frac{(1+(x^1)^2)y^1}{2}\frac{dx^2}{dt} - x^1(m - x^1n) = n.$$

If the geodesic in (H_3, g_2) is given as (4.9), then above equations reduce to

$$\frac{dy^1}{dt} + (uty^2 + \frac{y^3}{2})v = l,$$

$$\frac{dy^2}{dt} - (\frac{uty^2}{2} + \frac{y^3}{2})u - \frac{uvty^1}{2} = m - unt,$$

$$\frac{dy^3}{dt} + \frac{(1+u^2t^2)y^2}{2}u + \frac{(1+u^2t^2)y^1}{2}v - ut(m - unt) = n,$$

i.e. we have the following system

$$(**) : \begin{cases} \frac{dy_1}{dt} + utvy_2 + \frac{v}{2}y_3 = l, \\ \frac{dy_2}{dt} - \frac{ut}{2}y_1 - \frac{u^2t}{2}y_2 - \frac{u}{2}y_3 = m - unt, \\ \frac{dy_3}{dt} + \frac{(1+u^2t^2)v}{2}y_1 + \frac{(1+u^2t^2)u}{2}y_2 - ut(m - unt) = n. \end{cases}$$

Remark 4.5 *When $u = v = 0$, a particular solution for the above system is obtained by*

$$y_1(t) = lt, \ y_2(t) = mt, \ y_3(t) = nt.$$

By virtue of this remark, we can express the theorem below.

Theorem 4.6 *If the curve from (H_3, g_2) reduces to the origin point $(0,0,0)$, then the geodesics from (TH_3, g_2^s) are the curves $\gamma(t) = (0,0,0,lt,mt,nt)$.*

We know that if a geodesic being in a fiber of the tangent bundle (TM, g^s) of a Riemannian manifold (M, g) defined by $x^h = c^h$, $\forall h = 1, ..., n$, where c^h is a constant, then the geodesic is expressed as $x^h = c^h$, $y^h = a^h t + b^h$, according to the induced coordinates $\{x^h, y^h\}_{h=1,...n}$, where a^h, b^h, c^h are constants. When we assume (TH_3, g_2^s) this result becomes to Theorem 4.6 , since if x^i are constants, we get $\frac{Dy^i}{dt} = \frac{dy^i}{dt}$, $i = 1,2,3$, and equations (4.4)-(4.6) with the conditions $\frac{dy^1}{dt}(0) = l$, $\frac{dy^2}{dt}(0) = m$, $\frac{dy^3}{dt}(0) = n$ lead to $x^i = 0$, $i = 1,2,3$.

Remark 4.7 *When $m = n = 0$, a special solution of the system $(**)$ is given by*

$$y_1(t) = lt, \ y_2(t) = 0, \ y_3(t) = 0.$$

Having in mind above remark and equations (4.9), we conclude the final theorem of this paper.

Theorem 4.8 *One of the geodesics of the tangent bundle (TH_3, g_2^s) has the form $\tilde{\gamma}(t) = (ut, vt, 0, lt, 0, 0)$.*

Acknowledgment

The author would like to thank the anonymous referees for their valuable suggestions and comments.

References

[1] Altunbaş M. Killing magnetic curves in non-flat Lorentzian-Heisenberg spaces. Filomat 2022; 36 (8): 2645-2655.

[2] Altunbaş M, Gezer A, Şimşek R. A study concerning Berger type deformed Sasaki metric on the tangent bundle. Journal of Mathematical Physics, Analysis, Geometry 2019; 15 (4): 435-447. https://doi.org/10.15407/mag15.04.435

[3] Batat W, Rahmani S. Isometries, geodesics and Jacobi fields of Lorentzian Heisenberg group. Mediterranean Journal of Mathematics 2011; 8: 411-430. https://doi.org/10.1007/s00009-010-0070-z

[4] Dida HM, A. Ikemakhen A. A class of metrics on tangent bundles of pseudo-Riemannian manifolds. Archivum Mathematicum 2011; 47 (4): 293-308.

[5] Druta SL, Piu P. Geodesicity and isoclinity properties for the tangent bundle of the Heisenberg manifold with Sasaki metric. Turkish Journal of Mathematics 2012; 36: 331-343. https://doi.org/10.48550/arXiv.1002.3237

[6] Gezer A, Altunbaş M. Some notes concerning Riemannian metrics of Cheeger Gromoll type. Journal of Mathematical Analysis and Applications 2012; 396: 119-132. https://doi.org/10.1016/j.jmaa.2012.06.011

[7] Lotarets L. Geodesics of fiberwise cigar soliton deformation of the Sasaki metric. Turkish Journal of Mathematics 2022; 46: 130-144. https://doi.org/10.3906/mat-2107-99

[8] Lutz R. Quelques remarques sur la geometrie metrique des structures de contact. Coll. Travaux en cours Hermann, Paris, 1984.

[9] Mikeš, J. (editor). Differential geometry of special mappings. Olomouc, Palacky Univ. 2019.

[10] Rahmani N, Rahmani S. Lorentzian geometry of the Heisenberg group. Geometriae Dedicata 2006; 118: 133-140. https://doi.org/10.1007/s10711-005-9030-3

[11] Rýparová L, Mikeš J, Sabykanov A. On geodesic bifurcations of product spaces. Journal of Mathematical Sciences 2019; 239 (1): 86-91 https://doi.org/10.1007/s10958-019-04290-1

[12] Rýparová L, Mikeš J. Bifurcation of closed geodesics. Geometry, Integrability and Quantization 2018; 19: 188-192. https://doi.org/10.7546/giq-19-2018-188-192

[13] Rýparová L, Mikeš J. On geodesic bifurcations. Geometry, Integrability and Quantization 2017; 18: 217-224. https://doi.org/10.7546/giq-18-2017-217-224

[14] Salimov AA, Kazimova S. Geodesics of the Cheeger-Gromoll metric. Turkish Journal of Mathematics 2009; 33: 99-105. https://doi.org/10.3906/mat-0804-24

[15] Sasaki S. On the differential geometry of the tangent bundle of Riemannian manifolds. Tohoku Mathematics Journal 1958; 10: 238-354.

[16] Sato S. Geodesics on the tangent bundles over space forms. Tensor New Series 1978; 32: 5-10.

[17] Wood JC. Harmonic morphisms, foliations and Gauss maps. Contemporary Mathematics 1986; 49: 145-184.

[18] Yampolsky A, On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kahlerian manifold. Journal of Mathematical Physics, Analysis, Geometry 2012; 8 (2): 177-189.

[19] Yano K, Ishihara S. Tangent and Cotangent Bundles. New York, M. Dekker Inc., 1973.

[20] Zagane A. Geodesics on tangent bundles with horizontal Sasaki gradient metric. Transactions of National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences, Issue Mathematics 2020; 40 (4): 188-197.