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ORHAN, HALİT; AKTAŞ, İBRAHİM; and ARIKAN, HAVA (2023) "On a new subclass of biunivalent functions associated with the $(p,q)$-Lucas polynomials and bi-Bazilevic type functions of order $\rho+i\xi$," Turkish Journal of Mathematics: Vol. 47: No. 1, Article 7. https://doi.org/10.55730/1300-0098.3348
Available at: https://journals.tubitak.gov.tr/math/vol47/iss1/7

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On a new subclass of biunivalent functions associated with the \((p, q)\)-Lucas polynomials and bi-Bazilevič type functions of order \(\rho + i\xi\)

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Abstract: Using \((p, q)\)-Lucas polynomials and bi-Bazilevič type functions of order \(\rho + i\xi\), we defined a new subclass of biunivalent functions. We obtained coefficient inequalities for functions belonging to the new subclass. In addition to these results, the upper bound for the Fekete-Szegö functional was obtained. Finally, for some special values of parameters, several corollaries were presented.

Key words: Bazilevič functions, Lucas polynomial, analytic functions, univalent functions, biunivalent functions

1. Introduction

Let \(A\) denote the class of functions of the form

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \]  

which are analytic in the unit disk \(U = \{z \in \mathbb{C} : |z| < 1\}\) and normalized by the conditions \(f(0) = 0\) and \(f'(0) = 1\). Let \(S\) be the subclass of \(A\) consisting of functions univalent in \(A\). It is known that if \(f \in S\), then there exists the inverse function \(f^{-1}\). Because of the normalization \(f(0) = 0\), \(f^{-1}\) is defined in some neighborhood of the origin.

If the functions \(f\) and \(g \in A\), then \(f\) is said to be subordinate to \(g\) if there exists a Schwarz function \(w \in \Theta\), where

\[ \Theta = \{w : w(0) = 0 \text{ and } |w(z)| < 1 \ (z \in U)\}, \]

such that

\[ f(z) = g(w(z)) \quad (z \in U). \]

This subordination is shown by

\[ f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U). \]

If \(g\) is univalent function in \(U\), then this subordination is equivalent to

\[ f(0) = g(0), \quad f(U) \subset g(U). \]

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2010 AMS Mathematics Subject Classification: 30C45, 05A15, 30D15

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Let $\mathcal{P}$ denote the class of functions of the form

$$t(z) = 1 + t_1z + t_2z^2 + t_3z^3 + \cdots \quad (z \in U)$$

which are analytic and $\Re(t(z)) > 0$. Here the function $t(z)$ is called Carathéodory function.

We now turn to the Koebe one-quarter theorem (see [11]), which ensures that the image of $U$ under every function in the normalized univalent function class $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, clearly, every such univalent function has an inverse $f^{-1}$ which satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots := g(w).$$

A function $f \in \mathcal{A}$ is called biunivalent function in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. The class of biunivalent functions defined in the open unit disk $U$ is denoted by $\Sigma$. Comprehensive information and some interesting examples of the class $\Sigma$ can be found in the pioneering work [22] written by Srivastava et al. in 2010. As indicated in [22], the following examples can be given for functions in the class $\Sigma$:

$$\frac{z}{1-z}, -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function and also the functions

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are not biunivalent although they are univalent. Several important coefficient estimates of the functions in the class $\Sigma$ were given by many authors. For example, Lewin gave a bound for second coefficient of the class $\Sigma$ as $|a_2| \leq 1.51$ in [17], while, motivated by Lewin’s work, in [9] Brannan and Clunie presented a conjecture that $|a_2| \leq \sqrt{2}$. In the literature, one of the most important open problems for the class $\Sigma$ is the coefficient estimates on $|a_n|, n \in \mathbb{N}, n \geq 3$, (see [22]). In recent years, Brannan and Taha studied certain subclasses of the class $\Sigma$ and gave some coefficient estimates. In addition, motivated by the pioneering paper of Srivastava et al. [22], the authors in [1, 4, 5, 13–15, 20, 22, 28, 29] and the references therein defined some subclasses of the class $\Sigma$ and they gave nonsharp estimates on initial coefficients of mentioned subclasses. These subclasses were defined by using some polynomials such as Faber, Fibonacci, Lucas, Chebyshev, Pell, Lucas-Lehmer, orthogonal polynomials and their generalizations. Special polynomials and their generalizations are of great importance in a variety of branches such as physics, engineering, architecture, nature, art, number theory, combinatorics and numerical analysis. These polynomials have been studied in several papers from a theoretical point of view (see, for example, [25, 27–29, 31] and the references therein). In addition, some subclasses were also defined by making use of certain differential operators like Sălăgean, Hohlov, and Frasin.
This paper is organized as follows: The rest of this section is devoted to some basic definitions and preliminaries. Section 2 deals with initial coefficient estimates on new subclass introduced, while we investigate Fekete-Szegö problem for this new class in Section 3.

For $f(z)$ given by (1.1) and $g(z)$ defined by
\[ g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad b_k \geq 0 \]
the Hadamard product (or convolution) $(f \ast g)(z)$ of the functions $f(z)$ and $g(z)$ is defined by
\[ (f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z), \quad (z \in U). \]

Let $f \in A$. In [19], Sălăgean considered the following differential operator:
\[
\begin{align*}
D^0 f(z) &= f(z) \\
D^1 f(z) &= D f(z) = zf'(z) \\
&\vdots \\
D^\tau f(z) &= D(D^{\tau-1} f(z)). \quad (\tau \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).
\end{align*}
\]
Note that
\[ D^\tau f(z) = z + \sum_{k=2}^{\infty} k^\tau a_k z^k \quad (\tau \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \quad (1.2) \]

Consider the function
\[ f_\delta(z) = \int_0^z \left( \frac{1+r}{1-r} \right)^\delta \frac{1}{1-r^2} dr = z + \sum_{k=2}^{\infty} b_k(\delta) z^k, \quad \delta > 0, \; z \in U, \quad (1.3) \]
where
\[ b_2(\delta) = \delta \quad \text{and} \quad b_3(\delta) = \frac{1}{3} (2\delta^2 + 1). \]

It is worth mentioning that for $\delta < 1$, the function $zf_\delta'(z)$ is starlike with two slits. Moreover, since $zf_\delta'(z)$ is the Koebe function, all functions $f_\delta$ for $0 \leq \delta \leq 1$ are univalent and convex. More details about the function $f_\delta$ can be found in [26].

For $f \in A$, given by (1.1), we define the function $h_\delta$ ($\delta > 0$) as follows:
\[ h_\delta(z) = (f \ast f_\delta)(z) = z + \sum_{k=2}^{\infty} b_k(\delta) a_k z^k = (f_\delta \ast f)(z), \quad z \in U. \quad (1.4) \]

For $D^\tau f(z)$ given by (1.2) and $h_\delta(z)$ given by (1.4), we define the function $F(z)$ as follows:
\[ F(z) = D^\tau h_\delta(z) = z + \sum_{k=2}^{\infty} b_k(\delta) k^\tau a_k z^k. \quad (1.5) \]
In that case every such function \( F(z) \) has an inverse \( F^{-1}(z) \), which satisfies
\[
F^{-1}(w) = w - b_2(\delta) 2^\gamma a_2 w^2 + (b_2^2(\delta) 2^{2\gamma+1} a_2^2 - b_3(\delta) 3^\gamma a_3)w^3 \\
- (5b_2^3(\delta) 2^{3\gamma} a_2^3 - 5b_2 (\delta) 2^\gamma b_3(\delta) 3^\gamma a_2 a_3 + b_4(\delta) 4^\gamma a_4) w^4 + \cdots := G(w).
\]

The following is the definition of \((p,q)\)-Lucas polynomials introduced by Lee and Asci [16] and it is related to our study.

**Definition 1.1** [16] Let \( p(x) \) and \( q(x) \) be polynomials with real coefficients. The \((p,q)\)-Lucas Polynomials \( L_{p,q,n}(x) \) are defined by the recurrence relation
\[
L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x) \quad (n \geq 2),
\]
from which the first few Lucas polynomials can be expressed as below:
\[
L_{p,q,0}(x) = 2, \quad L_{p,q,1}(x) = p(x), \quad L_{p,q,2}(x) = p^2(x) + 2q(x).
\]  

(1.6)

For the special cases of \( p(x) \) and \( q(x) \), the \((p,q)\)-Lucas polynomials reduce to the special polynomials below: \( L_{x,1,n}(x) \equiv L_n(x) \) Lucas Polynomials, \( L_{2x,1,n}(x) \equiv D_n(x) \) Pell-Lucas Polynomials, \( L_{1,2x,n}(x) \equiv J_n(x) \) Jacobsthal-Lucas Polynomials, \( L_{3x,-2,n}(x) \equiv F_n(x) \) Fermat-Lucas Polynomials, \( L_{2x,-1,n}(x) \equiv T_n(x) \) Chebyshev Polynomials of the first kind.

**Lemma 1.2** [16] Let \( G_{\{L_n(x)\}}(z) \) be the generating function of the \((p,q)\)-Lucas Polynomials Sequence \( L_{p,q,n}(x) \). Then,
\[
G_{\{L_n(x)\}}(z) = \sum_{n=0}^{\infty} L_{p,q,n}(x) z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}
\]

and
\[
\Psi_{\{L_n(x)\}}(z) = G_{\{L_n(x)\}}(z) - 1 = 1 + \sum_{n=1}^{\infty} L_{p,q,n}(x) z^n = \frac{1 + q(x)z^2}{1 - p(x)z - q(x)z^2}.
\]

**Definition 1.3** [24] For \( \rho \geq 0, \xi \in \mathbb{R}, \rho + i\xi \neq 0, \) and \( F \in \mathcal{A} \), let \( \mathcal{B}(\rho, \xi, \delta, \tau) \) denote the class of Bazilević type function if and only if
\[
\Re \left[ \frac{zF'(z)}{F(z)} \left( \frac{F(z)}{z} \right)^{\rho+i\xi} \right] > 0.
\]

Many researchers have worked different subclasses of the famous Bazilević functions of type \( \rho \) from various view points (see [3] and [23]). In the literature, there are not many papers for \((p,q)\)-Lucas polynomials associated with Bazilević type functions of order \( \rho + i\xi \). One of the main goals of this paper is to contribute to this kind of studies. For this purpose, motivated by the very recent work of Ala Amourah et al. [6] (also see [18]), we introduce the new subclass \( \mathcal{B}(\rho, \xi, \delta, \tau) \) of biunivalent functions associated with bi-Bazilević type function and \((p,q)\)-Lucas polynomials.
Definition 1.4 For $F \in \Sigma$, $\rho \geq 0$, $\xi \in \mathbb{R}$, $\rho + i\xi \neq 0$, let $\bar{B}(\rho, \xi, \delta, \tau)$ denote the class of bi-Bazilevič type function of order type $\rho + i\xi$ if and only if

$$
\left[ \left( \frac{zF'(z)}{F(z)} \right) \left( \frac{F(z)}{z} \right) \right]^{\rho + i\xi} < \Psi_{\{L_n(x)\}}(z), \quad z \in \mathcal{U}
$$

(1.7)

and

$$
\left[ \left( \frac{wG'(w)}{G(w)} \right) \left( \frac{G(w)}{w} \right) \right]^{\rho + i\xi} < \Psi_{\{L_n(x)\}}(w), \quad w \in \mathcal{U},
$$

(1.8)

where $\Psi_{L_p,q_n(x)}(z) \in \mathcal{P}$ and the function $G$ is described as $G(w) = F^{-1}(w)$.

Remark 1.5 Note that, by specializing the parameters $\rho, \xi, \delta$ and $\tau$, we obtain the following subclasses studied by various authors.

1. $\bar{B}(\rho, \xi, 1, 0) \equiv B(\rho, \xi)$ (Ala Amourah et al. [6]).
2. $\bar{B}(\rho, 0, 1, 0) \equiv B(\rho)$ (Altınkaya et al. [2])

The class $\bar{B}(0, 0, \delta, \tau) = S_{\Sigma}^\ast$ is defined as follows:

Definition 1.6 A function $F \in \Sigma$ is said to be in the class $S_{\Sigma}^\ast$, if the following subordinations hold

$$
\left( \frac{zF'(z)}{F(z)} \right) < \Psi_{\{L_n(x)\}}(z), \quad z \in \mathcal{U}
$$

and

$$
\left( \frac{wG'(w)}{G(w)} \right) < \Psi_{\{L_n(x)\}}(w), \quad w \in \mathcal{U},
$$

where $G(w) = F^{-1}(w)$.

2. Coefficient estimates for the function class $\bar{B}(\rho, \xi, \delta, \tau)$

In this section, we propose to find the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the class $\bar{B}(\rho, \xi, \delta, \tau)$ which is introduced in Definition (1.4). We first state the following theorem.

Theorem 2.1 Let the function $F(z)$ given by (1.5) be in the class $\bar{B}(\rho, \xi, \delta, \tau)$. Then,

$$
|a_2| \leq \frac{1}{b_2(\delta) 2^\tau} \frac{|p(x)| \sqrt{2 |p(x)|}}{\sqrt{(\rho + 1)^2 + \xi^2 |(\rho + i\xi) p^2(x) + 4q(x) (\rho + i\xi + 1)|}}
$$

and

$$
|a_3| \leq \frac{1}{b_3(\delta) 3^\tau} \left\{ \frac{p^2(x)}{(\rho + 1)^2 + \xi^2} + \frac{|p(x)|}{\sqrt{(\rho + 2)^2 + \xi^2}} \right\}.
$$
Proof  Let \( F(z) \in \mathcal{B}(\rho, \xi, \delta, \tau) \). Then, there exist two analytic functions \( \gamma, \varphi : \mathcal{U} \to \mathcal{U} \) such that \( \gamma(0) = \varphi(0) = 0 \), \( |\gamma(z)| < 1 \) and \( |\varphi(w)| < 1 \). Thus, we can write from (1.7) and (1.8) that

\[
\left[ \left( \frac{zF'(z)}{F(z)} \right) \right]^{\rho + i\xi} = \Psi_{L_n(z)}(\gamma(z)) \quad (z \in \mathcal{U}) \tag{2.1}
\]

and

\[
\left[ \left( \frac{wG'(w)}{G(w)} \right) \right]^{\rho + i\xi} = \Psi_{L_n(z)}(\varphi(w)) \quad (w \in \mathcal{U}). \tag{2.2}
\]

It is well known that the following inequalities

\[ |\gamma(z)| = |\gamma_1 z + \gamma_2 z^2 + \cdots| < 1 \]

and

\[ |\varphi(w)| = |\varphi_1 w + \varphi_2 w^2 + \cdots| < 1, \]

imply that

\[ |\gamma_j| \leq 1 \quad \text{and} \quad |\varphi_j| \leq 1 \quad (j \in \mathbb{N}). \]

It can be easily seen that

\[
\Psi_{L_n(z)}(\gamma(z)) = 1 + L_{p,q,1}(x)\gamma_1 z + \left[ L_{p,q,1}(x)\gamma_2 + L_{p,q,2}(x)\gamma_1^2 \right] z^2 + \cdots \tag{2.3}
\]

and

\[
\Psi_{L_n(z)}(\varphi(w)) = 1 + L_{p,q,1}(x)\varphi_1 w + \left[ L_{p,q,1}(x)\varphi_2 + L_{p,q,2}(x)\varphi_1^2 \right] w^2 + \cdots. \tag{2.4}
\]

By taking into account the equalities (2.3) and (2.4) in the equalities (2.1) and (2.2), respectively, we deduce

\[
\left[ \left( \frac{zF'(z)}{F(z)} \right) \right]^{\rho + i\xi} = 1 + L_{p,q,1}(x)\gamma_1 z + \left[ L_{p,q,1}(x)\gamma_2 + L_{p,q,2}(x)\gamma_1^2 \right] z^2 + \cdots \tag{2.5}
\]

and

\[
\left[ \left( \frac{wG'(w)}{G(w)} \right) \right]^{\rho + i\xi} = 1 + L_{p,q,1}(x)\varphi_1 w + \left[ L_{p,q,1}(x)\varphi_2 + L_{p,q,2}(x)\varphi_1^2 \right] w^2 + \cdots. \tag{2.6}
\]

It follows from (2.5) and (2.6) that

\[
(\rho + i\xi + 1) b_2(\delta) 2^\tau a_2 = L_{p,q,1}(x)\gamma_1, \tag{2.7}
\]

\[
(\rho + i\xi + 2) \left[ (\rho + i\xi - 1) b_2^2(\delta) 2^{2\tau-1} a_2^2 + b_3(\delta) 3^\tau a_3 \right] = L_{p,q,1}(x)\gamma_2 + L_{p,q,2}(x)\gamma_1^2 \tag{2.8}
\]

and

\[
- (\rho + i\xi + 1) b_2(\delta) 2^\tau a_2 = L_{p,q,1}(x)\varphi_1, \tag{2.9}
\]

\[
(\rho + i\xi + 2) \left[ (\rho + i\xi + 3) b_2^2(\delta) 2^{2\tau-1} a_2^2 - b_3(\delta) 3^\tau a_3 \right] = L_{p,q,1}(x)\varphi_2 + L_{p,q,2}(x)\varphi_1^2. \tag{2.10}
\]
respectively. From (2.7) and (2.9), we get

\[ \gamma_1 = -\varphi_1 \]  

(2.11)

and

\[ 2(\rho + i\xi + 1)^2 b_2^2(\delta) 2^{2\tau} a_2^2 = L_{p,q,1}^2(x) \left( \gamma_1^2 + \varphi_1^2 \right). \]  

(2.12)

Also, adding (2.8) to (2.10) yields

\[ (\rho + i\xi + 1) (\rho + i\xi + 1) b_2^2(\delta) 2^{2\tau} a_2^2 = L_{p,q,1}(x) (\gamma_2 + \varphi_2) + L_{p,q,2}(x) \left( \gamma_1^2 + \varphi_1^2 \right). \]  

(2.13)

Now, using (2.12) in (2.13) implies that

\[ (\rho + i\xi + 1) \left[ (\rho + i\xi + 1) \frac{2L_{p,q,2}(x)(\rho + i\xi + 1)}{L_{p,q,1}(x)} \right] b_2^2(\delta) 2^{2\tau} a_2^2 = L_{p,q,1}(x) (\gamma_2 + \varphi_2) \]

and so, we can write that

\[ a_2^2 = \frac{L_{p,q,1}^3(x) (\gamma_2 + \varphi_2)}{b_2^2(\delta) 2^{2\tau} (\rho + i\xi + 1) \left[ (\rho + i\xi + 2) L_{p,q,1}(x) - 2L_{p,q,2}(x) (\rho + i\xi + 1) \right].} \]  

(2.14)

Considering (1.6) in (2.14), we can write that

\[ |a_2| \leq \frac{1}{b_2(\delta) 2^\tau \sqrt{(\rho + 1)^2 + \xi^2}} \frac{|\rho(x)| 2 |p(x)|}{\sqrt{2(\rho + 1)|p(x)| + |p(x)| (\rho + i\xi)(\rho + i\xi + 1)}}. \]

In order to prove the estimate on \(|a_3|\), let us subtract (2.10) from (2.8). As a result of this computation, we have

\[ (\rho + i\xi + 2) \left[ 2b_3(\delta) 3^\tau a_3 - b_2^2(\delta) 2^{2\tau + 1} a_2^2 \right] = L_{p,q,1}(x) (\gamma_2 - \varphi_2) + L_{p,q,2}(x) \left( \gamma_1^2 - \varphi_1^2 \right), \]

and since (2.11), we get

\[ 2(\rho + i\xi + 2) b_3(\delta) 3^\tau a_3 = L_{p,q,1}(x) (\gamma_2 - \varphi_2) + (\rho + i\xi + 2) b_2^2(\delta) 2^{2\tau + 1} a_2^2. \]

Thus, it is easily obtained that

\[ a_3 = \frac{L_{p,q,1}(x) (\gamma_2 - \varphi_2)}{2b_3(\delta) 3^\tau (\rho + i\xi + 2)} + \frac{b_2^2(\delta) 2^{2\tau} a_2^2}{b_3(\delta) 3^\tau}. \]  

(2.15)

By virtue of (2.11) and (2.12), we can write from (2.15) that

\[ a_3 = \frac{L_{p,q,1}(x)}{2b_3(\delta) 3^\tau (\rho + i\xi + 1)^2} \left( \gamma_1^2 + \varphi_1^2 \right) + \frac{L_{p,q,1}(x)}{2b_3(\delta) 3^\tau (\rho + i\xi + 2)} (\gamma_2 - \varphi_2), \]

and

\[ |a_3| \leq \frac{p^2(x)}{b_3(\delta) 3^\tau |\rho + i\xi + 1|^2} + \frac{p(x)}{b_3(\delta) 3^\tau |\rho + i\xi + 2|} = \frac{1}{b_3(\delta) 3^\tau} \left\{ \frac{p^2(x)}{(\rho + 1)^2 + \xi^2} + \frac{p(x)}{\sqrt{(\rho + 2)^2 + \xi^2}} \right\}. \]

The proof is thus completed. \hfill \Box

Putting \( \xi = 0 \), in Theorem 2.1, we get:
Corollary 2.2 Let the function $F(z)$ given by (1.5) be in the class $B(\rho, 0, \delta, \tau)$. Then,

$$|a_2| \leq \frac{1}{b_2(\delta)} \frac{|p(x)| \sqrt{2|p(x)|}}{(\rho + 1)^{3/2}|\rho p^2(x) + 4q(x)(\rho + 1)|}$$

and

$$|a_3| \leq \frac{1}{b_3(\delta)} \left\{ \frac{p^2(x)}{(\rho + 1)^2} + \frac{|p(x)|}{\rho + 2} \right\}$$

Remark 2.3 For the certain special values of the parameters in Theorem 2.1 and Corollary 2.2, respectively, we obtain some earlier results as follows:

i. By giving $\delta = 1$ and $\tau = 0$ in Theorem 2.1, we have the results by [6, Theorem 2.1].

ii. Letting $\tau = 0$ and $\delta = 1$ in Corollary 2.2, we have the results given by [6, Corollary 2.2].

iii. Taking $\rho = \xi = \tau = 0$ and $\delta = 1$ in Corollary 2.2, we get the results given by [3, Corollary 1].

3. Fekete-Szegö inequality for the class $B(\rho, \xi, \delta, \tau)$

In geometric function theory, the Fekete-Szegö inequality is an inequality for the coefficients of univalent analytic functions founded by Fekete and Szegö [12], related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete-Szegö problem. This problem have been handled by many authors for some function classes (see [7, 8, 21, 30]).

The Fekete-Szegö inequality states that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

is a univalent analytic function on the unit disk $U$ and $\lambda \in [0, 1)$, then

$$|a_3 - \lambda a_2^2| \leq 1 + 2e^{-2\lambda}.$$ 

In the limit case when $\lambda \to 1^-$, an elementary inequality is obtained given by $|a_3 - a_2^2| \leq 1$. It is known that the coefficient functional

$$\varsigma_\lambda(f) = a_3 - \lambda a_2^2$$

for the normalized analytic functions $f$ in the unit disk $U$ plays an important role in function theory.

In this section, we aim to provide Fekete-Szegö inequalities for functions in the class $B(\rho, \xi, \delta, \tau)$.

Theorem 3.1 Let $F$ given by (1.5) be in the class $B(\rho, \xi, \delta, \tau)$ and $\lambda \in \mathbb{R}$. Then,

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{p(x)}{b_3(\delta)} \frac{1}{\sqrt{(\rho + 2)^2 + \xi^2}}, & |\varsigma_\lambda(f)| \leq \frac{1}{2\sqrt{(\rho + 2)^2 + \xi^2}} \frac{1}{\sqrt{(\rho + 2)^2 + \xi^2}}, \\ \frac{b_3(\delta)}{2p(x)|\varsigma_\lambda(f)|} & |\varsigma_\lambda(f)| \geq \frac{1}{2\sqrt{(\rho + 2)^2 + \xi^2}}, \end{cases}$$

(3.1)

where

$$\varsigma_\lambda(f) = \frac{(b_3^2(\delta)2^{2\tau} - \lambda b_3(\delta)3^{3\tau}) L_{p,q,1}(x)}{b_3^2(\delta)2^{2\tau} (\rho + i\xi + 1) [(\rho + i\xi + 2) L_{p,q,1}(x) - 2L_{p,q,2}(x)(\rho + i\xi + 1)].}$$
Proof In order to prove the inequality (3.1), consider (2.14) and (2.15). It follows that

\[
a_3 - \lambda a_2^2 = \frac{\left( b_2^3(\delta) 2^{2\tau} - \lambda b_3(\delta) 3^\tau \right) L_{p,q,1}^3(x) \gamma_2 + \varphi_2}{b_2(\delta) 2^{2\tau} b_3(\delta) 3^\tau (\rho + i\xi + 1) \left[ (\rho + i\xi + 2) L_{p,q,1}^2(x) - 2 L_{p,q,2}(x)(\rho + i\xi + 1) \right]}
+ \frac{L_{p,q,1}(x) \gamma_2 - \varphi_2}{2 b_3(\delta) 3^\tau (\rho + i\xi + 2)}
+ \frac{L_{p,q,1}(x)}{b_3(\delta) 3^\tau} \left[ \left( h(\lambda) + \frac{1}{2(\rho + i\xi + 2)} \right) \gamma_2 + \left( h(\lambda) - \frac{1}{2(\rho + i\xi + 2)} \right) \varphi_2 \right].
\]

As a result, by virtue of (2.13), we deduce the desired result given in (3.1). □

By putting some special values to the parameters in Theorem 3.1, we arrive at the following corollaries.

Taking \( \xi = 0 \) in Theorem 3.1, we get

**Corollary 3.2** Let \( \mathcal{F} \) given by (1.5) be in the class \( \overline{B}(\rho,0,\delta,\tau) \). Then,

\[
|a_3 - \lambda a_2^2| \leq \begin{cases} 
\frac{p(x)}{(\rho^2 + 2)^2 + \xi^2}, & |s(\lambda)| \leq \frac{1}{2(\rho^2 + 2)}, \\
\frac{2^\tau x(1)}{2 b_3(\delta) 3^\tau}, & |s(\lambda)| \geq \frac{1}{2(\rho^2 + 2)},
\end{cases}
\]

where

\[
s(\lambda) = \frac{\left[ b_2^3(\delta) 2^{2\tau} - \lambda b_3(\delta) 3^\tau \right] L_{p,q,1}^2(x)}{b_2^2(\delta) 2^{2\tau}(\rho + 1) \left[ (\rho + 2) L_{p,q,1}^2(x) - 2 L_{p,q,2}(x)(\rho + 1) \right]}.
\]

It is important to mention here that the Fekete-Szegö functional will become second Hankel determinant \( H_2(1) \) for \( \lambda = 1 \). Taking \( \lambda = 1 \) in Theorem 3.1, we have

**Corollary 3.3** If \( \mathcal{F} \in \overline{B}(\rho,\xi,\delta,\tau) \), then

\[
|a_3 - a_2^2| \leq \begin{cases} 
\frac{p(x)}{b_3(\delta) 3^\tau \sqrt{[\rho + 2]^2 + \xi^2}}, & |h(1)| \leq \frac{1}{2 \sqrt{[\rho + 2]^2 + \xi^2}}, \\
\frac{2^\tau x(1)}{b_3(\delta) 3^\tau}, & |h(1)| \geq \frac{1}{2 \sqrt{[\rho + 2]^2 + \xi^2}},
\end{cases}
\]

where

\[
h(1) = \frac{\left[ b_2^3(\delta) 2^{2\tau} - b_3(\delta) 3^\tau \right] L_{p,q,1}^2(x)}{b_2^2(\delta) 2^{2\tau}(\rho + i\xi + 1) \left[ (\rho + i\xi + 2) L_{p,q,1}^2(x) - 2 L_{p,q,2}(x)(\rho + i\xi + 1) \right]}
\]

By choosing \( \rho = 0 = \xi \) and \( \lambda = 1 \) in Theorem 3.1, we obtain the following result

**Corollary 3.4** Let \( \mathcal{F} \) given by (1.5) be in the class \( \overline{B}(0,0,\delta,\tau) \). Then,

\[
|a_3 - a_2^2| \leq \begin{cases} 
\frac{p(x)}{2 b_3(\delta) 3^\tau}, & |s(1)| \leq \frac{1}{4}, \\
\frac{2^\tau x(1)}{b_3(\delta) 3^\tau}, & |s(1)| \geq \frac{1}{4},
\end{cases}
\]

where

\[
s(1) = \frac{\left[ b_1(\delta) 3^\tau - b_2^3(\delta) 2^{2\tau} \right] p^2(x)}{4^\tau + 1 b_2^2(\delta) q(x)}.
\]
Remark 3.5 Theorem 3.1 reduces to the following earlier results for special values of parameters:

i. For $\delta = 1$ and $\tau = 0$, we have the results given by [6, Theorem 3.1].

ii. For $\delta = \lambda = 1$ and $\tau = 0$, we have the results given by [6, Corollary 3.2].

iii. For $\delta = 1$ and $\tau = \xi = 0$, we have the results given by [6, Corollary 3.3].

iv. For $\delta = \lambda = 1$ and $\rho = \tau = \xi = 0$, we have the results given by [6, Corollary 3.4].

4. Conclusion

In the present investigation, we have defined a new subclass of analytic biunivalent function class $\Sigma$ by using $(p, q)$-Lucas polynomial and bi-Bazilevič type functions of order $\rho + i\xi$. Then, we have investigated certain properties such as nonsharp initial coefficient estimates and Fekete-Szegö problem for this subclass. Also, we have derived corresponding results for the some special values of the parameters. Our results generalize the recent papers [2, 3] and [6]. In the future, Hankel determinant problem for the subclass introduced here can be handled by researchers.

Acknowledgments

The authors are thankful to the referees for their helpful comments and suggestions.

References


