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On a new subclass of biunivalent functions associated with the (p,q)-Lucas polynomials and bi-Bazilevič type functions of order $\rho + i\xi$

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Abstract: Using (p,q)-Lucas polynomials and bi-Bazilevič type functions of order $\rho + i\xi$, we defined a new subclass of biunivalent functions. We obtained coefficient inequalities for functions belonging to the new subclass. In addition to these results, the upper bound for the Fekete-Szegö functional was obtained. Finally, for some special values of parameters, several corollaries were presented.

Key words: Bazilevič functions, Lucas polynomial, analytic functions, univalent functions, biunivalent functions

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathcal{A}$. It is known that if $f \in \mathcal{S}$, then there exists the inverse function f^{-1}. Because of the normalization $f(0) = 0$, f^{-1} is defined in some neighborhood of the origin.

If the functions f and $g \in \mathcal{A}$, then f is said to be subordinate to g if there exists a Schwarz function $w \in \Theta$, where

$$\Theta = \{w : w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \mathcal{U})\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathcal{U}).$$

This subordination is shown by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \mathcal{U}).$$

If g is univalent function in $\mathcal{U}$, then this subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathcal{U}) \subset g(\mathcal{U}).$$

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Let $\mathcal{P}$ denote the class of functions of the form

$$t(z) = 1 + t_1 z + t_2 z^2 + t_3 z^3 + \cdots \qquad (z \in \mathcal{U})$$

which are analytic and $\Re(t(z)) > 0$. Here the function $t(z)$ is called Carathéodory function.

We now turn to the Koebe one-quarter theorem (see [11]), which ensures that the image of $\mathcal{U}$ under every function in the normalized univalent function class $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, clearly, every such univalent function has an inverse f^{-1} which satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots := g(w).$$

A function $f \in \mathcal{A}$ is called biunivalent function in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. The class of biunivalent functions defined in the open unit disk $\mathcal{U}$ is denoted by Σ. Comprehensive information and some interesting examples of the class Σ can be found in the pioneering work [22] written by Srivastava et al. in 2010. As indicated in [22], the following examples can be given for functions in the class Σ:

$$\frac{z}{1-z}, -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function and also the functions

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are not biunivalent although they are univalent. Several important coefficient estimates of the functions in the class Σ were given by many authors. For example, Lewin gave a bound for second coefficient of the class Σ as $|a_2| \leq 1.51$ in [17], while, motivated by Lewin's work, in [9] Brannan and Clunie presented a conjecture that $|a_2| \leq \sqrt{2}$. In the literature, one of the most important open problems for the class Σ is the coefficient estimates on $|a_n|, n \in \mathbb{N}, n \geq 3$, (see [22]). In recent years, Brannan and Taha studied certain subclasses of the class Σ and gave some coefficient estimates. In addition, motivated by the pioneering paper of Srivastava et al. [22], the authors in [1, 4, 5, 13–15, 20, 22, 28, 29] and the references therein defined some subclasses of the class Σ and they gave nonsharp estimates on initial coefficients of mentioned subclasses. These subclasses were defined by using some polynomials such as Faber, Fibonacci, Lucas, Chebyshev, Pell, Lucas-Lehmer, orthogonal polynomials and their generalizations. Special polynomials and their generalizations are of great importance in a variety of branches such as physics, engineering, architecture, nature, art, number theory, combinatorics and numerical analysis. These polynomials have been studied in several papers from a theoretical point of view (see, for example, [25, 27–29, 31] and the references therein). In addition, some subclasses were also defined by making use of certain differential operators like Sălăgean, Hohlov, and Frasin.

This paper is organized as follows: The rest of this section is devoted to some basic definitions and preliminaries. Section 2 deals with initial coefficient estimates on new subclass introduced, while we investigate Fekete-Szegö problem for this new class in Section 3.

For $f(z)$ given by (1.1) and $g(z)$ defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad b_k \geq 0$$

the Hadamard product (or convolution) $(f * g)(z)$ of the functions $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in \mathcal{U}).$$

Let $f \in \mathcal{A}$. In [19], Sălăgean considered the following differential operator:

$$\begin{aligned} \mathcal{D}^0 f(z) &= f(z) \\ \mathcal{D}^1 f(z) &= \mathcal{D} f(z) = z f'(z) \\ &\vdots \\ \mathcal{D}^\tau f(z) &= \mathcal{D}(\mathcal{D}^{\tau-1} f(z)). \quad (\tau \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \end{aligned}$$

Note that

$$\mathcal{D}^\tau f(z) = z + \sum_{k=2}^{\infty} k^\tau a_k z^k \qquad (\tau \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.2}$$

Consider the function

$$f_\delta(z) = \int_0^z \left(\frac{1+r}{1-r}\right)^\delta \frac{1}{1-r^2} dr = z + \sum_{k=2}^{\infty} b_k(\delta) z^k, \quad \delta > 0, \ z \in \mathcal{U}, \tag{1.3}$$

where

$$b_2(\delta) = \delta \quad \text{and} \quad b_3(\delta) = \frac{1}{3}\left(2\delta^2 + 1\right).$$

It is worth mentioning that for $\delta < 1$, the function $z f_\delta'(z)$ is starlike with two slits. Moreover, since $z f_\delta'(z)$ is the Koebe function, all functions f_δ for $0 \leq \delta \leq 1$ are univalent and convex. More details about the function f_δ can be found in [26].

For $f \in \mathcal{A}$, given by (1.1), we define the function h_δ $(\delta > 0)$ as follows:

$$h_\delta(z) = (f * f_\delta)(z) = z + \sum_{k=2}^{\infty} b_k(\delta) a_k z^k = (f_\delta * f)(z), \quad z \in \mathcal{U}. \tag{1.4}$$

For $\mathcal{D}^\tau f(z)$ given by (1.2) and $h_\delta(z)$ given by (1.4), we define the function $\mathcal{F}(z)$ as follows:

$$\mathcal{F}(z) = \mathcal{D}^\tau h_\delta(z) = z + \sum_{k=2}^{\infty} b_k(\delta) k^\tau a_k z^k. \tag{1.5}$$

In that case every such function $\mathcal{F}(z) \in \mathcal{S}$ has an inverse $\mathcal{F}^{-1}(z)$, which satisfies

$$\begin{aligned}\mathcal{F}^{-1}(w) =& w - b_2(\delta)\, 2^{\tau} a_2 w^2 + (b_2^2(\delta)\, 2^{2\tau+1} a_2^2 - b_3(\delta)\, 3^{\tau} a_3) w^3 \\ & - \left(5 b_2^3(\delta)\, 2^{3\tau} a_2^3 - 5 b_2(\delta)\, 2^{\tau} b_3(\delta)\, 3^{\tau} a_2 a_3 + b_4(\delta)\, 4^{\tau} a_4\right) w^4 + \cdots := G(w).\end{aligned}$$

The following is the definition of (p,q)-Lucas polynomials introduced by Lee and Ascı [16] and it is related to our study.

Definition 1.1 *[16] Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The (p,q)-Lucas Polynomials $L_{p,q,n}(x)$ are defined by the recurrence relation*

$$\mathcal{L}_{p,q,n}(x) = p(x) L_{p,q,n-1}(x) + q(x) L_{p,q,n-2}(x) \quad (n \geq 2),$$

from which the first few Lucas polynomials can be expressed as below:

$$L_{p,q,0}(x) = 2, \qquad L_{p,q,1}(x) = p(x), \qquad L_{p,q,2}(x) = p^2(x) + 2q(x). \tag{1.6}$$

For the special cases of $p(x)$ and $q(x)$, the (p,q)- Lucas polynomials reduce to the special polynomials below: $L_{x,1,n}(x) \equiv L_n(x)$ Lucas Polynomials, $L_{2x,1,n}(x) \equiv \mathcal{D}_n(x)$ Pell-Lucas Polynomials, $L_{1,2x,n}(x) \equiv J_n(x)$ Jacobsthal-Lucas Polynomials, $L_{3x,-2,n}(x) \equiv F_n(x)$ Fermat-Lucas Polynomials, $L_{2x,-1,n}(x) \equiv T_n(x)$ Chebyshev Polynomials of the first kind.

Lemma 1.2 *[16] Let $\mathcal{G}_{\{L_n(x)\}}(z)$ be the generating function of the (p,q)-Lucas Polynomials Sequence $L_{p,q,n}(x)$. Then,*

$$\mathcal{G}_{\{L_n(x)\}}(z) = \sum_{n=0}^{\infty} L_{p,q,n}(x) z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}$$

and

$$\Psi_{\{L_n(x)\}}(z) = \mathcal{G}_{\{L_n(x)\}}(z) - 1 = 1 + \sum_{n=1}^{\infty} L_{p,q,n}(x) z^n = \frac{1 + q(x)z^2}{1 - p(x)z - q(x)z^2}.$$

Definition 1.3 *[24] For $\rho \geq 0$, $\xi \in \mathbb{R}$, $\rho + i\xi \neq 0$, and $\mathcal{F} \in \mathcal{A}$, let $\mathcal{B}(\rho, \xi, \delta, \tau)$ denote the class of Bazilevič type function if and only if*

$$\mathcal{R}e\left[\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)\left(\frac{\mathcal{F}(z)}{z}\right)^{\rho+i\xi}\right] > 0.$$

Many researchers have worked different subclasses of the famous Bazilevič functions of type ρ from various view points (see [3] and [23]). In the literature, there are not many papers for (p,q)-Lucas polynomials associated with Bazilevič type functions of order $\rho + i\xi$. One of the main goals of this paper is to contribute to this kind of studies. For this purpose, motivated by the very recent work of Ala Amourah et al. [6] (also see [18]), we introduce the new subclass $\widetilde{\mathcal{B}}(\rho, \xi, \delta, \tau)$ of biunivalent functions associated with bi-Bazilevič type function and (p,q)-Lucas polynomials.

Definition 1.4 *For $\mathcal{F} \in \Sigma$, $\rho \geq 0$, $\xi \in \mathbb{R}$, $\rho + i\xi \neq 0$, let $\widetilde{\mathcal{B}}(\rho, \xi, \delta, \tau)$ denote the class of bi-Bazilevič type function of order type $\rho + i\xi$ if and only if*

$$\left[\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)\left(\frac{\mathcal{F}(z)}{z}\right)^{\rho+i\xi}\right] \prec \Psi_{\{L_n(x)\}}(z), \qquad z \in \mathcal{U} \tag{1.7}$$

and

$$\left[\left(\frac{wG'(w)}{G(w)}\right)\left(\frac{G(w)}{w}\right)^{\rho+i\xi}\right] \prec \Psi_{\{L_n(x)\}}(w), \qquad w \in \mathcal{U}, \tag{1.8}$$

where $\Psi_{Lp,q,n(x)}(z) \in \mathcal{P}$ and the function G is described as $G(w) = \mathcal{F}^{-1}(w)$.

Remark 1.5 *Note that, by specializing the parameters ρ, ξ, δ and τ, we obtain the following subclasses studied by various authors.*

1. $\widetilde{\mathcal{B}}(\rho, \xi, 1, 0) \equiv \mathcal{B}(\rho, \xi)$ *(Ala Amourah et al.[6]).*
2. $\widetilde{\mathcal{B}}(\rho, 0, 1, 0) \equiv \mathcal{B}(\rho)$ *(Altınkaya et al. [2])*

The class $\widetilde{\mathcal{B}}(0, 0, \delta, \tau) = \mathcal{S}^*_\Sigma$ is defined as follows:

Definition 1.6 *A function $\mathcal{F} \in \Sigma$ is said to be in the class $\mathcal{S}^*_\Sigma$, if the following subordinations hold*

$$\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) \prec \Psi_{\{L_n(x)\}}(z), \quad z \in \mathcal{U}$$

and

$$\left(\frac{wG'(w)}{G(w)}\right) \prec \Psi_{\{L_n(x)\}}(w), \quad w \in \mathcal{U},$$

where $G(w) = \mathcal{F}^{-1}(w)$.

2. Coefficient estimates for the function class $\widetilde{\mathcal{B}}(\rho, \xi, \delta, \tau)$

In this section, we propose to find the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the class $\widetilde{\mathcal{B}}(\rho, \xi, \delta, \tau)$ which is introduced in Definition (1.4). We first state the following theorem.

Theorem 2.1 *Let the function $\mathcal{F}(z)$ given by (1.5) be in the class $\widetilde{\mathcal{B}}(\rho, \xi, \delta, \tau)$. Then,*

$$|a_2| \leq \frac{1}{b_2(\delta)\, 2^\tau} \frac{|p(x)|\sqrt{2\,|p(x)|}}{\sqrt{\sqrt{(\rho+1)^2 + \xi^2}\,|(\rho + i\xi)\,p^2(x) + 4q(x)\,(\rho + i\xi + 1)|}}$$

and

$$|a_3| \leq \frac{1}{b_3(\delta)\, 3^\tau}\left\{\frac{p^2(x)}{(\rho+1)^2 + \xi^2} + \frac{|p(x)|}{\sqrt{(\rho+2)^2 + \xi^2}}\right\}.$$

Proof Let $\mathcal{F}(z) \in \widetilde{\mathcal{B}}(\rho, \xi, \delta, \tau)$. Then, there exist two analytic functions $\gamma, \varphi : \mathcal{U} \to \mathcal{U}$ such that $\gamma(0) = \varphi(0) = 0$, $|\gamma(z)| < 1$ and $|\varphi(w)| < 1$. Thus, we can write from (1.7) and (1.8) that

$$\left[\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)\left(\frac{\mathcal{F}(z)}{z}\right)^{\rho+i\xi}\right] = \Psi_{\{L_n(x)\}}(\gamma(z)) \qquad (z \in \mathcal{U}) \tag{2.1}$$

and

$$\left[\left(\frac{wG'(w)}{G(w)}\right)\left(\frac{G(w)}{w}\right)^{\rho+i\xi}\right] = \Psi_{\{L_n(x)\}}(\varphi(w)) \qquad (w \in \mathcal{U}). \tag{2.2}$$

It is well known that the following inequalities

$$|\gamma(z)| = \left|\gamma_1 z + \gamma_2 z^2 + \cdots\right| < 1$$

and

$$|\varphi(w)| = \left|\varphi_1 w + \varphi_2 w^2 + \cdots\right| < 1,$$

imply that

$$|\gamma_j| \leq 1 \qquad \text{and} \qquad |\varphi_j| \leq 1 \qquad (j \in \mathbb{N}).$$

It can be easily seen that

$$\Psi_{\{L_n(x)\}}(\gamma(z)) = 1 + L_{p,q,1}(x)\gamma_1 z + \left[L_{p,q,1}(x)\gamma_2 + L_{p,q,2}(x)\gamma_1^2\right] z^2 + \cdots \tag{2.3}$$

and

$$\Psi_{\{L_n(x)\}}(\varphi(w)) = 1 + L_{p,q,1}(x)\varphi_1 w + \left[L_{p,q,1}(x)\varphi_2 + L_{p,q,2}(x)\varphi_1^2\right] w^2 + \cdots. \tag{2.4}$$

By taking into acount the equalities (2.3) and (2.4) in the equalities (2.1) and (2.2), respectively, we deduce

$$\left[\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)\left(\frac{\mathcal{F}(z)}{z}\right)^{\rho+i\xi}\right] = 1 + L_{p,q,1}(x)\gamma_1 z + \left[L_{p,q,1}(x)\gamma_2 + L_{p,q,2}(x)\gamma_1^2\right] z^2 + \cdots \tag{2.5}$$

and

$$\left[\left(\frac{wG'(w)}{G(w)}\right)\left(\frac{G(w)}{w}\right)^{\rho+i\xi}\right] = 1 + L_{p,q,1}(x)\varphi_1 w + \left[L_{p,q,1}(x)\varphi_2 + L_{p,q,2}(x)\varphi_1^2\right] w^2 + \cdots \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$(\rho + i\xi + 1)\, b_2(\delta)\, 2^\tau a_2 = L_{p,q,1}(x)\gamma_1, \tag{2.7}$$

$$(\rho + i\xi + 2)\left[(\rho + i\xi - 1)\, b_2^2(\delta)\, 2^{2\tau-1} a_2^2 + b_3(\delta)\, 3^\tau a_3\right] = L_{p,q,1}(x)\gamma_2 + L_{p,q,2}(x)\gamma_1^2 \tag{2.8}$$

and

$$-(\rho + i\xi + 1)\, b_2(\delta)\, 2^\tau a_2 = L_{p,q,1}(x)\varphi_1, \tag{2.9}$$

$$(\rho + i\xi + 2)\left[(\rho + i\xi + 3)\, b_2^2(\delta)\, 2^{2\tau-1} a_2^2 - b_3(\delta)\, 3^\tau a_3\right] = L_{p,q,1}(x)\varphi_2 + L_{p,q,2}(x)\varphi_1^2, \tag{2.10}$$

respectively. From (2.7) and (2.9), we get

$$\gamma_1 = -\varphi_1 \tag{2.11}$$

and

$$2\left(\rho+i\xi+1\right)^2 b_2^2\left(\delta\right) 2^{2\tau} a_2^2 = L_{p,q,1}^2(x)\left(\gamma_1^2+\varphi_1^2\right). \tag{2.12}$$

Also, adding (2.8) to (2.10) yields

$$\left(\rho+i\xi+2\right)\left(\rho+i\xi+1\right) b_2^2\left(\delta\right) 2^{2\tau} a_2^2 = L_{p,q,1}(x)\left(\gamma_2+\varphi_2\right)+L_{p,q,2}(x)\left(\gamma_1^2+\varphi_1^2\right). \tag{2.13}$$

Now, using (2.12) in (2.13) implies that

$$\left(\rho+i\xi+1\right)\left[\left(\rho+i\xi+2\right)-\frac{2L_{p,q,2}(x)\left(\rho+i\xi+1\right)}{L_{p,q,1}^2(x)}\right] b_2^2\left(\delta\right) 2^{2\tau} a_2^2 = L_{p,q,1}(x)\left(\gamma_2+\varphi_2\right)$$

and so, we can write that

$$a_2^2 = \frac{L_{p,q,1}^3(x)\left(\gamma_2+\varphi_2\right)}{b_2^2\left(\delta\right) 2^{2\tau}\left(\rho+i\xi+1\right)\left[\left(\rho+i\xi+2\right)L_{p,q,1}^2(x)-2L_{p,q,2}(x)\left(\rho+i\xi+1\right)\right]}. \tag{2.14}$$

Considering (1.6) in (2.14), we can write that

$$|a_2| \le \frac{1}{b_2\left(\delta\right) 2^{\tau}} \frac{|p(x)|\sqrt{2\,|p(x)|}}{\sqrt{\sqrt{(\rho+1)^2+\xi^2}\,\left|\left(\rho+i\xi\right)p^2(x)+4q(x)\left(\rho+i\xi+1\right)\right|}}.$$

In order to prove the estimate on $|a_3|$, let us subtract (2.10) from (2.8). As a result of this computation, we have

$$\left(\rho+i\xi+2\right)\left[2b_3(\delta)3^{\tau}a_3 - b_2^2\left(\delta\right) 2^{2\tau+1} a_2^2\right] = L_{p,q,1}(x)\left(\gamma_2-\varphi_2\right)+L_{p,q,2}(x)\left(\gamma_1^2-\varphi_1^2\right),$$

and since (2.11), we get

$$2\left(\rho+i\xi+2\right) b_3(\delta)3^{\tau}a_3 = L_{p,q,1}(x)\left(\gamma_2-\varphi_2\right)+\left(\rho+i\xi+2\right) b_2^2\left(\delta\right) 2^{2\tau+1} a_2^2.$$

Thus, it is easily obtained that

$$a_3 = \frac{L_{p,q,1}(x)\left(\gamma_2-\varphi_2\right)}{2b_3(\delta)3^{\tau}\left(\rho+i\xi+2\right)} + \frac{b_2^2\left(\delta\right) 2^{2\tau} a_2^2}{b_3(\delta)3^{\tau}}. \tag{2.15}$$

By virtue of (2.11) and (2.12), we can write from (2.15) that

$$a_3 = \frac{L_{p,q,1}^2(x)}{2b_3(\delta)3^{\tau}\left(\rho+i\xi+1\right)^2}\left(\gamma_1^2+\varphi_1^2\right) + \frac{L_{p,q,1}(x)}{2b_3(\delta)3^{\tau}\left(\rho+i\xi+2\right)}\left(\gamma_2-\varphi_2\right)$$

and

$$|a_3| \le \frac{p^2(x)}{b_3(\delta)3^{\tau}\left|\rho+i\xi+1\right|^2} + \frac{p(x)}{b_3(\delta)3^{\tau}\left|\rho+i\xi+2\right|} = \frac{1}{b_3\left(\delta\right)3^{\tau}}\left\{\frac{p^2(x)}{\left(\rho+1\right)^2+\xi^2} + \frac{p(x)}{\sqrt{\left(\rho+2\right)^2+\xi^2}}\right\}$$

The proof is thus completed. □

Putting $\xi = 0$, in Theorem 2.1, we get:

Corollary 2.2 *Let the function $\mathcal{F}(z)$ given by (1.5) be in the class $\widetilde{\mathcal{B}}(\rho,0,\delta,\tau)$. Then,*

$$|a_2| \leq \frac{1}{b_2(\delta)\,2^{\tau}} \frac{|p(x)|\sqrt{2\,|p(x)|}}{\sqrt{(\rho+1)\,|\rho p^2(x)+4q(x)\,(\rho+1)|}}$$

and

$$|a_3| \leq \frac{1}{b_3(\delta)\,3^{\tau}} \left\{ \frac{p^2(x)}{(\rho+1)^2} + \frac{|p(x)|}{\rho+2} \right\}$$

Remark 2.3 *For the certain special values of the parameters in Theorem 2.1 and Corollary 2.2, respectively, we obtain some earlier results as follows:*

i. *By giving $\delta = 1$ and $\tau = 0$ in Theorem 2.1, we have the results by [6, Theorem 2.1].*

ii. *Letting $\tau = 0$ and $\delta = 1$ in Corallary 2.2, we have the results given by [6, Corollary 2.2].*

iii. *Taking $\rho = \xi = \tau = 0$ and $\delta = 1$ in Corollary 2.2, we get the results given by [3, Corollary 1].*

3. Fekete-Szegö inequality for the class $\widetilde{\mathcal{B}}(\rho,\xi,\delta,\tau)$

In geometric function theory, the Fekete-Szegö inequality is an inequality for the coefficients of univalent analytic functions founded by Fekete and Szegö [12], related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete-Szegö problem. This problem have been handled by many authors for some function classes (see [7, 8, 21, 30]).

The Fekete-Szegö inequality states that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

is a univalent analytic function on the unit disk $\mathcal{U}$ and $\lambda \in [0,1)$, then

$$\left|a_3 - \lambda a_2^2\right| \leq 1 + 2e^{\frac{-2\lambda}{(1-\lambda)}}.$$

In the limit case when $\lambda \to 1^-$, an elementary inequality is obtained given by $\left|a_3 - a_2^2\right| \leq 1$. It is known that the coefficient functional

$$\varsigma_\lambda(f) = a_3 - \lambda a_2^2$$

for the normalized analytic functions f in the unit disk $\mathcal{U}$ plays an important role in function theory.

In this section, we aim to provide Fekete-Szegö inequalities for functions in the class $\widetilde{\mathcal{B}}(\rho,\xi,\delta,\tau)$.

Theorem 3.1 *Let $\mathcal{F}$ given by (1.5) be in the class $\widetilde{\mathcal{B}}(\rho,\xi,\delta,\tau)$ and $\lambda \in \mathbb{R}$. Then,*

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} \frac{p(x)}{b_3(\delta)3^{\tau}\sqrt{(\rho+2)^2+\xi^2}}, & |h(\lambda)| \leq \frac{1}{2\sqrt{(\rho+2)^2+\xi^2}} \\ \frac{2p(x)|h(\lambda)|}{b_3(\delta)3^{\tau}}, & |h(\lambda)| \geq \frac{1}{2\sqrt{(\rho+2)^2+\xi^2}} \end{cases}, \tag{3.1}$$

where

$$h(\lambda) = \frac{\left(b_2^2(\delta)2^{2\tau} - \lambda b_3(\delta)3^{\tau}\right) L_{p,q,1}^2(x)}{b_2^2(\delta)2^{2\tau}\,(\rho+i\xi+1)\left[(\rho+i\xi+2)\,L_{p,q,1}^3(x) - 2L_{p,q,2}(x)\,(\rho+i\xi+1)\right]}.$$

Proof In order to prove the inequality (3.1), consider (2.14) and (2.15). It follows that

$$
\begin{aligned}
a_3-\lambda a_2^2 &= \frac{\left(b_2^2\left(\delta\right)2^{2\tau}-\lambda b_3\left(\delta\right)3^{\tau}\right)L_{p,q,1}^3(x)\left(\gamma_2+\varphi_2\right)}{b_2^2\left(\delta\right)2^{2\tau}b_3\left(\delta\right)3^{\tau}\left(\rho+i\xi+1\right)\left[\left(\rho+i\xi+2\right)L_{p,q,1}^2(x)-2L_{p,q,2}(x)\left(\rho+i\xi+1\right)\right]} \\
&+\frac{L_{p,q,1}(x)\left(\gamma_2-\varphi_2\right)}{2b_3\left(\delta\right)3^{\tau}\left(\rho+i\xi+2\right)} \\
&=\frac{L_{p,q,1}(x)}{b_3\left(\delta\right)3^{\tau}}\left[\left(h(\lambda)+\frac{1}{2\left(\rho+i\xi+2\right)}\right)\gamma_2+\left(h(\lambda)-\frac{1}{2\left(\rho+i\xi+2\right)}\right)\varphi_2\right].
\end{aligned}
$$

As a result, by virtue of (2.13), we deduce the desired result given in (3.1). □

By putting some special values to the parameters in Theorem 3.1, we arrive at the following corollaries. Taking $\xi=0$ in Theorem 3.1, we get

Corollary 3.2 *Let $\mathcal{F}$ given by (1.5) be in the class $\widetilde{\mathcal{B}}(\rho,0,\delta,\tau)$. Then,*

$$
\left|a_3-\lambda a_2^2\right|\leq\begin{cases}\frac{p(x)}{(\rho+2)b_3(\delta)3^{\tau}}, & |s(\lambda)|\leq\frac{1}{2(\rho+2)}\\ \frac{2p(x)|s(\lambda)|}{b_3(\delta)3^{\tau}}, & |s(\lambda)|\geq\frac{1}{2(\rho+2)}\end{cases},
$$

where

$$
s(\lambda)=\frac{\left[b_2^2\left(\delta\right)2^{2\tau}-\lambda b_3\left(\delta\right)3^{\tau}\right]L_{p,q,1}^2(x)}{b_2^2\left(\delta\right)2^{2\tau}(\rho+1)\left[(\rho+2)L_{p,q,1}^2(x)-2L_{p,q,2}(x)(\rho+1)\right]}
$$

It is important to mention here that the Fekete-Szegö functional will become second Hankel determinant $H_2(1)$ for $\lambda=1$. Taking $\lambda=1$ in Theorem 3.1, we have

Corollary 3.3 *If $\mathcal{F}\in\widetilde{\mathcal{B}}(\rho,\xi,\delta,\tau)$, then*

$$
\left|a_3-a_2^2\right|\leq\begin{cases}\frac{p(x)}{b_3(\delta)3^{\tau}\sqrt{(\rho+2)^2+\xi^2}}, & |h(1)|\leq\frac{1}{2\sqrt{(\rho+2)^2+\xi^2}}\\ \frac{2p(x)|h(1)|}{b_3(\delta)3^{\tau}}, & |h(1)|\geq\frac{1}{2\sqrt{(\rho+2)^2+\xi^2}}\end{cases},
$$

where

$$
h(1)=\frac{\left[b_2^2(\delta)2^{2\tau}-b_3(\delta)3^{\tau}\right]L_{p,q,1}^2(x)}{b_2^2(\delta)2^{2\tau}(\rho+i\xi+1)\left[(\rho+i\xi+2)L_{p,q,1}^2(x)-2L_{p,q,2}(x)(\rho+i\xi+1)\right]}
$$

By choosing $\rho=0=\xi$ and $\lambda=1$ in Theorem 3.1, we obtain the following result

Corollary 3.4 *Let $\mathcal{F}$ given by (1.5) be in the class $\widetilde{\mathcal{B}}(0,0,\delta,\tau)$. Then,*

$$
\left|a_3-a_2^2\right|\leq\begin{cases}\frac{p(x)}{2b_3(\delta)3^{\tau}}, & |s(1)|\leq\frac{1}{4}\\ \frac{2p(x)|s(1)|}{b_3(\delta)3^{\tau}}, & |s(1)|\geq\frac{1}{4}\end{cases},
$$

where

$$
s(1)=\frac{\left[b_3\left(\delta\right)3^{\tau}-b_2^2\left(\delta\right)2^{2\tau}\right]p^2(x)}{4^{\tau+1}b_2^2\left(\delta\right)q(x)}.
$$

Remark 3.5 *Theorem 3.1 reduces to the following earlier results for special values of parameters:*

i. *For* $\delta = 1$ *and* $\tau = 0$, *we have the results given by [6, Theorem 3.1].*

ii. *For* $\delta = \lambda = 1$ *and* $\tau = 0$, *we have the results given by [6, Corollary 3.2].*

iii. *For* $\delta = 1$ *and* $\tau = \xi = 0$, *we have the results given by [6, Corollary 3.3].*

iv. *For* $\delta = \lambda = 1$ *and* $\rho = \tau = \xi = 0$, *we have the results given by [6, Corollary 3.4].*

4. Conclusion

In the present investigation, we have defined a new subclass of analytic biunivalent function class Σ by using (p,q)-Lucas polynomial and bi-Bazilevič type functions of order $\rho + i\xi$. Then, we have investigated certain properties such as nonsharp initial coefficient estimates and Fekete-Szegö problem for this subclass. Also, we have derived corresponding results for the some special values of the parameters. Our results generalize the recent papers [2, 3] and [6]. In the future, Hankel determinant problem for the subclass introduced here can be handled by researchers.

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