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Jackson-type theorem in the weak L_1-space

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Abstract: The weak L_1-space meets in many areas of mathematics. For example, the conjugate functions of Lebesgue integrable functions belong to the weak L_1-space. The difficulty of working with the weak L_1-space is that the weak L_1-space is not a normed space. Moreover, infinitely differentiable (even continuous) functions are not dense in this space. Due to this, the theory of approximation was not produced in this space. In the present paper, we introduced the concept of the modulus of continuity of the functions from the weak L_1-space, studied its properties, found a criterion for convergence to zero of the modulus of continuity of the function from the weak L_1-space, and proved in this space an analogue of the Jackson-type theorem.

Key words: Modulus of continuity, weak space, Jackson-type theorem, best approximation

1. Introduction

Let $L_p(T)$, $1 \le p < \infty$, the space of all measurable 2π-periodic functions with finite $L_p(T)$-norm $||f||_p = \left(\pi^{-1} \int_T |f(x)|^p dx\right)^{1/p}$, and $L_\infty(T) \equiv C(T)$ the space of all continuous 2π-periodic functions with uniform norm $||f||_\infty = \max_{x\in T} |f(x)|$, where $T = [-\pi, \pi]$; let $E_n(f)_p$ be the best approximation of a function f in the metric $L_p(T)$ by trigonometric polynomials of order at most n, $n \in Z_+$; and let $\omega(f,\delta)_p = \sup_{0<h\le\delta} \|f(\cdot + h) - f(\cdot)\|_{L_p(T)}$, $\delta \ge 0$.

It was proved by D. Jackson that (see, for example, [8]) if $f \in L_p(T)$, $1 \le p \le \infty$, then

$$E_n(f)_p \le c \cdot \omega\left(f, \frac{\pi}{n+1}\right)_p, \quad n = 0, 1, 2, 3, ...,$$

where c is an absolute constant.

This central theorem gave impetus to the intensive development of approximation theory in the spaces L_p. Further, for the development of the theory of approximation in other function spaces, an analogue of Jackson's theorem in these spaces was obtained (see [1–7, 9–12, 14–20] and many references therein).

The space of all functions measurable on $[a,b]$ with bounded quasi-norm

$$\|f\|_{WL_1([a,b])} = \sup_{\lambda>0} \lambda \cdot m\{x \in [a,b] : |f(x)| \ge \lambda\} \tag{1.1}$$

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is called a weak L_1-space and is denoted by $WL_1([a,b])$, where m - is the Lebesgue measure.

Note that for any $f,g \in WL_1([a,b])$ the inequality

$$\|f+g\|_{WL_1([a,b])} \le \left(\sqrt{\|f\|_{WL_1([a,b])}} + \sqrt{\|g\|_{WL_1([a,b])}}\right)^2 \le 2\left(\|f\|_{WL_1([a,b])} + \|g\|_{WL_1([a,b])}\right) \tag{1.2}$$

shows that (1.1) is indeed a quasi-norm.

The weak L_1-space meets in many areas of mathematics. For example, the conjugate functions of Lebesgue integrable functions belong to the weak L_1-space (see, for example, [13]). The difficulty of working with the weak L_1-space is that the weak L_1-space is not a normed space. Moreover, infinitely differentiable (even continuous) functions are not dense in this space. Due to this, the theory of approximation was not produced in this space. In the present paper, we introduced the concept of the modulus of continuity $\omega_{weak}(f;\delta),\ \delta>0$ of the functions $f \in WL_1([a,b])$, studied its properties, found a criterion for convergence to zero of the modulus of continuity $\omega_{weak}(f;\delta)$ as $\delta \to 0$, and proved in this space an analogue of the Jackson-type theorem.

2. Modulus of continuity of functions from a weak L_1-space and its properties

For each function $f \in WL_1([a,b])$ we put

$$\omega_{weak}(f;\delta) = \sup_{0<h\le\delta} \|f(\cdot+h)-f(\cdot)\|_{WL_1([a,b-h])}$$

$$= \sup_{0<h\le\delta} (\sup_{\lambda>0} \lambda \cdot m\{x \in [a,b-h]:\ |f(x+h)-f(x)| \ge \lambda\}),\ 0<\delta\le b-a,$$

$$\omega^*_{weak}(f;\delta) = \sup_{0<h\le\delta} \|f(\cdot+h)-f(\cdot)\|_{WL_1([a,b])}$$

$$= \sup_{0<h\le\delta} (\sup_{\lambda>0} \lambda \cdot m\{x \in [a,b]:\ |f(x+h)-f(x)| \ge \lambda\}),\ \delta>0,$$

where in the second case f is assumed to be extended by periodicity with period $b-a$. The quantities $\omega_{weak}(f;\delta)$ and $\omega^*_{weak}(f;\delta)$ are called moduli of continuity of the function $f \in WL_1([a,b])$ ($\omega^*_{weak}(f;\delta)$ is the periodic modulus of continuity).

We note some properties of the modulus of continuity $\omega_{weak}(f;\delta)$.

Property 1. For any $f \in WL_1([a,b])$, the modulus of continuity $\omega_{weak}(f;\delta)$ is a nondecreasing function.

Property 2. For any $f \in WL_1([a,b])$ and $0<\delta\le b-a$

$$\omega_{weak}(f;\delta) \le 4||f||_{WL_1([a,b])}.$$

Property 3. For any $f,g \in WL_1([a,b])$ and $0<\delta\le b-a$

$$\omega_{weak}(f+g;\delta) \le \left(\sqrt{\omega_{weak}(f;\delta)} + \sqrt{\omega_{weak}(g;\delta)}\right)^2 \le 2(\omega_{weak}(f;\delta)+\omega_{weak}(g;\delta)).$$

Property 4. If $f \in WL_1([a,b])$, then for every $\delta_1,\delta_2>0,\ \delta_1+\delta_2\le b-a$

$$\omega_{weak}(f;\delta_1+\delta_2) \le \left(\sqrt{\omega_{weak}(f;\delta_1)} + \sqrt{\omega_{weak}(f;\delta_2)}\right)^2 \le 2(\omega_{weak}(f;\delta_1)+\omega_{weak}(f;\delta_2)).$$

Property 5. If $f \in WL_1([a,b])$, then for any $k \in N$ and $0 < \delta \leq \frac{b-a}{n}$

$$\omega_{weak}(f;k\delta) \leq k^2 \omega_{weak}(f;\delta),$$

where N is the set of natural numbers.

Property 1 is obvious, properties 2, 3, and 4 follow from (1.2), and property 5 follows from property 4. Indeed, for $k = 1$, property 5 is obvious. If property 5 holds for $k \in N$, then it follows from property 4 that

$$\omega_{weak}(f;(k+1)\delta) \leq (\sqrt{\omega_{weak}(f;k\delta)} + \sqrt{\omega_{weak}(f;\delta)})^2 \leq (k+1)^2 \omega_{weak}(f;\delta),$$

which means that property 5 holds for $k+1$. It follows from mathematical induction that property 5 holds for any $k \in N$.

But the equality

$$\lim_{\delta \to 0+} \omega_{weak}(f;\delta) = 0 \tag{2.1}$$

is generally not satisfied. For example, the function $f(x) = \frac{1}{x}$ belongs to the class $WL_1([0,1])$, but for any $\delta > 0$ we have $\omega_{weak}(f;\delta) = 1$, and, therefore, equality (2.1) is not satisfied for this function.

Theorem 2.1 *The modulus of continuity of the function $f \in WL_1([a,b])$ satisfies equality (2.1) iff*

$$\lim_{\lambda \to +\infty} \lambda \cdot m\{x \in [a,b] : |f(x)| \geq \lambda\} = 0. \tag{2.2}$$

Proof **Necessity.** Let the equality (2.1) holds. Let us prove that the equality (2.2) holds. Assume that the (2.2) does not hold. Then

$$\limsup_{\lambda \to +\infty} \lambda \cdot m\{x \in [a,b] : |f(x)| \geq \lambda\} = \alpha_0 > 0.$$

It follows that there is a sequence of numbers $\{\lambda_n\}_{n=1}^{\infty}$, such that $\lim_{n\to\infty} \lambda_n = +\infty$ and for any $n \in N$

$$m\{x \in [a,b] : |f(x)| \geq \lambda_n\} > \frac{\alpha_0}{2\lambda_n}. \tag{2.3}$$

Denote $\varepsilon_0 = \frac{\alpha_0}{16} > 0$. It follows from (2.1) that there exists a number $0 < \delta_0 < \frac{b-a}{2}$ such that for any $0 < h \leq \delta_0$ and $\lambda > 0$

$$\lambda \cdot m\{x \in [a, b-h] : |f(x+h) - f(x)| \geq \lambda\} \leq \varepsilon_0. \tag{2.4}$$

Denote

$$\Phi_n = \{(x,h) : \ 0 < h \leq \delta_0, \ x \in [a, b-h], \ |f(x+h) - f(x)| \geq \lambda_n/2\}.$$

It follows from (2.4) that for any $0 < h \leq \delta_0$

$$m\{x \in [a, b-h] : |f(x+h) - f(x)| \geq \lambda_n/2\} \leq \frac{2\varepsilon_0}{\lambda_n}.$$

This implies the estimate

$$m(\Phi_n) \leq \frac{2\varepsilon_0}{\lambda_n} \cdot \delta_0 = \frac{\alpha_0}{8\lambda_n} \cdot \delta_0, \tag{2.5}$$

where $m(\Phi_n)$ denotes the Lebesgue measure of the set Φ_n.

It follows from inclusions

$$\{x \in [a, b-h] : |f(x)| \geq \lambda_n \wedge |f(x+h)| < \lambda_n/2\} \subset \{x \in [a, b-h] : |f(x+h) - f(x)| \geq \lambda_n/2\},$$

$$\{x \in [a, b-h] : |f(x+h)| \geq \lambda_n \wedge |f(x)| < \lambda_n/2\} \subset \{x \in [a, b-h] : |f(x+h) - f(x)| \geq \lambda_n/2\},$$

that

$$2m(\Phi_n) \geq m\{(x,h) : \ 0 < h \leq \delta_0, \ x \in [a, b-h], \ |f(x)| \geq \lambda_n \wedge |f(x+h)| < \lambda_n/2\}$$

$$+m\{(x,h) : \ 0 < h \leq \delta_0, \ x \in [a, b-h], \ |f(x+h)| \geq \lambda_n \wedge |f(x)| < \lambda_n/2\}$$

$$= m\{(x,h) : \ 0 < h \leq \delta_0, \ x \in [a, b-h], \ |f(x)| \geq \lambda_n \wedge |f(x+h)| < \lambda_n/2\}$$

$$+m\{(x,h) : \ 0 < h \leq \delta_0, \ x \in [a+h, b], \ |f(x)| \geq \lambda_n \wedge |f(x-h)| < \lambda_n/2\}$$

$$= m\{(x,h) : x \in [a,b], \ 0 < h \leq \min\{\delta_0, b-x\}, \ \ |f(x)| \geq \lambda_n \wedge |f(x+h)| < \lambda_n/2\}$$

$$+m\{(x,h) : x \in [a,b], \ 0 < h \leq \min\{\delta_0, x-a\}, \ \ |f(x)| \geq \lambda_n \wedge |f(x-h)| < \lambda_n/2\}$$

Considering that for any $x \in [a,b]$

$$m\{h : 0 < h \leq \min\{\delta_0, b-x\}, \ |f(x+h)| < \lambda_n/2\}$$

$$+m\{h : 0 < h \leq \min\{\delta_0, x-a\}, \ |f(x-h)| < \lambda_n/2\}$$

$$= \min\{\delta_0, b-x\} - m\{h : 0 < h \leq \min\{\delta_0, b-x\}, \ |f(x+h)| \geq \lambda_n/2\}$$

$$+ \min\{\delta_0, x-a\} - m\{h : 0 < h \leq \min\{\delta_0, x-a\}, \ |f(x+h)| \geq \lambda_n/2\}$$

$$\geq \min\{\delta_0, b-x\} - \frac{2}{\lambda_n}\|f\|_{WL_1([a,b])} + \min\{\delta_0, x-a\} - \frac{2}{\lambda_n}\|f\|_{WL_1([a,b])}$$

$$\geq \delta_0 - \frac{4}{\lambda_n}\|f\|_{WL_1([a,b])},$$

due to inequality (2.3) we get

$$m(\Phi_n) \geq \frac{\alpha_0}{4\lambda_n} \cdot \left(\delta_0 - \frac{4}{\lambda_n}\|f\|_{WL_1([a,b])}\right). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\delta_0 - \frac{4}{\lambda_n}\|f\|_{WL_1([a,b])} \leq \delta_0/2.$$

But this is impossible due to the condition $\lim_{n\to\infty} \lambda_n = +\infty$. The resulting contradiction proves the validity of equality (2.2).

Sufficiency. Let the equality (2.2) holds. Let us prove that the equality (2.1) holds. Assume that the (2.1) is not held. Then there are the number ε_0 and the sequences of positive numbers $\{h_n\}$, $\{\lambda_n\}$ such that $\lim_{n\to\infty} h_n = 0$ and for any $n \in N$

$$\lambda_n \cdot m\{x \in [a, b-h_n] : |f(x+h_n) - f(x)| \geq \lambda_n\} > \varepsilon_0. \tag{2.7}$$

It follows from the inclusion

$$\{x \in [a, b-h_n] : |f(x+h_n) - f(x)| \geq \lambda_n\}$$

$$\subset \{x \in [a, b - h_n] : |f(x)| \geq \lambda_n/2\} \cup \{x \in [a, b - h_n] : |f(x + h_n)| \geq \lambda_n/2\}$$

that

$$m\{x \in [a, b - h_n] : |f(x + h_n) - f(x)| \geq \lambda_n\} \leq 2m\{x \in [a, b] : |f(x)| \geq \lambda_n/2\}.$$

From here and from (2.7) it follows that

$$m\{x \in [a, b] : |f(x)| \geq \lambda_n/2\} > \frac{\varepsilon_0}{2\lambda_n}. \tag{2.8}$$

Inequalities (2.2) and (2.8) show that the sequence $\{\lambda_n\}$ is bounded. Therefore the sequence $\{\lambda_n\}$ has a convergent subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$. Let

$$\lambda_0 = \lim_{k \to \infty} \lambda_{n_k}.$$

It follows from (2.7) that

$$\lambda_0 \geq \frac{\varepsilon_0}{b - a} > 0.$$

Therefore there exists $k_o \in N$ that for any $k > k_0$

$$\lambda_0/2 < \lambda_{n_k} < 2\lambda_0.$$

Then from inequality (2.7) we obtain that for $k > k_0$

$$m\{x \in [a, b - h_{n_k}] : |f(x + h_{n_k}) - f(x)| \geq \lambda_0/2\}$$

$$\geq m\{x \in [a, b - h_{n_k}] : |f(x + h_{n_k}) - f(x)| \geq \lambda_{n_k}\} > \frac{\varepsilon_0}{\lambda_{n_k}} > \frac{\varepsilon_0}{2\lambda_0}. \tag{2.9}$$

It follows from (2.2) that there exists $M_0 > 0$ such that

$$m\{x \in [a, b] : |f(x)| \geq M_0\} < \frac{\varepsilon_0}{8\lambda_0}. \tag{2.10}$$

Denote

$$f_1(x) = f(x), \quad for \quad |f(x)| \leq M_0; \quad f_1(x) = 0, \quad for \quad |f(x)| > M_0,$$

$$f_2(x) = 0, \quad for \quad |f(x)| \leq M_0; \quad f_2(x) = f(x), \quad for \quad |f(x)| > M_0.$$

Then for any $x \in [a, b]$ we have $f(x) = f_1(x) + f_2(x)$. It follows from the inclusion

$$\{x \in [a, b - h_{n_k}] : |f(x + h_{n_k}) - f(x)| \geq \lambda_0/2\}$$

$$\subset \{x \in [a, b - h_{n_k}] : |f_1(x + h_{n_k}) - f_1(x)| \geq \lambda_0/4\} \cup \{x \in [a, b - h_{n_k}] : |f_2(x + h_{n_k}) - f_2(x)| \geq \lambda_0/4\}$$

and from (2.10) that

$$m\{x \in [a, b - h_{n_k}] : |f(x + h_{n_k}) - f(x)| \geq \lambda_0/2\}$$

$$\leq m\{x \in [a, b - h_{n_k}] : |f_1(x + h_{n_k}) - f_1(x)| \geq \lambda_0/4\} + \frac{\varepsilon_0}{4\lambda_0}. \tag{2.11}$$

Then it follows from (2.9) and (2.11) that

$$m\{x \in [a, b-h_{n_k}] : |f_1(x+h_{n_k}) - f_1(x)| \geq \lambda_0/4\} \geq \frac{\varepsilon_0}{4\lambda_0}. \quad (2.12)$$

Since the function f_1 is bounded, it is Lebesgue integrable on $[a,b]$. It follows from Lebesgue's theorem that

$$\lim_{h\to 0+} \int_a^{b-h} |f_1(x+h) - f_1(x)|dx = 0.$$

Therefore

$$\lim_{k\to\infty} \int_a^{b-h_{n_k}} |f_1(x+h_{n_k}) - f_1(x)|dx = 0. \quad (2.13)$$

On the other hand, it follows from (2.12) that for any $k \in N$

$$\int_a^{b-h_{n_k}} |f_1(x+h_{n_k}) - f_1(x)|dx \geq \frac{\lambda_0}{4} \cdot m\{x \in [a, b-h_{n_k}] : |f_1(x+h_{n_k}) - f_1(x)| \geq \lambda_0/4\} \geq \frac{\varepsilon_0}{16}.$$

But this is impossible due to (2.13). The resulting contradiction proves the validity of equality (2.1). □

Definition 2.2 *Denote by $WA([a,b])$ the class of functions $f \in WL_1([a,b])$ satisfying condition* (2.2).

Theorem 2.1 shows that in the class of functions $WA([a,b])$ the modulus of continuity $\omega_{weak}(f;\delta)$ satisfies condition (2.1).

Note that properties 1-5 and theorem 2.1 also hold for the modulus of continuity $\omega^*_{weak}(f;\delta)$.

3. The best approximations and Jackson-type theorem in the weak L_1-space

Let $f \in WL_1([-\pi,\pi])$ be a 2π-periodic function. For any $n \in Z_+ = N \cup 0$ we write

$$E_n(f)_{weak} = \inf ||f - T_n||_{WL_1([-\pi,\pi])}$$

for the best approximation in $WL_1([-\pi,\pi])$ to f by trigonometric polynomials, where the infimum is taken over all trigonometric polynomials of order at most n.

Theorem 3.1 *For any function $f \in WA([-\pi,\pi])$ the inequality*

$$E_n(f)_{weak} \leq c \cdot \omega^*_{weak}(f; \frac{\pi}{n+1}), \quad n \in Z_+ \quad (3.1)$$

holds, where c is an absolute constant.

Proof From the condition $f \in WA([-\pi,\pi])$ it follows that

$$\lim_{\lambda\to+\infty} \lambda \cdot m\{x \in [-\pi,\pi] : \ |f(x)| \geq \lambda\} = 0.$$

Therefore, for any $\varepsilon > 0$ and $n \in N$ there exist $\lambda_n(\varepsilon) > 1$ such that for any $\lambda \geq \lambda_n(\varepsilon)$

$$\lambda \cdot m\{x \in [-\pi,\pi] : \ |f(x)| \geq \lambda\} < \varepsilon \cdot \omega^*_{weak}(f; \frac{\pi}{2n}) \quad (3.2)$$

Denote

$$f_1(x) = f(x), \quad \textbf{for} \quad |f(x)| \le \lambda_n(\varepsilon); \quad f_1(x) = 0, \quad \textbf{for} \quad |f(x)| > \lambda_n(\varepsilon),$$

$$f_2(x) = 0, \quad \textbf{for} \quad |f(x)| \le \lambda_n(\varepsilon); \quad f_2(x) = f(x), \quad \textbf{for} \quad |f(x)| > \lambda_n(\varepsilon).$$

Then it follows from inequality (3.2) that

$$||f_2||_{WL_1([-\pi,\pi])} = \sup_{\lambda>0} \lambda \cdot m\{x \in [-\pi,\pi] : |f_2(x)| \ge \lambda\}$$

$$= \sup_{\lambda>\lambda_n(\varepsilon)} \lambda \cdot m\{x \in [-\pi,\pi] : |f(x)| \ge \lambda\} \le \varepsilon \cdot \omega^*_{weak}(f; \frac{\pi}{2n}). \tag{3.3}$$

Denote

$$U_n(x) = \frac{3}{2\pi n(2n^2+1)} \cdot \int_{-\pi}^{\pi} f_1(t) \cdot \left[\frac{\sin n \cdot \frac{t-x}{2}}{\sin \frac{t-x}{2}}\right]^4 dt, \quad x \in [-\pi,\pi], \quad n \in N.$$

It is well known, that (see, for example, [8]) $U_n(x)$ is a trigonometric polynomial of order at most $2n-2$ and

$$U_n(x) - f_1(x) = \frac{3}{\pi n(2n^2+1)} \cdot \int_0^{\pi/2} [f_1(x+2t) + f_1(x-2t) - 2f_1(x)] \cdot \left[\frac{\sin nt}{\sin t}\right]^4 dt$$

Then for any $\lambda > 0$ we have

$$m\{x \in [-\pi,\pi] : \ |U_n(x) - f_1(x)| \ge \lambda\}$$

$$\le m\left\{x \in [-\pi,\pi] : \ \int_0^{\pi/2} |f_1(x+2t) + f_1(x-2t) - 2f_1(x)| \cdot \left|\frac{\sin nt}{\sin t}\right|^4 dt \ge \lambda \cdot \frac{\pi n(2n^2+1)}{3}\right\}$$

$$\le m\left\{x \in [-\pi,\pi] : \ \int_0^{\pi/2} |f_1(x+2t) - f_1(x)| \cdot \left|\frac{\sin nt}{\sin t}\right|^4 dt \ge \lambda \cdot \frac{\pi n(2n^2+1)}{6}\right\}$$

$$+ m\left\{x \in [-\pi,\pi] : \ \int_0^{\pi/2} |f_1(x) - f_1(x-2t)| \cdot \left|\frac{\sin nt}{\sin t}\right|^4 dt \ge \lambda \cdot \frac{\pi n(2n^2+1)}{6}\right\}$$

$$\le m\left\{x \in [-\pi,\pi] : \ \int_0^{\pi/2n} |f_1(x+2t) - f_1(x)| \cdot \left|\frac{\sin nt}{\sin t}\right|^4 dt \ge \lambda \cdot \frac{\pi n(2n^2+1)}{12}\right\}$$

$$+ m\left\{x \in [-\pi,\pi] : \ \int_{\pi/2n}^{\pi/2} |f_1(x+2t) - f_1(x)| \cdot \left|\frac{\sin nt}{\sin t}\right|^4 dt \ge \lambda \cdot \frac{\pi n(2n^2+1)}{12}\right\}$$

$$+ m\left\{x \in [-\pi,\pi] : \ \int_0^{\pi/2n} |f_1(x) - f_1(x-2t)| \cdot \left|\frac{\sin nt}{\sin t}\right|^4 dt \ge \lambda \cdot \frac{\pi n(2n^2+1)}{12}\right\}$$

$$+ m\left\{x \in [-\pi,\pi] : \ \int_{\pi/2n}^{\pi/2} |f_1(x) - f_1(x-2t)| \cdot \left|\frac{\sin nt}{\sin t}\right|^4 dt \ge \lambda \cdot \frac{\pi n(2n^2+1)}{12}\right\}$$

$$= J_1 + J_2 + J_3 + J_4. \tag{3.4}$$

It follows from the inequalities

$$|\sin nt| \le n|\sin t|, \quad 0 \le t \le \pi/2n;$$

$$|\sin nt| \le 1, \quad |\sin t| \ge \frac{2}{\pi}t, \quad 0 \le t \le \pi/2$$

and from property 2.5 that

$$J_1 \le m\left\{x \in [-\pi,\pi]: \int_0^{\pi/2n} |f_1(x+2t) - f_1(x)|dt \ge \lambda \cdot \frac{\pi(2n^2+1)}{12n^3}\right\}$$

$$\le m\left\{x \in [-\pi,\pi]: \int_0^{\pi/2n} |f_1(x+2t) - f_1(x)|dt \ge \lambda \cdot \frac{\pi}{6n}\right\}$$

$$\le \frac{3}{\lambda} \sup_{0\le h\le \pi/n} \sup_{\mu>0} \mu \cdot m\{x \in [-\pi,\pi]: |f_1(x+h) - f_1(x)| \ge \mu\} = \frac{3}{\lambda}\omega^*_{weak}(f_1;\frac{\pi}{n})$$

$$\le \frac{12}{\lambda}\omega^*_{weak}(f_1;\frac{\pi}{2n}), \tag{3.5}$$

$$J_2 \le m\left\{x \in [-\pi,\pi]: \int_{\pi/2n}^{\pi/2} |f_1(x+2t) - f_1(x)|\frac{dt}{t^4} \ge \lambda \cdot \frac{16}{\pi^4} \cdot \frac{\pi(2n^2+1)}{12n^3}\right\}$$

$$\le m\left\{x \in [-\pi,\pi]: \int_{\pi/2n}^{\pi/n} |f_1(x+2t) - f_1(x)|\frac{dt}{t^4} \ge \lambda \cdot \frac{8n^3}{3\pi^3}\right\}.$$

Let $n = 2^m + b$, where $0 \le b < 2^m$. Then, we have

$$J_2 \le m\left\{x \in [-\pi,\pi]: \sum_{k=0}^{m} \int_{2^k\cdot\pi/2n}^{2^{k+1}\cdot\pi/2n} |f_1(x+2t) - f_1(x)|\frac{dt}{t^4} \ge \lambda \cdot \frac{8n^3}{3\pi^3}\right\}$$

$$\le \sum_{k=0}^{m} m\left\{x \in [-\pi,\pi]: \int_{2^k\cdot\pi/2n}^{2^{k+1}\cdot\pi/2n} |f_1(x+2t) - f_1(x)|\frac{dt}{t^4} \ge \lambda \cdot \frac{8n^3}{3\pi^3(k+1)(k+2)}\right\}$$

$$\le \sum_{k=0}^{m} m\left\{x \in [-\pi,\pi]: \int_{2^k\cdot\pi/2n}^{2^{k+1}\cdot\pi/2n} |f_1(x+2t) - f_1(x)|dt \ge \lambda \cdot \frac{\pi\cdot 2^{4k}}{6n(k+1)(k+2)}\right\}$$

$$\le \sum_{k=0}^{m} \frac{3(k+1)(k+2)}{\lambda\cdot 2^{3k}} \sup_{0\le h\le 2^k\cdot\pi/2n} \sup_{\mu>0} \mu \cdot m\{x \in [-\pi,\pi]: |f_1(x+h) - f_1(x)| \ge \mu\}$$

$$= \sum_{k=0}^{m} \frac{3(k+1)(k+2)}{\lambda\cdot 2^{3k}} \cdot \omega^*_{weak}(f_1;\frac{\pi}{2n}\cdot 2^k)$$

$$\le \frac{3}{\lambda}\sum_{k=0}^{m} \frac{(k+1)(k+2)}{2^k} \cdot \omega^*_{weak}(f_1;\frac{\pi}{2n})$$

$$\le \frac{48}{\lambda} \cdot \omega^*_{weak}(f_1;\frac{\pi}{2n}). \tag{3.6}$$

Similarly,

$$J_3 \leq \frac{12}{\lambda} \omega^*_{weak}(f_1; \frac{\pi}{2n}), \tag{3.7}$$

$$J_4 \leq \frac{48}{\lambda} \cdot \omega^*_{weak}(f_1; \frac{\pi}{2n}). \tag{3.8}$$

It follows from (3.4), (3.5), (3.6), (3.7), and (3.8) that the inequality

$$m\{x \in [-\pi, \pi] : \ |U_n(x) - f_1(x)| \geq \lambda\} \leq \frac{120}{\lambda} \cdot \omega^*_{weak}(f_1; \frac{\pi}{2n}).$$

holds for any $\lambda > 0$. This shows that

$$||U_n - f_1||_{WL_1([-\pi,\pi])} \leq 120 \cdot \omega^*_{weak}(f_1; \frac{\pi}{2n}). \tag{3.9}$$

It follows from (3.3), (3.9) and from properties 2.2, 2.3 that

$$||U_n - f_1||_{WL_1([-\pi,\pi])} \leq 120 \cdot \left(\sqrt{\omega^*_{weak}(f; \frac{\pi}{2n})} + \sqrt{\omega^*_{weak}(f_2; \frac{\pi}{2n})} \right)^2$$

$$\leq 120(1 + 2\sqrt{\varepsilon})^2 \cdot \omega^*_{weak}(f; \frac{\pi}{2n}). \tag{3.10}$$

From (1.2), (3.3), and (3.10) we have

$$||U_n - f||_{WL_1([-\pi,\pi])} \leq \left(\sqrt{||U_n - f_1||_{WL_1([-\pi,\pi])}} + \sqrt{||f_2||_{WL_1([-\pi,\pi])}} \right)^2$$

$$\leq \left(\sqrt{120}(1 + 2\sqrt{\varepsilon}) + \sqrt{\varepsilon} \right)^2 \cdot \omega^*_{weak}(f; \frac{\pi}{2n}).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$||U_n - f||_{WL_1([-\pi,\pi])} \leq 120 \cdot \omega^*_{weak}(f; \frac{\pi}{2n}). \tag{3.11}$$

Since U_n is a trigonometric polynomial of order at most $2n - 2$, the inequality (3.1) follows from (3.11). □

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