Turkish Journal of Mathematics

Volume 47 | Number 1 Article 10

1-1-2023

On the properties of solutions for nonautonomous third-order stochastic differential equation with a constant delay

AYMAN MOHAMMED MAHMOUD

DOAA ALI MOHAMED BAKHIT

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 135 – 158
© TÜBİTAK
doi:10.55730/1300-0098.3351

On the properties of solutions for nonautonomous third-order stochastic differential equation with a constant delay

Ayman Mohammed MAHMOUD, Doaa Ali Mohamed BAKHIT*
Department of Mathematics, Faculty of Science, New Valley University, El Kharga, Egypt

Received: 24.06.2021 • **Accepted/Published Online:** 15.11.2022 • **Final Version:** 13.01.2023

Abstract: In this work, complete Lyapunov functionals (LFs) are constructed and used for the established conditions on the nonlinear functions appearing in the main equation, to guarantee stochastically asymptotically stable (SAS), uniformly stochastically bounded (USB) and uniformly exponentially asymptotically stable (UEAS) in probability of solutions to the nonautonomous third-order stochastic differential equation (SDE) with a constant delay as

$$\dddot{x}(t) + a(t)f(x(t), \dot{x}(t))\ddot{x}(t) + b(t)\varphi(x(t))\dot{x}(t) + c(t)\psi(x(t-r))$$
$$+ g(t, x)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)).$$

In Section 4, we give two numerical examples as an application to illustrate the results.

Key words: (DDE), (SDE), (SDDE), (SAS), (USB), (UEAS)

1. Introduction

Stochastic differential equation (SDE) is typically a dynamical system endowing random components that models the evolution over time of particular phenomena that is subject to uncertainty. For instance the evolution of a financial asset, risk assessment in insurance policy, etc.

To the best of our knowledge, SDE of the third-order with or without time-varying delays naturally appears in multiple applications, where deterministic models are perturbed by the white noise or its generalizations [13, 23, 27, 28]. In most cases, SDEs are understood as a continuous time limit of the corresponding SDEs. This understanding of SDEs is ambiguous and must be complemented by a proper mathematical definition of the corresponding integral. Such a mathematical definition was first proposed by Kiyosi It$\hat{o}$, leading to what is known today as the It$\hat{o}$ formula (IF). Mathematically, stochastic delay differential equations (SDDEs) were first introduced by It$\hat{o}$ and Nisio [15], in which the existence and uniqueness of the solutions have been investigated.

More than one hundred years ago, Lyapunov introduced the concept of stability of a dynamical system and created a very powerful tool known as the Lyapunov's second method (LSM) in the study of stability and boundedness. A manifest advantage of this method is that it does not require the knowledge of solutions for equations and thus has exhibited a great power in applications. In general, many results have been obtained on uniformly stochastically stable (USS) and USB of solutions for delay differential equations (DDEs) by using LSM. See, for example [1, 2, 9, 10, 14, 20, 29–32, 34–41], and the references cited in therein.

*Correspondence: doaa.ali@sci.nvu.edu.eg; doaa_math90@yahoo.com
2010 *AMS Mathematics Subject Classification:* 34D20, 35B5, 34K50

In the last decades, SDDEs have attracted a great interest in the literature because of their applications in characterizing many problems in physics, mechanics, electrical engineering, biology, ecology and so on. See [17, 24–26]. Some related papers will be presented on the kind of SDDEs. See, for example [3–8, 11, 12, 16, 19, 21, 22, 42].

Recently, Mahmoud and Tunç [22] investigated new criteria for USB and UEAS for a certain third-order SDDE as

$$\dddot{x}(t) + \Phi(x,\dot{x})\ddot{x} + G(\dot{x}(t-r)) + F(x(t-r)) + \sigma x(t-h(t))\dot{\omega}(t) = P(t,x,\dot{x},\ddot{x}).$$

The purpose of this paper is to establish the sufficient conditions for SAS of the zero solution, USB and UESA of all solution to third-order SDE with a constant delay as

$$\begin{aligned} &\dddot{x}(t) + a(t)f(x(t),\dot{x}(t))\ddot{x}(t) + b(t)\varphi(x(t))\dot{x}(t) + c(t)\psi(x(t-r)) \\ &\quad + g(t,x)\dot{\omega}(t) = p(t,x(t),\dot{x}(t),\ddot{x}(t)), \end{aligned} \tag{1.1}$$

where $a(t)$, $b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty)$, also $f(x,y), \varphi(x), g(t,x)$, and $\psi(x)$ are continuous functions with $\varphi(0)=\psi(0)=0$, $\omega(t)\in\mathbb{R}^n$ is standard Brownian motion, $\dot{\omega}(t)=\frac{d\omega}{dt}$. In this study we will consider the derivatives $a'(t)=\frac{da(t)}{dt}$, $b'(t)=\frac{db(t)}{dt}, c'(t)=\frac{dc(t)}{dt}$, $\psi'=\frac{d\psi}{dx}$ and $f_x(x,y)=\frac{\partial f(x,y)}{\partial x}$.

Remark 1.1 *We will give the following remarks:*

(*1*) *In [37], Sadek investigated the asymptotic stability of DDE. Comparing his equation to (1.1), we find* $f(x(t),\dot{x}(t))=1$ *and* $\varphi(x(t))=1$ *with* $p=0$, *then (1.1) can be reduced to Sadek's equation without the stochastic term.*

(*2*) *The obtained results in [3, 6, 7, 12, 42] are on second-order SDDE.*

(*3*) *In (1.1), if we let* $a(t)=a,\ b(t)=b,\ f(x(t),\dot{x}(t))=1,\ \varphi(x(t))=1,\ c(t)=1$ *and* $g(t,x)=\sigma x(t)$, *we note that the equation studied by Ademola [8] represents a special case from (1.1) in this study.*

(*4*) *In 2015, Abou El-Ela and his students have begun studding the behaviour of SDDE, see [3–6] and then Mahmoud et al. continued the studding of the SDDE, see [19, 21, 22]. Our paper represents a generalization of all the above studding. For example, in (1.1), let* $a(t)=b(t)=c(t)=1$, $\varphi(x(t))\dot{x}(t)=G(\dot{x}(t-r))$ *and* $g(t,x)=\sigma x(t-h(t))$, *it tends to the SDDE studied in [22].*

2. Stability result

Let $B(t)=(B_1(t),\ldots,B_m(t))$ be an m-dimensional Brownian motion defined on the probability space. Consider a nonautonomous n-dimensional SDDE

$$dx(t) = M(t,x(t),x(t-r))dt + N(t,x(t),x(t-r))dB(t),\ \forall\, t\geq 0, \tag{2.1}$$

with initial data $\{x(\theta): -r<\theta<0\}, x_0\in\mathcal{C}([-r,0];\mathbb{R}^n)$. Suppose that $M:\mathbb{R}^+\times\mathbb{R}^{2n}\to\mathbb{R}^n$ and $N:\mathbb{R}^+\times\mathbb{R}^{2n}\to\mathbb{R}^{n\times m}$ satisfy the local Lipschitz condition and the linear growth condition. Hence, for any given initial value $x(0)=x_0\in\mathbb{R}^n$, it is therefore known that Equation (2.1) has a unique continuous solution on

$t \geq 0$, which is known as $x(t;x_0)$ in this section. Suppose that $M(t,0,0)=0$ and $N(t,0,0)=0$, for all $t \geq 0$. Hence, the SDDE admits the zero solution $x(t;0) \equiv 0$ for any given initial value $x_0 \in \mathcal{C}([-r,0];\mathbb{R}^n)$.

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n;\mathbb{R}^+)$ denote the family of nonnegative functions $W(t,x_t)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, which are once continuously differentiable in t and twice continuously differentiable in x.

By IF we have

$$dW(t,x_t) = \mathcal{L}W(t,x_t)dt + W_x(t,x_t)N(t,x_t)dB(t),$$

where

$$\begin{aligned} \mathcal{L}W(t,x_t) &= W_t(t,x_t) + W_x(t,x_t)M(t,x_t) \\ &\quad + \frac{1}{2} trace[N^T(t,x_t)W_{xx}(t,x_t)N(t,x_t)], \end{aligned} \tag{2.2}$$

where $W_x = (W_{x_1},\ldots,W_{x_n})$ and $W_{xx} = (W_{x_i x_j})_{n\times n}$, $i, j = 0,1,2,...,n$. Moreover, let $\mathcal{K}$ denote the family of all continuous nondecreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0)=0$ and $\rho(r)>0$, if $r>0$. Here, we will use the diffusion operator $\mathcal{L}W(t,x_t)$ defined in (2.2) to replace $W'(t,x_t) = \frac{d}{dt}W(t,x_t)$.

Theorem 2.1 [8, 13] *Assume that there exist $W \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n;\mathbb{R}^+)$ and ρ_1, ρ_2, $\rho_3 \in \mathcal{K}$ such that*

$$\rho_1(\|x\|) \leq W(t,x_t) \leq \rho_2(\|x\|),$$

and

$$\mathcal{L}W(t,x_t) \leq -\rho_3(\|x\|), \quad \forall\,(t,x_t) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then, the zero solution of the SDDE (2.1) is SAS.

Now, we can give the following main theorem.

Theorem 2.2 *In addition to the basic fundamental assumptions imposed on the functions f, φ and ψ appearing in Equation (1.1), suppose that there exist the positive constants a_1, a_2, b_1, b_2, f_1 L_0, L, $c_1, \varphi_2, \alpha, \sigma$ and ψ_0, and the negative constant a_0 such that the following conditions are satisfied:*

(i) $a_1 \leq a(t) \leq a_2$, $b_1 \leq b(t) \leq b_2 \leq 1$, *and* $c_1 \leq c(t) \leq 1$, *for all* $t \geq 0$.

(ii) $a'(t) \leq a_0$, $-\sigma \leq b'(t) \leq c'(t) \leq a'(t) \leq 0$, $t \geq 0$.

(iii) $1 \leq f(x,y) \leq f_1$, *with* $f_x(x,y) < 0$ *for all* x, y, $1 \leq \varphi(x) \leq \varphi_2$,
$\frac{\psi(x)}{x} \geq L_0$, $x \neq 0$, *and* $|\psi'(x)| \leq L$, *such that* $\psi(x) sign x > 0$, $\sup\{|\psi'(x)|\} = \frac{\psi_0}{2}$ *with* $a_1 b_1 - \psi_0 > 0$.

(iv) $|b_2 \varphi'(x) y| \leq \Delta$, *for all* $(x,y,z) \in D$ *where* D *is a domain in* xyz*-space containing the origin.*

(v) $\{\mu b_1 - \psi'(x)\} \geq \Delta = \mu b_1 - \frac{\psi_0}{2}$, *for some* $\Delta > 0$, *with* $\mu = \frac{a_1 b_1 + \psi_0}{4 b_1}$.

(vi) $g(t,x) \leq \alpha x$, *such that* $\alpha^2 \leq 2c_1 L_0 - a_1 - b_1 - 2$.

(vii) $\Delta \geq 3 + b_1 + \mu a_0 f_1$ *and* $2\Delta \geq b_1$.

Therefore, the zero solution of (1.1) *is SAS, provided that*

$$r < \min\left\{\frac{2c_1L_0 - a_1 - b_1 - 2 - \alpha^2}{2L}, \frac{\Delta - \mu a_0 f_1 - b_1 - 3}{4(\mu+1)L}, \frac{2\Delta - b_1}{2b_1}\right\}.$$

Proof of Theorem 2.2.

By considering $p = 0$, (1.1) is equivalent to the following

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -a(t)f(x,y)z - b(t)\varphi(x)y - c(t)\psi(x) + c(t)\int_{t-r}^{t}\psi'(x(s))y(s)ds\\ &\quad - g(t,x)\dot{\omega}(t).\end{aligned} \tag{2.3}$$

Define LF $W_1 = W_1(x_t, y_t, z_t)$ of the system (2.3) as the following

$$\begin{aligned}W_1(x_t,y_t,z_t) =& \mu c(t)\int_0^x \psi(\xi)d\xi + c(t)\psi(x)y + \mu a(t)\int_0^y f(x,\eta)\eta d\eta + \mu yz\\ &+ \frac{z^2}{2} + xz + b(t)\varphi(x)\frac{y^2}{2} + x^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds,\end{aligned} \tag{2.4}$$

where λ is a positive constant to be determined later in the proof.

Differentiating (2.4) and applying IF, then the time derivative of the function $W_1(t, x_t, y_t, z_t)$, along the system (2.3), becomes

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t) =& \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + \mu a'(t)\int_0^y f(x,\eta)\eta d\eta\\ &+ \mu a(t)y\int_0^y f_x(x,\eta)\eta d\eta + c(t)\psi'(x)y^2 + \mu z^2 - \mu b(t)\varphi(x)y^2\\ &- a(t)f(x,y)z^2 + yz - a(t)f(x,y)xz - b(t)\varphi(x)xy - c(t)\psi(x)x\\ &+ 2xy + c(t)\big(\mu y + z + x\big)\int_{t-r}^{t}\psi'(x(s))y(s)ds + b'(t)\varphi(x)\frac{y^2}{2}\\ &+ b(t)\varphi'(x)\frac{y^3}{2} + \lambda r y^2 - \lambda\int_{t-r}^{t} y^2(s)ds + \frac{1}{2}g^2(t,x).\end{aligned}$$

From the assumptions $(i)-(iii)$ and (vi), we have

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t) \le& \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + b'(t)\varphi(x)\frac{y^2}{2} + \mu z^2 + 2xy + yz\\ &- \big(\mu b_1 - \psi'(x)\big)y^2 + \frac{1}{2}b_2\varphi'(x)y^3 + \frac{1}{2}\mu a_0 f_1 y^2 - a_1 z^2 - a_1 xz\\ &- b_1 xy - c_1 L_0 x^2 + \big(\mu y + z + x\big)\int_{t-r}^{t}\psi'(x(s))y(s)ds\\ &+ \lambda r y^2 - \lambda\int_{t-r}^{t} y^2(s)ds + \frac{1}{2}\alpha^2 x^2.\end{aligned}$$

Using the assumptions (iv) and (v), we have $\{\mu b_1 - \psi'(x)\} \geq \Delta$ and $|b_2\varphi'(x)y| \leq \Delta$. Therefore, the above equation becomes

$$\begin{aligned}\mathcal{L}W_1(x_t, y_t, z_t) \leq & \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + \frac{1}{2}b'(t)\varphi(x)y^2 + \mu z^2 + 2xy + yz \\ & - \frac{1}{2}\Delta y^2 + \frac{1}{2}\mu a_0 f_1 y^2 - a_1 z^2 - a_1 xz - b_1 xy - c_1 L_0 x^2 + \frac{1}{2}\alpha^2 x^2 \\ & + \left(\mu y + z + x\right)\int_{t-r}^t \psi'(x(s))y(s)ds + \lambda r y^2 - \lambda\int_{t-r}^t y^2(s)ds.\end{aligned}$$

Now, let the function $R(t,x,y)$ be known as

$$R = \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + \frac{1}{2}b'(t)\varphi(x)y^2. \tag{2.5}$$

1. If $c'(t) = 0$, then Equation (2.5) becomes

$$R = \frac{1}{2}b'(t)\varphi(x)y^2 \leq 0, \text{by } (ii). \tag{2.6}$$

2. If $c'(t) < 0$, then $R(t,x,y)$ can be written as

$$R = \mu c'(t)R_1(t,x,y),$$

where

$$R_1 = \int_0^x \psi(\xi)d\xi + \frac{1}{\mu}\psi(x)y + \frac{b'(t)}{2\mu c'(t)}\varphi(x)y^2.$$

The function $R_1(t,x,y)$ can be represented as follows:

$$R_1 = \int_0^x \psi(\xi)d\xi + \frac{b'(t)\varphi(x)}{2\mu c'(t)}\left(y + \frac{c'(t)}{b'(t)\varphi(x)}\psi(x)\right)^2 - \frac{c'(t)}{2\mu b'(t)\varphi(x)}\psi^2(x).$$

By using assumption (ii), we have $0 \leq \frac{c'(t)}{b'(t)} \leq 1$, and also from (iii), we get

$$R_1 \geq \int_0^x (1 - \frac{\psi'(\xi)}{\mu})\psi(\xi)d\xi.$$

Since $\mu = \frac{a_1 b_1 + \psi_0}{4b_1}$, and by assumption (iii), we obtain

$$1 - \frac{\psi'(\xi)}{\mu} \geq \frac{1}{\mu}\left(\frac{a_1 b_1 + \psi_0}{4b_1} - \frac{\psi_0}{2}\right) = \left\{\frac{a_1 b_1 + \psi_0(1 - 2b_1)}{4\mu b_1}\right\}.$$

Since $b_1 \leq 1$, it follows that

$$R_1 \geq \frac{a_1 b_1 - \psi_0}{4b_1\mu}\int_0^x \psi(\xi)d\xi.$$

From the condition (iii), we have $a_1b_1-\psi_0>0$; therefore, we conclude that $R_1\geq 0$,

and since $c'(t)<0$, then the function $R(t,x,y)$ becomes

$$R=\mu c'(t)R_1<0.$$

Hence, from the two cases (1) and (2) it can be concluded that $R\leq 0$ for all x, y and $t\geq 0$. Thus, we have

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t)\leq&\mu z^2+2xy+yz-\frac{1}{2}\Delta y^2+\frac{1}{2}\mu a_0f_1y^2-a_1z^2-a_1xz\\&-b_1xy-c_1L_0x^2+\big(\mu y+z+x\big)\int_{t-r}^{t}\psi'(x(s))y(s)ds\\&+\lambda ry^2-\lambda\int_{t-r}^{t}y^2(s)ds+\frac{1}{2}\alpha^2x^2.\end{aligned}$$

By applying the inequality $2|pq|\leq p^2+q^2$, and using the assumption $|\psi'(x)|\leq L$, we obtain

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t)\leq&-\left\{\frac{2c_1L_0-a_1-b_1-2-\alpha^2}{2}-\frac{L}{2}r\right\}x^2\\&-\left\{\frac{\Delta-3-\mu a_0f_1-b_1}{2}-\frac{\mu L}{2}r-\lambda r\right\}y^2\\&-\left\{\frac{a_1}{2}-\mu-\frac{1}{2}-\frac{L}{2}r\right\}z^2\\&+\left\{\frac{1}{2}L(\mu+2)-\lambda\right\}\int_{t-r}^{t}y^2(s)ds.\end{aligned}\tag{2.7}$$

Take $\lambda=\frac{1}{2}L(\mu+2)$, and since $\mu=\frac{a_1b_1+\psi_0}{4b_1}$, so $\frac{a_1}{2}-\mu=\frac{a_1b_1-\psi_0}{4b_1}=\frac{\Delta}{b_1}>0$, by (i), then we get

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t)\leq&-\left\{\frac{2c_1L_0-2-a_1-b_1-\alpha^2}{2}-\frac{c_2L}{2}r\right\}x^2\\&-\left\{\frac{\Delta-3-\mu a_0f_1-b_1}{2}-L(\mu+1)r\right\}y^2\\&-\left\{\frac{2\Delta-b_1}{2b_1}-\frac{L}{2}r\right\}z^2.\end{aligned}$$

Therefore, suppose that

$$r<\min\left\{\frac{2c_1L_0-a_1-b_1-2-\alpha^2}{2L},\frac{\Delta-\mu a_0f_1-b_1-3}{4(\mu+1)L},\frac{2\Delta-b_1}{2b_1}\right\}.$$

Then, for positive constant δ_1, we can write

$$\mathcal{L}W_1(t,x_t,y_t,z_t)\leq\delta_1(x^2+y^2+z^2).\tag{2.8}$$

Since $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ is positive, we have

$$\begin{aligned}W_1(x_t,y_t,z_t) \geq &\mu c(t)\int_0^x \psi(\xi)d\xi + c(t)\psi(x)y + \mu a(t)\int_0^y f(x,\eta)\eta d\eta\\ &+ \mu yz + \frac{z^2}{2} + xz + b(t)\varphi(x)\frac{y^2}{2} + x^2.\end{aligned}$$

From the assumptions $(i)-(iii)$, we get

$$W_1(x_t,y_t,z_t) \geq \mu c_1\int_0^x \psi(\xi)d\xi + c_1\psi(x)y + \mu a_1\frac{y^2}{2} + \mu yz + \frac{z^2}{2} + xz + b_1\frac{y^2}{2} + x^2.$$

Now, we can write the above equation as

$$\begin{aligned}W_1(x_t,y_t,z_t) \geq &\frac{1}{2b_1}\big\{b_1 y + c_1\psi(x)\big\}^2 + \big(\mu y + \frac{z}{2}\big)^2 + \frac{\mu}{2}\big(a_1 - 2\mu\big)y^2 + \big(x + \frac{z}{2}\big)^2\\ &+ \frac{c_1}{2b_1 y^2}\left[4\int_0^x \psi(\xi)\bigg\{(\mu b_1 - c_1\psi'(\xi))\eta d\eta\bigg\}d\xi\right].\end{aligned}$$

Since $\mu b_1 - c_1\psi'(\xi) = \frac{a_1b_1+(1-2c_1)\psi_0}{4} > \frac{a_1b_1-\psi_0}{4} > 0,$ by (i); therefore, we conclude

$$\begin{aligned}W_1(x_t,y_t,z_t) \geq &\frac{1}{2b_1}\big\{b_1 y + c_1\psi(x)\big\}^2 + \big(\mu y + \frac{z}{2}\big)^2 + \frac{\mu}{2}\big(a_1 - 2\mu\big)y^2\\ &+ \big(x + \frac{z}{2}\big)^2 + \frac{c_1(a_1b_1 - \psi_0)}{4}\int_0^x \psi(\xi)d\xi.\end{aligned} \tag{2.9}$$

Since $a_1 - 2\mu = \frac{a_1b_1-\psi_0}{2b_1} > 0,$ by (iii), we get

$$W_1(x_t,y_t,z_t) \geq \delta_2(x^2 + y^2 + z^2) \ \ for\ some\ \ \delta_2 > 0. \tag{2.10}$$

By applying the assumptions $(i)-(iii)$, we have

$$\begin{aligned}W_1(x_t,y_t,z_t) \leq &\frac{1}{2}\mu L x^2 + Lxy + \frac{1}{2}\mu a_2 f_1 y^2 + \mu yz + \frac{z^2}{2} + xz + x^2\\ &+ \frac{1}{2}b_2\varphi_2 y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds.\end{aligned}$$

Since

$$\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \leq \|y\|^2\int_{t-r}^{t}(\theta - t + r)d\theta = \frac{r^2}{2}\|y\|^2. \tag{2.11}$$

Therefore, from the inequality $pq \leq \frac{1}{2}(p^2+q^2),$ we obtain

$$\begin{aligned}W_1(x_t,y_t,z_t) \leq &\frac{1}{2}\bigg\{(\mu+1)L + 3\bigg\}\|x\|^2 + \frac{1}{2}\bigg\{L + \mu(a_2 f_1 + 1)\\ &+ b_2\varphi_2 + \lambda r^2\bigg\}\|y\|^2 + \frac{1}{2}(\mu+2)\|z\|^2.\end{aligned} \tag{2.12}$$

Then, we get

$$W_1(x_t, y_t, z_t) \leq \delta_3(x^2 + y^2 + z^2) \;\; for \;\; some \; \delta_3 > 0. \tag{2.13}$$

Hence, from the results (2.8), (2.10) and (2.13), we find that all conditions of Theorem 2.1 are satisfied. The proof of Theorem 2.2 is now complete.

3. Boundedness result

Assuming Equation (2.1) and the function $W(t, x_t)$ chosen from $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ satisfies the following.

Assumption 3.1 [18, 33] *We assume that for any solution $x(t)$ of (2.1) and for any fixed $0 \leq t_0 \leq T < \infty$, the following condition hold:*

$$E^{x_0}\Big\{\int_{t_0}^{T} W_{x_i}^2(t, x_t))N_{ik}^2(t, x_t)dt\Big\} < \infty, \;\; 1 \leq i \leq n, 1 \leq k \leq m. \tag{3.1}$$

Theorem 3.1 [18, 33] *Suppose that there exists a function $W(t, x_t)$ in $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ satisfying Assumption 3.1, such that for all $(t, x_t) \in \mathbb{R}^+ \times \mathbb{R}^n$:*

$$(i)\|x\|^{q_1} \leq W \leq \|x\|^{q_2},$$

$$(ii)\mathcal{L}W \leq -\nu(t)\|x\|^n + \beta(t),$$

$$(iii)W - W^{n/q_2} \leq \gamma,$$

where $\nu, \beta \in C(\mathbb{R}^+; \mathbb{R}^+)$, q_1, q_2, n are positive constants, $q_1 \geq 1$ and γ is a nonnegative constant. Then all solutions of (2.1) satisfy

$$E^{x_0}\|x(t; t_0, x_0)\| \leq \Big\{W(t_0, x_0)e^{-\int_{t_0}^{t}\nu(s)ds} + \int_{t_0}^{t}\big(\gamma\nu(u) + \beta(u)\big)e^{-\int_{u}^{t}\nu(s)ds}du\Big\}^{1/q_1},$$

for all $t \geq t_0$.

Definition 3.1 [18, 33] *Let $M(t, 0, 0) = 0$ and $N(t, 0, 0) = 0$. We say that the zero solution of (2.1) is ν-UEAS in probability, if there exists a positive continuous function $\nu(t)$ such that $\int_{t_0}^{t}\nu(s)ds \to \infty$ as $t \to \infty$ and constants Γ, $C \in \mathbb{R}^+$ such that any solution $x(t; t_0, x_0)$ of (2.1) satisfies the following*

$$E^{x_0}\|x(t, t_0, x_0)\| \leq C(\|x_0\|, t_0)\big(e^{-\int_{t_0}^{t}\nu(s)ds}\big)^{\Gamma}, \;\; for \;\; all \; t \geq t_0,$$

where the constant C may depend on t_0 and x_0. The zero solution of (2.1) is said to be ν-UEAS in probability, if C is independent of t_0.

Corollary 3.1 [18, 33] *Suppose that the hypotheses of Theorem 3.1 hold. In addition*

$$\int_{t_0}^{t}\{\gamma\nu(u) + \beta(u)\}e^{-\int_{u}^{t}\nu(s)ds}du \leq \mathcal{M}, \;\; for \; all \; t \geq t_0 \geq 0, \tag{3.2}$$

for some positive constant $\mathcal{M}$, then all solutions of (2.1) are USB.

Corollary 3.2 [18, 33] *Suppose* $M(t,0,0)=0$ *and* $N(t,0,0)=0$. *Assume*

$$\int_{t_0}^{t}\{\gamma\nu(u)+\beta(u)\}e^{\int_{t_0}^{u}\nu(s)ds}du\leq\mathcal{M},\ \textit{for all } t\geq t_0\geq 0, \tag{3.3}$$

for some positive constant $\mathcal{M}$ *and*

$$\int_{t_0}^{t}\nu(s)ds\to\infty,\ \textit{as } t\to\infty. \tag{3.4}$$

If the hypotheses of Theorem 3.1 hold, then the zero solution of (2.1) is ν*-UEAS in probability with* $\Gamma=1/q_1$.

Theorem 3.2 *If the conditions* $(i)-(vii)$ *of Theorem 2.2 hold. In addition, we assume that the following conditions are satisfied:*

$(viii)$ $\alpha^2\leq\frac{2c_1L_0(a_1b_1-\psi_0+1)-a_1-b_1-2}{a_1+1}$.

(ix) $|p(t,x(t),\dot{x}(t),\ddot{x}(t))|\leq m,\quad m>0.$

Then,

(1) *All solutions of Equation (1.1) are USB, provided that*

$$r<\min\left\{\frac{2c_1L_0(a_1b_1-\psi_0+1)-(a_1+b_1+2)-(a_1+1)\alpha^2}{2(a_1b_1-\psi_0+1)L},\right.$$
$$\left.\frac{\Delta+a_1\psi_0-\mu a_0f_1-(b_1+3)}{2L(2a_1^2+a_1+\mu+2+a_1b_1-\psi_0)L},\frac{2\Delta-b_1}{2Lb_1(a_1+1)}\right\}.$$

(2) *The zero solution of (1.1) is* $\nu-$ *UEAS in probability.*

Proof of Theorem 3.2.
In this case, we have $p\neq 0$ and the equivalent system is

$$\begin{aligned}
\dot{x}&=y,\\
\dot{y}&=z,\\
\dot{z}&=-a(t)f(x,y)z-b(t)\varphi(x)y-c(t)\psi(x)+c(t)\int_{t-r}^{t}\psi'(x(s))y(s)ds\\
&\quad-g(t,x)\dot{\omega}(t)+p(t,x,y,z).
\end{aligned}$$

Consider the function

$$W(x_t,y_t,z_t)=W_1(x_t,y_t,z_t)+W_2(x_t,y_t,z_t),$$

where W_1 is defined as (2.4) and we can define Lyapunov functional W_2 as the following

$$\begin{aligned}
W_2(x_t,y_t,z_t)=&c(t)a_1^2\int_0^x\psi(\xi)d\xi+a(t)a_1^2\int_0^y f(x,\eta)\eta d\eta+a_1c(t)\psi(x)y\\
&+\frac{b_1}{2}(a_1b_1-\psi_0)x^2\ +(a_1b_1-\psi_0)x(z+a_1y)\\
&+a_1^2yz+\frac{\psi_0}{2}c(t)y^2+\frac{a_1}{2}z^2.
\end{aligned} \tag{3.5}$$

Differentiating (3.5) by applying IF (2.2) and using the assumptions of Theorem 2.2, we get

$$\begin{aligned}\mathcal{L}W_2(x_t,y_t,z_t) \leq & c'(t)a_1^2\int_0^x \psi(\xi)d\xi + a_1c'(t)\psi(x)y + \frac{\psi_0}{2}c'(t)y^2 - \frac{a_1\psi_0}{2}y^2\\ &+ a_1^2a'(t)\int_0^y f(x,\eta)\eta d\eta - (a_1b_1-\psi_0)c_1L_0x^2 + \frac{a_1}{2}g^2(t,x)\\ &+ \Big\{(a_1b_1-\psi_0)x + a_1^2y + a_1z\Big\}\int_{t-r}^t \psi'(x(s))y(s)ds\\ &+ \Big\{(a_1b_1-\psi_0)x + a_1^2y + a_1z\Big\}p(t,x(t),\dot{x}(t),\ddot{x}(t)).\end{aligned}$$

Now, let the function $R_2(t,x,y)$ take the form

$$\begin{aligned}R_2 =& c'(t)a_1^2\int_0^x \psi(\xi)d\xi + a_1c'(t)\psi(x)y + \frac{\psi_0}{2}c'(t)y^2\\ &+ a_1^2a'(t)\int_0^y f(x,\eta)\eta d\eta.\end{aligned} \tag{3.6}$$

We have here two cases:

Case 1. If $c'(t)=0$ and from (ii), then Equation (3.6) becomes

$$R_1 = a_1^2a'(t)\int_0^y f(x,\eta)\eta d\eta \leq a_1^2a_0\int_0^y f(x,\eta)\eta d\eta \leq 0.$$

Case 2. If $c'(t)<0$, then $R_2(t,x,y)$ can be written as

$$R_2 = c'(t)R_3(t,x,y), \tag{3.7}$$

where

$$R_3 = a_1^2\int_0^x \psi(\xi)d\xi + a_1\psi(x)y + \frac{\psi_0}{2}y^2 + a_1^2\frac{a'(t)}{c'(t)}\int_0^y f(x,\eta)\eta d\eta.$$

It follows that

$$\begin{aligned}R_3 =& \frac{a_1^2a'(t)}{c'(t)}\int_0^y f(x,\eta)\eta d\eta + \frac{1}{2\psi_0}(\psi_0y + a_1\psi(x))^2\\ &+ \frac{a_1^2}{2\psi_0y^2}\left[4\int_0^x \psi(\xi)d\xi\Big\{\int_0^y (\psi_0-\psi'(\xi)\Big\}\eta d\eta\right].\end{aligned}$$

By using the condition (ii), we have $0 \leq \frac{a'(t)}{c'(t)} \leq 1$, and since $\psi_0 - \psi'(\xi) \geq \frac{\psi_0}{2}$, by (iii), we find

$$R_3 \geq \frac{1}{2\psi_0}\Big(\psi_0y + a_1\psi(x)\Big)^2 + \frac{a_1^2}{2}\int_0^x \psi(\xi)d\xi \geq 0.$$

Then, we conclude

$$R_2 = c'(t)R_3(t,x,y) \leq 0.$$

Therefore, on combining the two cases for $c'(t)$, we obtain $R_2(t,x,y) \le 0$, for all x, y, and $t \ge 0$. Thus, we get

$$\begin{aligned}\mathcal{L}W_2(x_t,y_t,z_t) \le &-\frac{a_1\psi_0}{2}y^2-(a_1b_1-\psi_0)c_1L_0x^2+\frac{a_1}{2}g^2(t,x)\\&+\Big\{(a_1b_1-\psi_0)x+a_1^2y+a_1z\Big\}\int_{t-r}^{t}\psi'(x(s))y(s)ds\\&+\Big\{(a_1b_1-\psi_0)x+a_1^2y+a_1z\Big\}p(t,x,y,z).\end{aligned}$$

Now, from the assumptions of Theorem 3.2 and by applying the fact that $|pq| \le \frac{1}{2}(p^2+q^2)$, we conclude

$$\begin{aligned}\mathcal{L}W_2(x_t,y_t,z_t) \le &-\left\{\frac{2(a_1b_1-\psi_0)c_1L_0-a_1\alpha^2}{2}-\frac{(a_1b_1-\psi_0)L}{2}\right\}x^2\\&-\left\{\frac{a_1\psi_0}{2}-\frac{a_1^2L}{2}r\right\}y^2+\frac{1}{2}a_1Lrz^2\\&+\left\{\frac{(a_1b_1-\psi_0)L}{2}+\frac{(a_1^2+a_1)L}{2}\right\}\int_{t-r}^{t}y^2(s)ds\\&+\Big\{(a_1b_1-\psi_0)|x|+a_1^2|y|+a_1|z|\Big\}m.\end{aligned}\tag{3.8}$$

Now, from Equation (1.1), Lyapunov functional W_1 in (2.4), and by using Equation (2.7), we conclude

$$\begin{aligned}\mathcal{L}W_1(t,x_t,y_t,z_t) \le &-\left\{\frac{2c_1L_0-a_1-b_1-2-\alpha^2}{2}-\frac{L}{2}r\right\}x^2\\&-\left\{\frac{\Delta-3-\mu a_0f_1-b_1}{2}-\frac{\mu L}{2}r-\lambda r\right\}y^2\\&-\left\{\frac{a_1}{2}-\mu-\frac{1}{2}-\frac{L}{2}r\right\}z^2+\left\{\frac{1}{2}L(\mu+2)-\lambda\right\}\int_{t-r}^{t}y^2(s)ds\\&+\big(|x|+\mu|y|+|z|\big)m.\end{aligned}\tag{3.9}$$

By combining Equations (3.8) and (3.9), we obtain

$$\begin{aligned}\mathcal{L}W(t,x_t,y_t,z_t) \le &-\left\{\frac{2c_1L_0(a_1b_1-\psi_0+1)-(a_1+b_1+2)-(a_1+1)\alpha^2}{2}\right.\\&\left.-\frac{(a_1b_1-\psi_0+1)L}{2}r\right\}x^2-\left\{\frac{a_1\psi_0+\Delta-\mu a_0f_1-(b_1+3)}{2}\right.\\&\left.-\frac{a_1^2}{2}Lr-\lambda r\right\}y^2-\left\{\frac{2\Delta-b_1}{2b_1}-\frac{(a_1+1)L}{2}r\right\}z^2\\&+\left\{\frac{(a_1b_1-\psi_0)L+(a_1^2+a_1)L+(\mu+2)L}{2}-\lambda\right\}\int_{t-r}^{t}y^2(s)ds\\&+\big\{(a_1b_1-\psi_0+1)|x|+(a_1^2+\mu)|y|+(a_1+1)|z|\big\}m.\end{aligned}$$

Choose $\lambda = \frac{\{(a_1b_1-\psi_0)+a_1^2+a_1+\mu+2\}L}{2}$; therefore, the above equation becomes

$$\begin{aligned}\mathcal{L}W(t,x_t,y_t,z_t) \leq -\bigg\{&\frac{2c_1L_0(a_1b_1-\psi_0+1)-(a_1+b_1+2)-(a_1+1)\alpha^2}{2}\\&-\frac{(a_1b_1-\psi_0+1)L}{2}r\bigg\}x^2 - \bigg\{\frac{a_1\psi_0+\Delta-\mu a_0 f_1-(b_1+3)}{2}\\&-\frac{(2a_1^2+a_1b_1-\psi_0+a_1+\mu+2)L}{2}r\bigg\}y^2\\&-\bigg\{\frac{2\Delta-b_1}{2b_1}-\frac{(a_1+1)L}{2}r\bigg\}z^2\\&+\big\{(a_1b_1-\psi_0+1)|x|+(a_1^2+\mu)|y|+(a_1+1)|z|\big\}m.\end{aligned}$$

If

$$\begin{aligned}r < \min\bigg\{&\frac{2c_1L_0(a_1b_1-\psi_0+1)-(a_1+b_1+2)-(a_1+1)\alpha^2}{2(a_1b_1-\psi_0+1)L},\\&\frac{\Delta+a_1\psi_0-\mu a_0 f_1-(b_1+3)}{2L(2a_1^2+a_1+\mu+2+a_1b_1-\psi_0)L},\frac{2\Delta-b_1}{2Lb_1(a_1+1)}\bigg\}.\end{aligned}$$

Thus, we have the following

$$\begin{aligned}\mathcal{L}W &\leq -H(x^2+y^2+z^2)+kH(|x|+|y|+|z|)\\&= -\frac{H}{2}(x^2+y^2+z^2)-\frac{H}{2}\{(|x|-k)^2+(|y|-k)^2+(|z|-k)^2\}+\frac{3H}{2}k^2\\&\leq -\frac{H}{2}(x^2+y^2+z^2)+\frac{3H}{2}k^2,\ \text{ for some } k,H>0,\end{aligned}$$

where

$$k = m\max\{a_1b_1-\psi_0+1, a_1^2+\mu, a_1+1\}.$$

By the conditions $(i)-(iii)$ of Theorem 2.2, we have

$$\begin{aligned}W_2(x_t,y_t,z_t) \geq & a_1^2c_1\int_0^x \psi(\xi)d\xi + \frac{c_1}{2\psi_0}\big(\psi_0 y + a_1\psi(x)\big)^2 - \frac{a_1^2c_1}{2\psi_0}\psi^2(x) + \frac{a_1}{2}(z+a_1y)^2\\&+\frac{(a_1b_1-\psi_0)}{2b_1}\bigg\{b_1x+(z+a_1y)\bigg\}^2 - \frac{(a_1b_1-\psi_0)}{2b_1}(z+a_1y)^2.\end{aligned}$$

Therefore, we get

$$\begin{aligned}W_2(x_t,y_t,z_t) \geq & \frac{c_1a_1^2}{2\psi_0y^2}\bigg[4\int_0^x\psi(\xi)d\xi\bigg\{\int_0^y\big(\psi_0-\psi'(\xi)\big)\bigg\}\eta d\eta\bigg] + \frac{c_1}{2\psi_0}(\psi_0y+a_1\psi(x))^2\\&+\frac{(a_1b_1-\psi_0)}{2b_1}\bigg\{b_1x+(z+a_1y)\bigg\}^2 + \frac{\psi_0}{2b_1}(z+a_1y)^2.\end{aligned}$$

From the condition (iii), we obtain

$$\begin{aligned} W_2(x_t,y_t,z_t) \geq & \frac{c_1a_1^2}{2}\int_0^x \psi(\xi)d\xi + \frac{c_1}{2\psi_0}\big(\psi_0 y + a_1\psi(x)\big)^2 \\ & + \frac{(a_1b_1-\psi_0)}{2b_1}\big\{b_1x + (z+a_1y)\big\}^2 + \frac{\psi_0}{2b_1}(z+a_1y)^2. \end{aligned} \tag{3.10}$$

Therefore, by both Equations (2.9) and (3.10), we conclude

$$\begin{aligned} W(x_t,y_t,z_t) \geq & \frac{1}{2b_1}\Big\{b_1y + c_1\psi(x)\Big\}^2 + \big(\mu y + \frac{z}{2}\big)^2 + \frac{\mu}{2}(a_1-2\mu)y^2 + (x+\frac{z}{2})^2 \\ & + \frac{c_1(a_1b_1-\psi_0+2a_1^2)}{4}\int_0^x \psi(\xi)d\xi + \frac{c_1}{2\psi_0}\Big(\psi_0 y + a_1\psi(x)\Big)^2 \\ & + \frac{(a_1b_1-\psi_0)}{2b_1}\Big\{b_1x + (z+a_1y)\Big\}^2 + \frac{\psi_0}{2b_1}(z+a_1y)^2. \end{aligned}$$

Therefore, for positive constant δ_4, we have

$$W(x_t,y_t,z_t) \geq \delta_4(x^2+y^2+z^2). \tag{3.11}$$

From (3.5) and using the conditions $(i)-(iii)$ of Theorem 2.2, we find

$$\begin{aligned} W_2(x_t,y_t,z_t) \leq & a_1^2Lx^2 + a_2a_1^2f_1y^2 + a_1Lxy + \frac{b_1}{2}(a_1b_1-\psi_0)x^2 \\ & + (a_1b_1-\psi_0)x(z+a_1y) + a_1^2yz + \frac{\psi_0}{2}y^2 + \frac{a_1}{2}z^2. \end{aligned}$$

By using the fact that $2pq \leq (p^2+q^2)$, then the last inequality becomes

$$\begin{aligned} W_2(x_t,y_t,z_t) \leq & \left\{\frac{(a_1^2+a_1)L + (a_1b_1-\psi_0)(a_1+b_1+1)}{2}\right\}\|x\|^2 \\ & + \left\{\frac{a_1^2a_2f_1 + a_1L + \psi_0 + a_1^2 + a_1(a_1b_1-\psi_0)}{2}\right\}\|y\|^2 \\ & + \left\{\frac{(a_1b_1-\psi_0) + a_1 + a_1^2}{2}\right\}\|z\|^2. \end{aligned} \tag{3.12}$$

Hence, by combining the two inequalities (2.12) and (3.12), we obtain

$$\begin{aligned} W(x_t,y_t,z_t) \leq & \left\{\frac{(\mu+1+a_1^2+a_1)L + (a_1b_1-\psi_0)(a_1+b_1+1)+3}{2}\right\}\|x\|^2 \\ & + \left\{\frac{a_1^2a_2f_1 + (a_1+1)L + \psi_0 + \mu(a_2f_1+1)}{2}\right. \\ & \left. + \frac{a_1^2 + b_2\varphi_2 + a_1(a_1b_1-\psi_0) + \lambda r^2}{2}\right\}\|y\|^2 \\ & + \left\{\frac{(a_1b_1-\psi_0) + (a_1+a_1^2) + (\mu+2)}{2}\right\}\|z\|^2. \end{aligned}$$

Consequently, for positive constant δ_5, we conclude

$$W \leq \delta_5(\|x\|^2 + \|y\|^2 + \|z\|^2). \tag{3.13}$$

Thus, the assumptions (ii) of Theorem 3.1 is satisfied by taking $\nu(t) = H/2$, $\beta(t) = (3H/2)k^2$ and $n = 2$. From inequalities (3.11) and (3.13), we see that the LF $W(x_t, y_t, z_t)$ also satisfies the condition (i) of Theorem 3.1. As well as we can test that the condition (iii) of Theorem 3.1 is satisfied with $q_1 = q_2 = n = 2$ with $\gamma = 0$. Then, all conditions of Theorem 3.1 hold.

Therefore, with $\nu(t) = H/2, \beta(t) = (3H/2)k^2$ and $n = 2$, with $\gamma = 0$, we find that

$$\int_{t_0}^{t} \{\gamma\nu(u) + \beta(u)\}e^{-\int_u^t \nu(s)ds}du = (3H/2)k^2 \int_{t_0}^{t} e^{-\frac{H}{2}\int_u^t ds}du \leq 3k^2,$$

for all $t \geq t_0 \geq 0$. Thus, condition (3.2) holds. Now, since

$$\begin{aligned}
&g^T = (00 \ \ -g(t,x)),\\
&W_x = (W_1)_x + (W_2)_x\\
&\quad = \mu c(t)\psi(x) + 2x + z + c(t)\psi'(x)y + b(t)\varphi'(x)\frac{y^2}{2} + c(t)a_1^2\psi(x)\\
&\quad + a_1c(t)\psi'(x)y + b_1(a_1b_1 - \psi_0)x + (a_1b_1 - \psi_0)(z + a_1y),\\
&W_y = (W_1)_y + (W_2)_y\\
&\quad = c(t)\psi(x) + \mu a(t)f(x,y)y + \mu z + b(t)\varphi(x)y + a_1^2a(t)f(x,y)y\\
&\quad + a_1c(t)\psi(x) + (a_1b_1 - \psi_0)a_1x + a_1^2z + \psi_0c(t)y,\\
&W_z = (W_1)_z + (W_2)_z = \mu y + z + x + (a_1b_1 - \psi_0)x + a_1^2y + a_1z.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|W_{x_i}(t,x)N_{ik}(t,x)| \leq &\alpha\left[\left\{\frac{\mu + 3 + 2(a_1b_1 - \psi_0) + a_1^2 + a_1}{2}\right\}x^2 + \left(\frac{\mu + a_1^2}{2}\right)y^2\right.\\
&\left. + \left(\frac{a_1 + 1}{2}\right)z^2\right] := \vartheta(t).
\end{aligned}$$

Therefore, condition (3.1) is satisfied. Hence, by Corollary 3.1 all solutions of (1.1) are USB and satisfy

$$E^{x_0}\|x(t,t_0,x_0)\| \leq \{Cx_0^2 + 3k^2\}^{1/2}, \text{ for all } t \geq t_0 \geq 0,$$

where C is a constant. Next,

$$\begin{aligned}
\int_{t_0}^{t} \{\gamma\nu(u) + \beta(u)\}e^{\int_{t_0}^u \nu(s)ds}du &= (3H/2)k^2 \int_{t_0}^{t} e^{\frac{H}{2}\int_{t_0}^u ds}du\\
&= 3k^2(e^{t-t_0} - 1) \leq \mathcal{M},
\end{aligned}$$

for all $t \geq t_0 \geq 0$, where $\mathcal{M}$ is a positive constant. Hence condition (3.3) is satisfied. We can see that condition (3.4) is also satisfied. By Corollary 3.2, we find that the zero solution of (1.1) is ν-UEAS in probability with $\Gamma = 1/2$.

4. Examples

Example 4.1 *With $p = 0$, consider the following SDDE of third-order*

$$\dddot{x}(t) + (-2\sin t + 13.5)\left(1 + \frac{3}{5 + 2x^5 + \dot{x}^4}\right)\ddot{x}(t) + \big(1 - \frac{1}{10 - t^4}\big)\big(1 + \frac{1}{8}e^{-x}\big)\dot{x}(t)$$
$$+ \big(1 - \frac{1}{8 - t^2}\big)\left\{20x^{\frac{1}{5}}(t-r) + \frac{1}{4}\sin x(t-r)\right\} + \frac{1}{4}xe^{-t}\dot{w}(t) = 0. \tag{4.1}$$

The equivalent system of (4.1) is

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= -(-2\sin t + 13.5)\big(1 + \frac{3}{5 + 2x^5 + y^4}\big)z\\
&\quad - \big(1 - \frac{1}{10 - t^4}\big)\big(1 + \frac{1}{8}e^{-x}\big)y - \big(1 - \frac{1}{8 - t^2}\big)\big(20x^{\frac{1}{5}} + \frac{1}{4}\sin x\big)\\
&\quad + \big(1 - \frac{1}{8 - t^2}\big)\int_{t-r}^{t}\left(4x^{\frac{-4}{5}}(s) + \frac{1}{4}\cos x(s)\right)y(s)ds\\
&\quad - \frac{1}{4}xe^{-t}\dot{w}(t).
\end{aligned} \tag{4.2}$$

If we compare system 2.3 with system 4.2, we get the following

$$a(t) = -2\sin t + 13.5,$$

we notice that

$$11.5 \le -2\sin t + 13.5 \le 15.5, \quad \textit{then } a_1 = 11.5, \ \ a_2 = 15.5.$$

And also

$$a'(t) = -2\cos t, \ \ \textit{it follows that } -2 \le a'(t) \le -1, \textit{so } \ a_0 = -1.$$

Figures 1 and 2 show the bounds of $a(t)$ with $t \in [-8\pi, 8\pi]$ and $a'(t)$ with $t \in [0, \frac{\pi}{3}]$.

The function

$$f(x,y) = 1 + \frac{3}{5 + 2x^5 + y^4}, \ \ \textit{since } 0 \le \frac{3}{5 + 2x^5 + y^4} \le \frac{3}{5},$$

therefore, we find

$$1 \le f(x,y) \le \frac{8}{5}, \ \ \textit{then } \ f_1 = \frac{8}{5},$$

and

$$f_x(x,y) = \frac{-30x^4}{(5 + 2x^5 + y^4)^2} < 0.$$

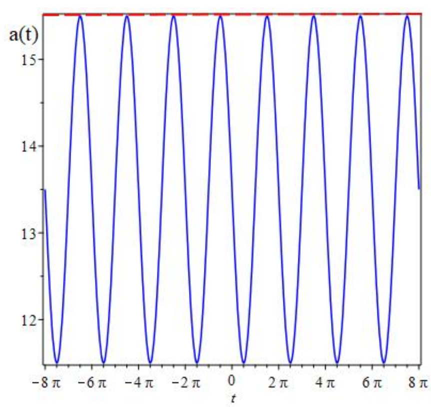

Figure 1. Trajectory of $a(t)$.

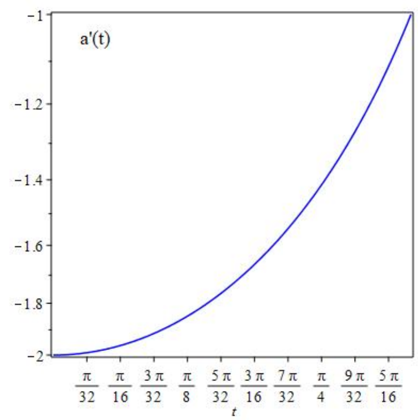

Figure 2. Trajectory of $a'(t)$.

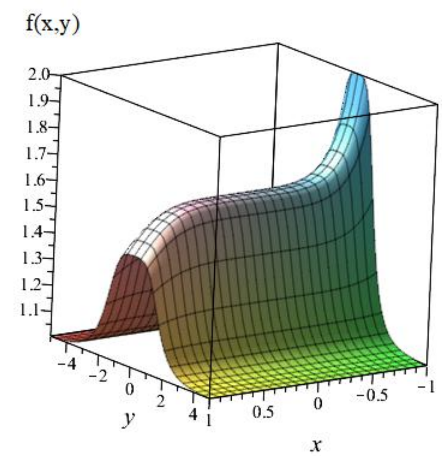

Figure 3. Trajectory of $f(x,y)$.

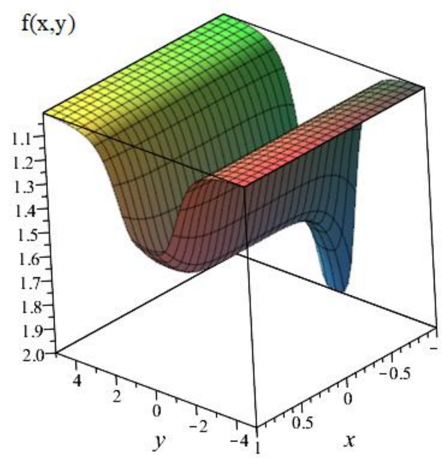

Figure 4. Trajectory of $f(x,y)$.

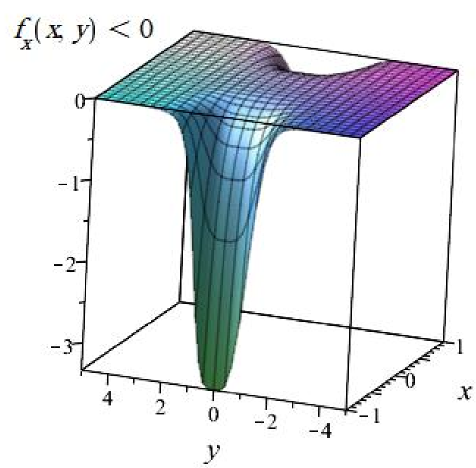

Figure 5. Trajectory of $f_x(x,y)$.

Figures 3 and 4 illustrate the path of the function $f(x,y)$, with $x \in [-1,1]$ and $y \in [-5.5]$ and also Figure 5 shows the behaviour of the function $f_x(x,y)$ with $x \in [-1,1]$ and $y \in [-5.5]$.

The function

$$b(t) = 1 - \frac{1}{10 - t^4}, \quad then \ \frac{9}{10} \le b(t) \le 1 \ \ and \ \ b_1 = 0.9.$$

The derivative of $b(t)$ in terms to t is

$$b'(t) = \frac{-4t^3}{(10 - t^4)^2} \le 0.$$

Now, the function

$$\psi(x) = 20x^{\frac{1}{5}} + \frac{1}{4}\sin x, \quad so\ we\ get \ \ \frac{\psi(x)}{x} = 20x^{\frac{-4}{5}} + \frac{1}{4x}\sin x.$$

We know that

$$-\frac{1}{4} \le \frac{1}{4x}\sin x \le \frac{1}{4},$$

thus, we get

$$\frac{\psi(x)}{x} \ge 8.31 = L_0.$$

And also

$$|\psi'(x)| = |4x^{-\frac{4}{5}} + \frac{1}{4}\cos x| \le 4.25 = L; \ \ therefore,\ we\ find$$

$$\sup\{|\psi'(x)|\} = 4.25 = \frac{\psi_0}{2}, \ \ then\ \ \psi_0 = 8.5.$$

Now, Figures 6 and 7 show the path of the functions $\psi'(x)$ and $\frac{\psi(x)}{x}$ with $x \in [1,3]$.

We have also the function

$$c(t) = 1 - \frac{1}{8 - t^2} \ \ so, \ \ \frac{7}{8} \le c(t) \le 1, \quad c_1 = \frac{7}{8}, \quad with\ c'(t) = \frac{-2t}{(8 - t^2)^2} \le 0.$$

The paths of $b(t)$ and $c(t)$ for $t \in [-10,10]$ are depicted in Figure 8.

The function

$$g(t,x) = \frac{1}{4}xe^{-t}, \ \ then\ g^2(t,x) = \frac{1}{16}x^2e^{-2t} \le \frac{1}{16}x^2.$$

Figures 9 and 10 show the trajectory of $g^2(t,x)$ for all t and $x,t \in [-4.4]$.

First, we get

$$2c_1L_0 - a_1 - b_1 - 2 = 0.14 > \frac{1}{16} = \alpha^2.$$

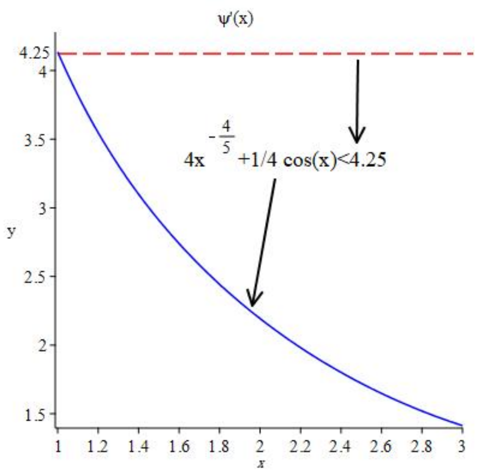

Figure 6. The path of $\psi'(x)$.

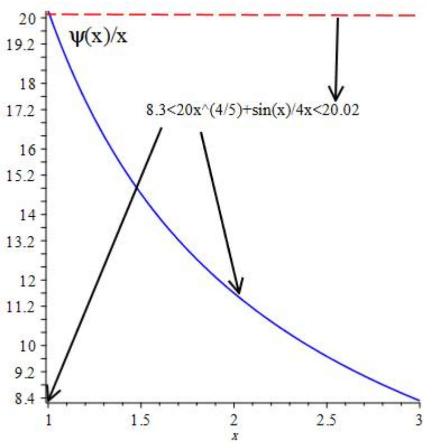

Figure 7. The path of $\frac{\psi(x)}{x}$.

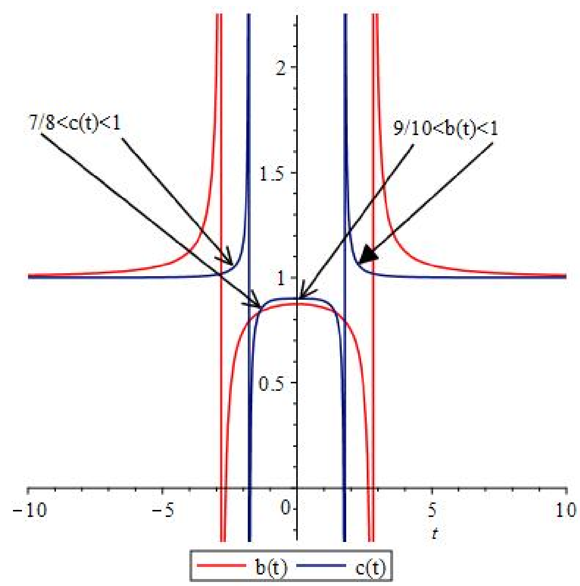

Figure 8. Trajectory of $b(t)$ and $c(t)$.

Second, with $\mu = \frac{a_1 b_1 + \psi_0}{4b_1} = 5.24$, *then we obtain*

$$3 + b_1 + \mu a_0 f_1 = -4.48, \ \ and \ \Delta = \frac{a_1 b_1 - \psi_0}{4} = 0.46 > 0,$$

it follows that

$$\Delta > 3 + b_1 + \mu a_0 f_1.$$

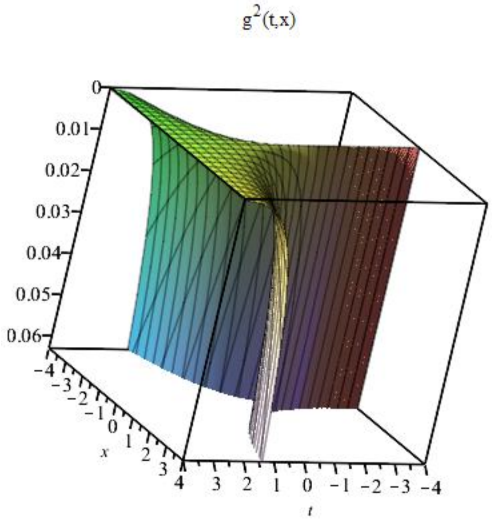

Figure 9. Trajectory of the function $g^2(t,x)$.

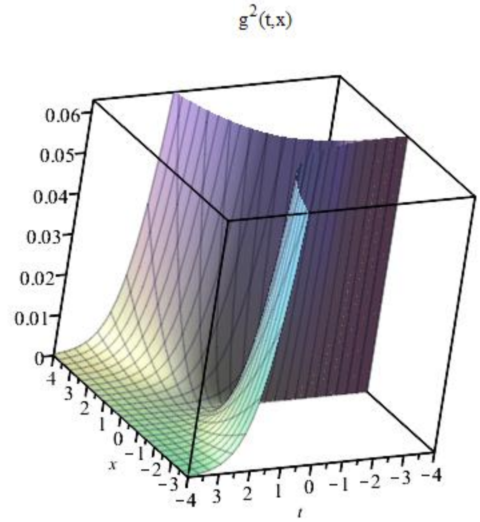

Figure 10. Trajectory of the function $g^2(t,x)$.

Finally, we conclude

$$\begin{aligned}\mathcal{L}W_1(t,x_t,y_t,z_t) \leq &-(0.04-2.13r)x^2-\{2.47-(\frac{\mu L}{2}+\lambda)r\}y^2\\ &-\{0.01-2.63r\}z^2+\left\{\frac{1}{2}L(\mu+2)-\lambda\right\}\int_{t-r}^{t}y^2(s)ds.\end{aligned}$$

Take $\lambda=\frac{L}{2}(\mu+2)=15.39>0$*; therefore, we get*

$$\mathcal{L}W_1(t,x_t,y_t,z_t)\leq -\big(0.04-2.13r\big)x^2-(2.47-29.15r)y^2+\{0.01-2.63r\}z^2.$$

Provided that

$$r<\min\{0.01,0.04,0.002\}\cong 0.002.$$

Hence, all conditions of Theorem 2.2 are satisfied, then the zero solution of (4.1) *is SAS.*

Example 4.2 *Consider here* $p\neq 0$*, then the SDDE* (4.1) *becomes*

$$\begin{aligned}\dddot{x}(t)&+(-2\sin t+13.5)\left(1+\frac{3}{5+2x^5+\dot{x}^4}\right)\ddot{x}(t)+\big(1-\frac{1}{10-t^4}\big)\big(1+\frac{1}{8}e^{-x}\big)\dot{x}(t)\\ &+\big(1-\frac{1}{8-t^2}\big)\left\{20x^{\frac{1}{5}}(t-r)+\frac{1}{4}\sin x(t-r)\right\}+\frac{1}{4}xe^{-t}\dot{w}(t)=p(t,x,\dot{x}(t),\ddot{x}(t)).\end{aligned}\tag{4.3}$$

We have

$$\frac{2c_1L_0(a_1b_1-\psi_0+1)-a_1-2}{a_1+1}=2.43>\frac{1}{16}=\alpha^2.$$

Thus, we obtain

$$\begin{aligned}\mathcal{L}W \leq &- (13.58 - 6.06r)x^2\\ &- \{51.35 - (24.4 + \lambda)r\}y^2 - \{0.01 - 26.56r\}z^2\\ &+ \left\{\frac{(a_1b_1 - \psi_0)L + (a_1^2 + a_1)L + (\mu + 2)L}{2} - \lambda\right\}\int_{t-r}^{t} y^2(s)ds\\ &+ \{(a_1b_1 - \psi_0 + 1)|x| + (a_1^2 + \mu)|y| + (a_1 + 1)|z|\}m.\end{aligned}$$

Take $\lambda = \frac{\{(a_1b_1-\psi_0)+a_1^2+a_1+\mu+2\}L}{2} = 324.79 > 0$ *and let* $m = 0.01$, *then we find*

$$\begin{aligned}\mathcal{L}W \leq &- \big(13.58 - 6.06r\big)x^2\\ &- (51.35 - 349.23r)y^2 - \big(0.01 - 26.56r\big)z^2\\ &+ 0.03|x| + 1.37|y| + 0.13|z|,\end{aligned}$$

with

$$r < \min\{1.12, 0.07, 0.0002\}.$$

If we take $H = 0.3$ *and* $m = 0.01$, *then we obtain*

$$k = 0.01\max\{2.58, 137.49, 12.5\} \cong 1.37.$$

Now, we can satisfy the condition (ii) of Theorem 3.1 by taking

$$\nu = 0.15 \ \ and \ \ \beta(t) = 0.84, \ \ with \ n = 2.$$

Then, since $q_1 = q_2 = n = 2$, *we get all assumptions of Theorem 3.1 are satisfied.*
It follows from the above estimates, the following inequality holds

$$\int_{t_0}^{t}\{\gamma\nu(u) + \beta(u)\}e^{\int_{t_0}^{u}\nu(s)ds}du \leq 5.63, \quad for\ all\ t \geq t_0 \geq 0.$$

Furthermore,

$$|W_{x_i}(t,x)N_{ik}(t,x)| \leq \frac{1}{4}\bigg(77.85x^2 + 68.75y^2 + 6.25z^2\bigg) := \vartheta(t).$$

Next,

$$E^{x_0}\|x(t,t_0,x_0)\| \leq \{x_0^2 + 5.63\}^{1/2}, \ for\ all\ t \geq t_0 \geq 0. \tag{4.4}$$

Hence condition (3.3) is satisfied. By Corollary 3.2, we conclude that the zero solution of (4.3) is ν*-UEAS in probability with* $\Gamma = 1/2$.

Simulation of the solutions: Here, by the numerical methods we will simulate the solutions of (4.1) and (4.3). Consequently, Figures 11 and 12 show the behaviour of the stochastic stability of the solution for (4.1) with the noise $\alpha = 0.25$ and $\alpha = 10$, respectively. Furthermore, Figures 13–15 illustrate the behaviour of the boundedness of the solutions for (4.3), with the noise $\alpha = 0.25, \alpha = 10$, and $\alpha = 100$, respectively. Hence, we get from our figures, the simulated solutions are SAS and USB which justifies our given results.

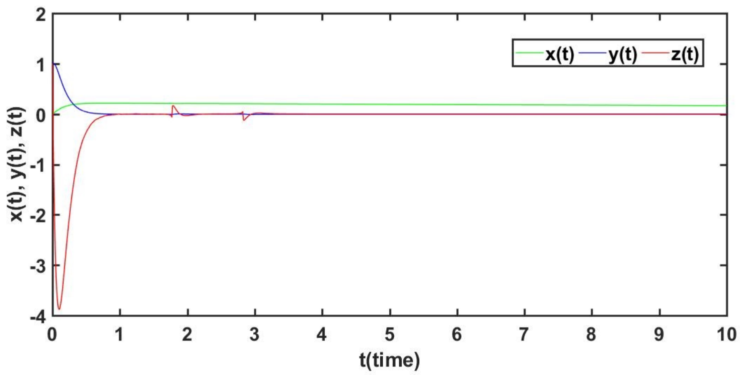

Figure 11. The behaviour for the stability of the solutions for (4.1), with $\alpha = 0.25.$.

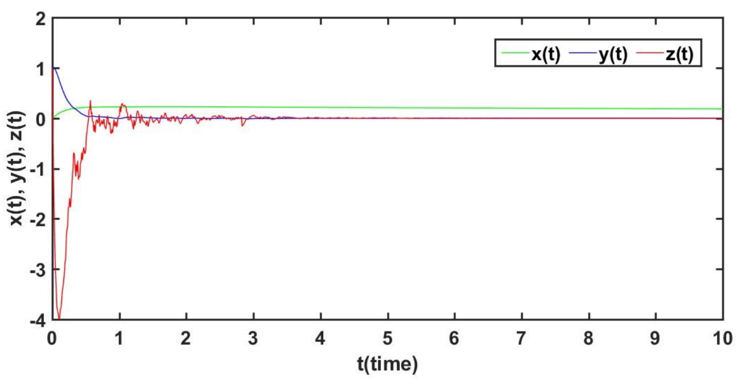

Figure 12. The behaviour for the stability of the solutions for (4.1), with $\alpha = 10.$.

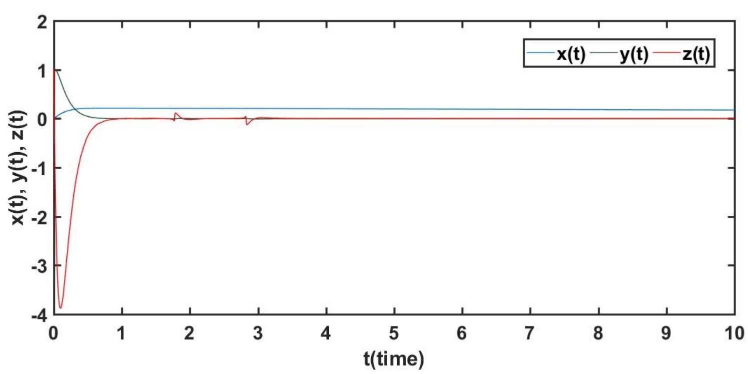

Figure 13. The path of the bonundedness for the solutions for (4.3), with $\alpha = 0.25.$.

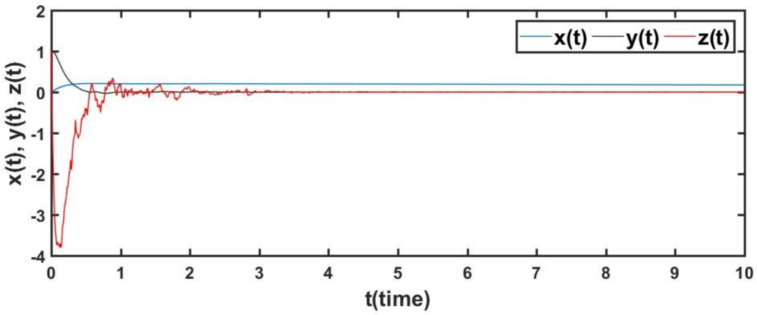

Figure 14. The path for the bonundedness of the solutions for (4.3), with $\alpha = 10.$.

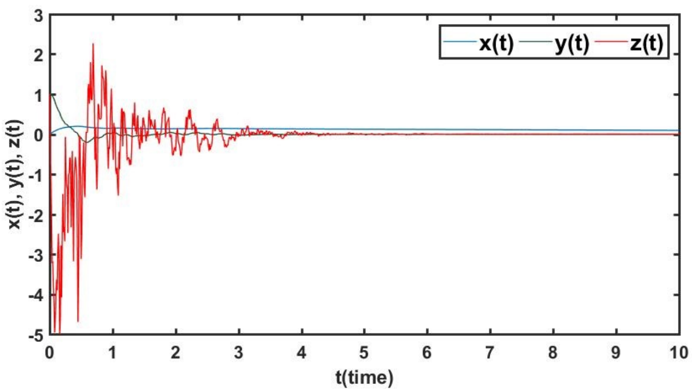

Figure 15. The path of the bonundedness of the solutions for (4.3), with $\alpha = 100.$.

5. Conclusion

The main results of the paper have discussed the following objects:
First: with $p = 0$, sufficiency criteria were established to study the SAS of the zero solution for (1.1).
Next: with $p \neq 0$, we established the sufficient conditions of the USB and UEAS in probability of solutions for (1.1).
Finally: two examples were given to illustrate feasibility of the established results and correctness of the main results. Lyapunove direct method was employed to set up the results.

The results obtained in this investigation extend many existing and exciting results on nonlinear nonautonomous third-order SDDE.

References

[1] Abdel-Razek MA, Mahmoud AM, Bakhit DAM. On stability and boundedness of solutions to a certain non-autonomous third-order functional differential equation with multiple deviating arguments. Differential Equations and Control Processes 2019; 1: 84-103.

[2] Abou-El-Ela AMA, Sadek AI, Mahmoud AM. Stability and boundedness of solutions of certain third-order nonlinear delay differential equation. International Journal on Automatic Control and System Engineering 2009; 9 (1): 9-15.

[3] Abou-El-Ela AMA, Sadek AI, Mahmoud AM. On the stability of solutions for certain second-order stochastic delay differential equations. Differential Equations and Control Processes 2015; 2: 1-13.

[4] Abou-El-Ela AMA, Sadek AI, Mahmoud AM, Farghaly ES. New stability and boundedness results for solutions of a certain third-order nonlinear stochastic differential equation. Asian Journal of Mathematics and Computer Research 2015; 5 (1): 60-70.

[5] Abou-El-Ela AMA, Sadek AI, Mahmoud AM, Farghaly ES. Stability of solutions for certain third-order nonlinear stochastic delay differential equations. Annal Journal of Applied Mathematics 2015; 31 (3): 253-261.

[6] Abou-El-Ela AMA, Sadek AI, Mahmoud AM, Taie ROA. On the stochastic stability and boundedness of solutions for stochastic delay differential equation of the second-order. Hindawi Publishing Corporation, Chinese Journal of Mathematics 2015; 2015: 1-8. https://doi.org/10.1155/2015/358936

[7] Abou-El-Ela AMA, Sadek AI, Mahmoud AM, Farghaly ES. Asymptotic stability of solutions for a certain non-autonomous second-order stochastic delay differential equation.Turkish Journal of Mathematics 2017; 41 (3): 576-584. https://doi.org/10.3906/mat-1508-62

[8] Ademola AT. Stability, boundedness and uniqueness of solutions to certain third-order stochastic delay differential equations. Differential Equations and Control Processes 2017; 2: 24-50.

[9] Ademola AT, Arawomo PO. Uniform stability and boundedness of solutions of nonlinear delay differential equations of third-order. Mathematical Journal of Okayama University 2013; 55: 157-166.

[10] Ademola AT, Arawomo PO, Ogunlaran OM, Oyekan EA. Uniform stability, boundedness and asymptotic behaviour of solutions of some third-order nonlinear delay differential equations. Differential Equations and Control Processes 2013; 4: 43-66.

[11] Ademola AT, Moyo S, Ogundare BS, Ogundiran MO, Adesina OA. Stability and boundedness of solutions to a certain second-order nonautonomous stochastic differential equation. Hindawi Publishing Corporation, International Journal of Analysis 2016; 2016: 1-11.

[12] Ademola AT, Akindeinde SO, Ogundare BS, Ogundiran MO, Adesina OA. On the stability and boundedness of solutions to certain second-order nonlinear stochastic delay differential equations. Journal of the Nigerian Mathematical Society 2019; 38 (2): 185-209. https://ojs.ictp.it/jnms/

[13] Arnold L. Stochastic Differential Equations: Theory and Applications. John Wiley and Sons, 1974.

[14] Graef JR, Tunç C. Global asymptotic stability and boundedness of certain multi-delay functional differential equations of third-order. Mathematical Methods in the Applied Sciences 2015; 38 (17): 3747-3752.

[15] It$\hat{o}$ K, Nisio M. On stationary solution of a stochastic differential equations. Journal of Mathematics of Kyoto University 1964; 4: 1-75.

[16] Li Z, Xu L. Exponential stability in mean square of stochastic functional differential equations with infinite delay. Acta Applicandae Mathematicae 2021; 174 (8): 1-18.

[17] Liu K. Stability of Infinite Dimensional Stochastic Differential Equation with Applications. Chapman & Hall/CRC: Boca Raton, 2006.

[18] Liu R, Raffoul Y. Boundedness and exponential stability of highly nonlinear stochastic differential equations. Electeronic Journal of Differential Equations 2009; 2009 (143): 1-10. http://ejde.math.txstate.edu Journal of Computational and Applied Mathematics 2006; 196: 87-93. https://doi.org/10.1016/j.cam.2005.08.023

[19] Mahmoud AM, Tunç C. A new result on the stability of a stochastic differential equation of third-order with a time-lag. Journal of the Egyptian Mathematical Society 2018; 26 (1): 202-210.

[20] Mahmoud AM, Bakhit DAM. Study of the stability behaviour and the boundedness of solutions to a certain third-order differential equation with a retarded argument. Annals of Applied Mathematics 2019; 35 (1): 99-110.

[21] Mahmoud AM, Tunç C. Asymptotic stability of solutions for a kind of third-order stochastic differential equations with delays. Miskolc Mathematical Notes 2019; 20 (1): 381-393. http://dx.doi.org/10.18514/MMN.2019.2800

[22] Mahmoud AM, Tunç C. Boundedness and exponential stability for a certain third-order stochastic delay differential equation. Dynamic Systems and Applications 2020; 29 (2): 288-302. http://dx.doi.org/10.46719/dsa20202927

[23] Mao X. Exponential stability for stochastic differential equations with respect to semimartingales. Stochastic Processes and Their Applications 1990; 35: 267-277.

[24] Mao X. Stability of Stochastic Differential Equations with respect to Semimartingales. Longman Scientific and Technical, 1991.

[25] Mao X. Exponential Stability of Stochastic Differential Equations. Marcel Dekker, 1994.

[26] Mao X. Stochastic Differential Equations and Applications. Horwood Publishing, Chichester, UK, 2007.

[27] Mao X, Yuan C, Zou J. Stochastic differential delay equations of population dynamics. Journal of Mathematical Analysis and Applications 2005; 304 (1): 296-320.

[28] Oksendal B. Stochastic Differential Equations, An introduction with Applications. Springer-Verlage, 2000.

[29] Omeike MO. New results on the stability of solution of some non-autonomous delay differential equations of the third-order. Differential Equations and Control Processes 2010; 1: 18-29.

[30] Omeike MO. Uniform ultimate boundedness of solutions of third-order nonlinear delay differential equations. Annals of the Alexandru Ioan Cuza University-Mathematics 2010; 2: 363-372.

[31] Omeike MO. Stability and boundedness of solutions of a certain system of third-order nonlinear delay differential equations. Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica 2015; 54 (1): 109-119.

[32] Oudjedi LD, Remili M. Boundedness and stability in third-order nonlinear vector differential equations with bounded delay. Afrika Mathematika 2018; 29: 899–909.

[33] Raffoul YN. Boundedness and exponential asymptotic stability in dynamical systems with applications to nonlinear differential equations with unbounded terms. Advances in Dynamical Systems and Applications 2007; 2 (1): 107-121.

[34] Remili M, Beldjerd D. On the asymptotic behavior of the solutions of third-order delay differential equations. Rendiconti del Circolo Matematico di Palermo 2014; 63 (3): 447-455. http://dx.doi.org/10.1007/s12215-014-0169-3

[35] Remili M, Oudjedi LD. Stability and boundedness of the solutions of non-autonomous third-order differential equations with delay. Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica 2014; 53 (2): 139-147.

[36] Remili M, Oudjedi LD. Uniform ultimate boundedness and asymptotic behaviour of third-order nonlinear delay differential equation. Afrika Matematika 2016; 27: 1227-1237.

[37] Sadek AI. On the stability of solutions of some non-autonomous delay differential equations of the third-order. Asymptotic Analysis 2005; 43: 1-7.

[38] Tunç C. On the stability and boundedness of solutions to third-order nonlinear differential equations with retarded argument. Nonlinear Dynamics 2009; 57: 97-106. http://dx.doi.org/10.1007/s11071-008-9423-6

[39] Tunç C. A new boundedness result to nonlinear differential equations of third order with finite lag. Communications in Applied Analysis 2009; 13 (1): 1-10.

[40] Tunç C. Bound of solutions to third-order nonlinear differential equations with bounded delay. Journal of the Franklin Institute 2010; 347 (2): 415-425. http://dx.doi.org/10.1016/j.jfranklin.2009.05.001

[41] Tunç C. On some qualitative behaviors of solutions to a kind of third-order nonlinear delay differential equations. Electronic Journal of Qualitative Theory of Differential Equations 2010; 12: 1-19.

[42] Tunç O, Tunç C. On the asymptotic stability of solutions of stochastic differential delay equations of second-order. Journal of Taibah University for Science 2019; 13 (1): 875-882.