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Forming coupled dispersionless equations of families of Bertrand curves

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Abstract: In this study, we establish a link of the coupled dispersionless (CD) equations system with the motion of Bertrand curve pairs. Moreover, we find the Lax equations that provide the integrability of these equations. By taking an appropriate choice of variables we show the link of the short pulse (SP) equation with the motion of Bertrand curve pairs via the reciprocal (hodograph) transformation. Finally, we prove that the conserved quantity of the corresponding coupled dispersionless equations obtained from each of these curve pairs is constant.

Key words: Bertrand curves, coupled dispersionless equations, Lax pair

1. Introduction

Curves and surfaces in 3-dimensional Euclidean space are fundamental subjects of differential geometry. It is known that curve pairs can be obtained by correlating the Frenet frames of the curves. One of the notable curve pairs is the Bertrand curve pair. In 1845, de Saint Venant brought forward an argument that whether a second curve with the principal normal vector field of a curve in Euclidean space would exist [30]. In the paper published by Bertrand in 1850, a second curve with the same principal normal vector field was required to exist. Bertrand stated that the condition of $\lambda\kappa + \mu\tau = 1$ for $\lambda, \mu \in R$ must be satisfied where the curvatures of the curve are denoted with κ and τ [6]. Afterwards, the Bertrand curve was named for the first curve and the Bertrand conjugate curve for the second curve [23]. Bertrand curves in different dimensional Euclidean spaces have been researched and many characteristic features have been given by [7, 8, 10, 19, 28, 33]. Also, the curve pairs in various spaces have been investigated with a variety of approaches [1–5, 11, 15, 26, 34]. Many problems modeling physical phenomena in nature are expressed by systems of differential equations. As a solution to a differential equation indicates a family of curves and a family of curves expresses the differential equation [9, 14, 24, 25, 29]. One of these differential equations is coupled dispersionless (CD) equation. Studies on coupled dispersionless equations play an important role in solving problems encountered in physics, mathematics, and various engineering disciplines. CD equations were first presented and solved by Konno and Oono using the inverse scattering transform method (IST) [20, 22]. The coupled dispersionless (CD) equations which are nonlinear differential equations are

$$\begin{aligned} u_{ys} &= \rho u, \\ \rho_s + u u_y &= 0 \end{aligned} \tag{1.1}$$

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where u is a real-valued function, the subscripts y and s denote partial differentiations [20, 22]. The generalized and complex versions of these CD equations were investigated in [17, 18]. The real and complex coupled dispersionless equations were obtained from the space curve family by Shen, Feng, and Ohta [32]. The Lax pair of these CD equations are expressed as

$$\psi_y = U\psi, \quad \psi_s = V\psi, \tag{1.2}$$

$$U = -i\lambda_1 \begin{pmatrix} \rho & u_y \\ u_y & -\rho \end{pmatrix}, \quad V = \begin{pmatrix} \frac{i}{4\lambda_1} & -\frac{1}{2}u \\ \frac{1}{2}u & -\frac{i}{4\lambda_1} \end{pmatrix}. \tag{1.3}$$

Here, $U_s - V_y + [U, V] = 0$ is provided [32]. Also, it is known that the CD equations system is equivalent to the short pulse (SP) equation [31]

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx} \tag{1.4}$$

with the reciprocal (hodograph) transformation $(y, s) \to (x, t)$ given by

$$\frac{\partial x}{\partial y} = \rho, \qquad \frac{\partial x}{\partial s} = -\frac{1}{2}q^2 \tag{1.5}$$

which is the transformation between the Lagrangian and the Eulerian coordinates [32].

The complex coupled dispersionless equation is obtained from the timelike curve according to the Darboux frame in Minkowski space [12]. In 1968 some formulas providing the integrability of nonlinear differential equations were expressed by P.D. Lax and these formulas are called Lax formulas. Using these Lax equations, the integrability conditions of these equations in Euclidean and Minkowski spaces were presented in [13, 21, 27]. In these regards, this study aims to research the coupled dispersionless equations of the motion of Bertrand curve pairs.

2. Preliminaries

Let $\gamma = \gamma(y)$ be a regular unit speed curve in Euclidean 3-space. If T, N, and B denote the tangent, principal normal, and binormal unit vectors at the point $\gamma(y)$ of the curve γ, respectively. Then the Frenet formulas are given

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_y = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where $\kappa = \langle T', N\rangle$ and $\tau = -\langle N, B'\rangle$ are the curvature and the torsion of the curve γ, respectively and "$'$" denotes the differentiation according to the parameter y. Let us give the definitions and theorems related to Bertrand curves which are the subject of many studies in differential geometry.

Definition 2.1 *Let γ and $\tilde{\gamma}$ be regular unit speed two curves in the Euclidean 3-space at any points $\gamma(y)$ and $\tilde{\gamma}(y)$. If the principal normal vectors of the curve γ and the curve $\tilde{\gamma}$ are linearly dependent, the $(\gamma, \tilde{\gamma})$ is called the Bertrand partner curves [8, 33].*

The Frenet frames of the curves γ and $\tilde{\gamma}$ will be denoted by $\{T, N, B\}$ and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$, respectively. The equation of the curve $\tilde{\gamma}$ is

$$\tilde{\gamma}(y) = \gamma(y) + \lambda(y)N(y) \tag{2.2}$$

where $\lambda(y)$ is a real-valued function [8, 33].

Theorem 2.2 *Let γ and $\tilde{\gamma}$ be Bertrand partner curves, then the angle between the tangent vectors of these curves at corresponding points is constant [8, 33].*

Theorem 2.3 *Let γ and $\tilde{\gamma}$ be Bertrand partner curves, then the relation between the curvature κ and the torsion τ of the curve γ is*

$$\lambda\kappa + \mu\tau = 1. \tag{2.3}$$

[33].

Theorem 2.4 *Let γ and $\tilde{\gamma}$ be Bertrand partner curves and the Frenet frames of the curves γ and $\tilde{\gamma}$ be $\{T, N, B\}$ and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$, respectively, then the relationship between the Frenet frames of the curves γ and $\tilde{\gamma}$ is*

$$\begin{aligned} \tilde{T} &= \cos\theta T - \sin\theta B, \\ \tilde{N} &= N, \\ \tilde{B} &= \sin\theta T + \cos\theta B, \end{aligned} \tag{2.4}$$

where $\left\langle T, \tilde{T}\right\rangle = \cos\theta$ [33].

Theorem 2.5 *Let γ and $\tilde{\gamma}$ be Bertrand partner curves and the curvature and torsion of the curves γ and $\tilde{\gamma}$ be κ, τ and $\tilde{\kappa}, \tilde{\tau}$, respectively, then the relationships between the curvatures and the torsions of these curves are*

$$\begin{aligned} \tilde{\kappa} &= \tfrac{\lambda\kappa - \sin\theta^2}{\lambda(1-\lambda\kappa)}, \\ \tilde{\tau} &= \tfrac{\sin\theta^2}{\lambda^2\tau}, \end{aligned} \tag{2.5}$$

where $\left\langle T, \tilde{T}\right\rangle = \cos\theta$ [33].

3. The link of the coupled dispersionless equations with Bertrand curves

In this part of the study, we investigate the correlation between the families of Bertrand curves and the equations of CD. Let us assume that

$$\gamma(y, s) : [0, l] \times [0, S] \to E^3$$

is a family of space curves, where $y \in [0, l]$ is the arc-length parameter and s represents the time. The time evolution of the orthonormal frame $\{T, N, B\}$ of the curve $\gamma(y, s)$ in matrix form is given as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \delta \\ -\beta & -\delta & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.1}$$

where α, β and δ are functions of y and s [32].

Theorem 3.1 *Let $\gamma(y, s)$ be a space curve family, then the coupled dispersionless equation corresponds to the set $\{\kappa, \tau, \alpha, \beta, \delta\} = \left\{c\rho, cu_y, -c^{-1}, u, 0\right\}$ [32].*

Corollary 3.2 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then we can write the equation*

$$c\left(\lambda\rho+\mu u_y\right)=1. \tag{3.2}$$

Also, the Frenet formulae for $\tilde{\gamma}$ is given as

$$\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix}_y=\begin{bmatrix}0 & \tilde{\kappa} & 0\\ -\tilde{\kappa} & 0 & \tilde{\tau}\\ 0 & -\tilde{\tau} & 0\end{bmatrix}\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix} \tag{3.3}$$

and also, the time evolution of the frame $\left\{\tilde{T},\tilde{N},\tilde{B}\right\}$ is written

$$\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix}_s=\begin{bmatrix}0 & \varepsilon & \eta\\ -\varepsilon & 0 & \xi\\ -\eta & -\xi & 0\end{bmatrix}\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix}, \tag{3.4}$$

where ε,η and ξ are functions of y and s.

Theorem 3.3 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves and θ be an angle between the tangent vectors of γ and $\tilde{\gamma}$. Then θ satisfies*

$$\theta=\arctan\left(\frac{\lambda c u_y}{\lambda c\rho-1}\right),$$

where $\lambda\neq\frac{1}{c\rho}=\frac{1}{\kappa}$.

Proof Considering the equations (2.1), $\kappa=c\rho$ and $\tau=cu_y$, if we take differentiation of the equation (2.2) with respect to arc-length parameter y, we get

$$\tilde{\gamma}'=\left(1-\lambda c\rho\right)T+\left(\lambda c u_y\right)B.$$

The norm of this equation is found as

$$\|\tilde{\gamma}'\|=\sqrt{1-2\lambda c\rho+\lambda^2c^2\left(\rho^2+u_y^2\right)}.$$

So, we obtain the Frenet vectors of $\tilde{\gamma}$ as

$$\begin{aligned}\tilde{T}&=\frac{(1-\lambda c\rho)T+\lambda c u_y B}{\sqrt{1-2\lambda c\rho+\lambda^2c^2\left(\rho^2+u_y^2\right)}},\\ \tilde{N}&=N,\\ \tilde{B}&=\frac{-\lambda c u_y T+(1-\lambda c\rho)B}{\sqrt{1-2\lambda c\rho+\lambda^2c^2\left(\rho^2+u_y^2\right)}}.\end{aligned} \tag{3.5}$$

By comparing this last equation (3.5) with the equation (2.4), we get

$$\sin\theta=\frac{-\lambda c u_y}{\sqrt{1-2\lambda c\rho+\lambda^2c^2\left(\rho^2+u_y^2\right)}}\text{ and }\cos\theta=\frac{1-\lambda c\rho}{\sqrt{1-2\lambda c\rho+\lambda^2c^2\left(\rho^2+u_y^2\right)}}.$$

Thus this easily completes the proof. □

We know $\mu = \frac{1-\lambda c\rho}{cu_y}$ from the equation (3.2) and if we substitute this equation into the equation (3.5), we can give the following corollary.

Corollary 3.4 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then the relationships between the Frenet frames' fields of γ and $\tilde{\gamma}$ are*

$$\begin{aligned} \tilde{T} &= \frac{\mu T + \lambda B}{\sqrt{\mu^2+\lambda^2}}, \\ \tilde{N} &= N, \\ \tilde{B} &= \frac{-\lambda T + \mu B}{\sqrt{\mu^2+\lambda^2}}. \end{aligned} \tag{3.6}$$

Theorem 3.5 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then the curvature and torsion of $\tilde{\gamma}$ are*

$$\tilde{\kappa} = \frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2+\mu^2\right)} \quad \text{and} \quad \tilde{\tau} = \frac{\mu u_y + \lambda\rho}{u_y\left(\lambda^2+\mu^2\right)}, \tag{3.7}$$

respectively.

Proof If we take derivative of the equation (2.2) and considering the equations $\kappa = c\rho$ and $\tau = cu_y$, then we get

$$\tilde{T}\frac{d\tilde{y}}{dy} = \left(1-\lambda c\rho\right)T + \lambda cu_y B.$$

From the inner product of this last equation with the vector $\tilde{T}$, we get the arc-length of the curve $\tilde{\gamma}$ as

$$\frac{d\tilde{y}}{dy} = \sqrt{\left(1-\lambda c\rho\right)^2 + \left(\lambda cu_y\right)^2}.$$

So, we obtain

$$\frac{d\tilde{y}}{dy} = cu_y\sqrt{\lambda^2+\mu^2}.$$

In this way, considering the equations (3.3) and (3.6), we find

$$\begin{aligned} \frac{d\tilde{T}}{d\tilde{y}} &= \frac{d\tilde{T}}{d\tilde{y}}\frac{d\tilde{y}}{dy} = \frac{c}{\sqrt{\lambda^2+\mu^2}}\left(\mu\rho - \lambda u_y\right)N, \\ \frac{d\tilde{T}}{d\tilde{y}} &= \frac{1}{u_y\left(\lambda^2+\mu^2\right)}\left(\mu\rho - \lambda u_y\right)N. \end{aligned}$$

From the inner product of this last equation with the vector $\tilde{N}$, we get the curvature of the curve $\tilde{\gamma}$ as

$$\tilde{\kappa} = \frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2 + \mu^2\right)}.$$

Similarly, considering the equations (3.3) and (3.6), we find

$$\begin{array}{l} \frac{d\tilde{B}}{d\tilde{y}} = \frac{d\tilde{B}}{d\tilde{y}}\frac{d\tilde{y}}{dy} = -\frac{c}{\sqrt{\lambda^2+\mu^2}}\left(\mu u_y + \lambda\rho\right)N, \\ \frac{d\tilde{B}}{d\tilde{y}} = -\frac{1}{u_y\left(\lambda^2+\mu^2\right)}\left(\mu u_y + \lambda\rho\right)N. \end{array}$$

From the inner product of this last equation with the vector $\tilde{N}$, we get the torsion of the curve $\tilde{\gamma}$ as

$$\tilde{\tau} = \frac{\mu u_y + \lambda\rho}{u_y\left(\lambda^2 + \mu^2\right)}.$$

□

Theorem 3.6 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$ denotes the Frenet frame of any Bertrand conjugate curve $\tilde{\gamma}$, then the following statement provides the CD equation:*

$$\begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix}_s = \begin{bmatrix} 0 & -\omega^{-1} & q \\ \omega^{-1} & 0 & 0 \\ -q & 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix}, \tag{3.8}$$

where $\omega \neq 0$ is constant and q is a real function.

Proof From the Frenet derivative formulae and the time evolution for the Frenet frame given in (3.3) and (3.4), respectively, we write

$$\begin{array}{l} \tilde{T}_y = \left(\frac{\mu\rho-\lambda u_y}{u_y\left(\lambda^2+\mu^2\right)}\right)\tilde{T}, \\ \tilde{N}_y = \left(-\frac{\mu\rho-\lambda u_y}{u_y\left(\lambda^2+\mu^2\right)}\right)\tilde{T} + \left(\frac{1}{cu_y\left(\lambda^2+\mu^2\right)}\right)\tilde{B}, \\ \tilde{B}_y = \left(-\frac{1}{cu_y\left(\lambda^2+\mu^2\right)}\right)\tilde{N}. \end{array} \quad \text{and} \quad \begin{array}{l} \tilde{T}_s = \varepsilon\tilde{N} + \eta\tilde{B}, \\ \tilde{N}_s = -\varepsilon\tilde{T} + \xi\tilde{B}, \\ \tilde{B}_s = -\eta\tilde{T} - \xi\tilde{N}. \end{array}$$

Thus, under favorable conditions $\tilde{T}_{sy} = \tilde{T}_{ys}, \tilde{N}_{sy} = \tilde{N}_{ys}, \tilde{B}_{sy} = \tilde{B}_{ys}$, we have

$$\varepsilon_y = \left(\frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2 + \mu^2\right)}\right)_s + \left(\frac{1}{cu_y\left(\lambda^2 + \mu^2\right)}\right)\eta, \tag{3.9}$$

$$\eta_y = \left(\frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2 + \mu^2\right)}\right)\xi - \left(\frac{1}{cu_y\left(\lambda^2 + \mu^2\right)}\right)\varepsilon, \tag{3.10}$$

$$\xi_y = \left(\frac{1}{cu_y\left(\lambda^2 + \mu^2\right)}\right)_s - \left(\frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2 + \mu^2\right)}\right)\eta. \tag{3.11}$$

By the hypothesis $\varepsilon = -\omega^{-1}$, $\eta = q$ and $\xi = 0$ are satisfied and then the equations (3.9)-(3.11) become

$$\left(\frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2+\mu^2\right)}\right)_s = -q\left(\frac{1}{cu_y\left(\lambda^2+\mu^2\right)}\right), \tag{3.12}$$

$$q_y = \omega^{-1}\left(\frac{1}{cu_y\left(\lambda^2+\mu^2\right)}\right), \tag{3.13}$$

$$\left(\frac{1}{cu_y\left(\lambda^2+\mu^2\right)}\right)_s = \left(\frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2+\mu^2\right)}\right)q, \tag{3.14}$$

respectively. If we take derivative of the equation (3.13), we have $\left(\frac{1}{cu_y(\lambda^2+\mu^2)}\right)_s = \omega q_{ys}$. By substituting this equation into the equation (3.14), we find

$$q_{ys} = pq, \tag{3.15}$$

where

$$p = \frac{1}{\omega}\left(\frac{\mu\rho - \lambda u_y}{u_y\left(\lambda^2+\mu^2\right)}\right). \tag{3.16}$$

By substituting the equations (3.13) and (3.16) into the equation (3.12), respectively, we have

$$p_s + qq_y = 0. \tag{3.17}$$

As a result, the equations (3.15) and (3.17) express the CD equations and this completes the proof. □

Corollary 3.7 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then there are the following relations for the curvature $\tilde{\kappa}$ and the torsion $\tilde{\tau}$ of the curve $\tilde{\gamma}$*

$$\tilde{\kappa} = \omega p \quad \textit{and} \quad \tilde{\tau} = \omega q_y. \tag{3.18}$$

Corollary 3.8 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves and $\tilde{\kappa}$ denotes the curvature of $\tilde{\gamma}$. If $\omega = 1$ in (3.8) under the setting of the CD equation then a transformation $(y,s) \to (x,t)$ given by*

$$x = \int_0^y \tilde{\kappa}\left(y', s'\right)dy' \quad , \quad t = s$$

provides the SP equation $q_{xt} = q + \frac{1}{6}\left(q^3\right)_{xx}$.

Proof Let $x = \int_0^y \tilde{\kappa}\left(y', s'\right)dy'$ $t = s$, and $\omega = 1$, we get $x = \int_0^y p\left(y', s'\right)dy'$ from Corollary (3.7). This implies that $\frac{\partial x}{\partial y} = p$ and $\frac{\partial x}{\partial s} = \int_0^y p_{s'}\left(y', s'\right)dy' = -\int_0^y \left(q\left(y', s'\right)q_{y'}\left(y', s'\right)\right)dy' = -\frac{1}{2}\int_0^y \left(q\left(y', s'\right)\right)_{y'}dy' = -\frac{1}{2}q^2$. Thus, it is easily seen that the reciprocal (hodograph) transformation (1.5) is satisfied and there is the relation $\partial_x(\partial_t - \frac{1}{2}u^2\partial_x)q = q$. This relation produces the SP equation (1.4) and this completes the proof. □

Now, let us give the Lax pair which provides integrability of CD equations by the following theorem.

Theorem 3.9 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then the Lax pair of the CD equations is*

$$\psi_y = P\psi, \psi_s = Q\psi \tag{3.19}$$

such that $P = -\tilde{\kappa} e_3 - \tilde{\tau} e_1$, $Q = -\varepsilon e_3 + \eta e_2$, *where* $\psi = \psi(y,s)$ *is a function with* $\mathrm{SO}(3)$ *value. Here* $\tilde{\kappa} = \left\langle \tilde{T}_y, \tilde{N} \right\rangle$, $\tilde{\tau} = -\left\langle \tilde{B}_y, \tilde{N} \right\rangle$, $\varepsilon = \left\langle \tilde{T}_s, \tilde{N} \right\rangle$, $\eta = \left\langle \tilde{T}_s, \tilde{B} \right\rangle$.

Proof The Lax pair of the CD equations is

$$P = -i\lambda_2 \begin{pmatrix} p & q_y \\ q_y & -p \end{pmatrix}, Q = \begin{pmatrix} \frac{i}{4\lambda_2} & \frac{-q}{2} \\ \frac{q}{2} & -\frac{i}{4\lambda_2} \end{pmatrix}. \tag{3.20}$$

Also, the compatibility condition $P_y - Q_s + PQ - QP = 0$ satisfies the CD equations [32]. The basis of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are

$$e_1 = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_2 = \frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, e_3 = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively and there is an isomorphism between the Lie algebras $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$. Under this isomorphism, the CD equations provide the equation (3.19). The functions P and Q are found as

$$\begin{aligned} P &= -\tilde{\kappa} e_3 - \tilde{\tau} e_1 = -\omega p e_3 - \omega q_y e_2 = -i\lambda_2 \begin{pmatrix} p & q_y \\ q_y & -p \end{pmatrix} \\ &= -i\lambda_2 \begin{pmatrix} \frac{1}{\omega}\left(\frac{\mu\rho - \lambda u_y}{u_y(\lambda^2+\mu^2)}\right) & \frac{1}{\omega}\left(\frac{1}{c u_y(\lambda^2+\mu^2)}\right) \\ \frac{1}{\omega}\left(\frac{1}{c u_y(\lambda^2+\mu^2)}\right) & -\frac{1}{\omega}\left(\frac{\mu\rho - \lambda u_y}{u_y(\lambda^2+\mu^2)}\right) \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} Q &= -\varepsilon e_3 + \eta e_2 = \omega^{-1} e_3 + q e_2 = \begin{pmatrix} \frac{i}{4\lambda_2} & \frac{-q}{2} \\ \frac{q}{2} & -\frac{i}{4\lambda_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{i}{4\lambda_2} & \frac{-1}{2}\int \frac{1}{\omega}\left(\frac{1}{c u_y(\lambda^2+\mu^2)}\right) dy \\ \frac{1}{2}\int \frac{1}{\omega}\left(\frac{1}{c u_y(\lambda^2+\mu^2)}\right) dy & -\frac{i}{4\lambda_2} \end{pmatrix} \end{aligned}$$

where $\omega = -2\lambda_2$, $\mu = \frac{1-\lambda c\rho}{c u_y}$. They provide a Lax pair of the CD equations as it is desired. □

Let us give the geometric interpretation of the conserved quantity of the CD equations by the following theorem.

Theorem 3.10 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then the conserved quantity of the CD equations is constant,*

$$I = p^2 + q_y^2$$

where $p = \frac{1}{\omega}\left(\frac{\mu\rho - \lambda u_y}{u_y(\lambda^2+\mu^2)}\right) = \frac{\tilde{\kappa}}{\omega}$, $q_y = \frac{1}{\omega}\left(\frac{1}{cu_y(\lambda^2+\mu^2)}\right) = \frac{\tilde{\tau}}{\omega}$, $\tilde{\kappa}$ *and* $\tilde{\tau}$ *are the Frenet frame curvatures of the space curve* $\tilde{\gamma}(y,s)$.

Proof Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then the conserved quantity of the CD equations with the curve γ is $I = \rho^2 + u_y^2$. From the equation (3.18), we find

$$\frac{d}{ds}\left(\tilde{\kappa}^2 + \tilde{\tau}^2\right) = \frac{d}{ds}\left((\omega p)^2 + (\omega q_y)^2\right) = \omega^2 \frac{d}{ds}\left(p^2 + q_y^2\right).$$

On the other hand, from the equation (3.12) and (3.14), we have $\tilde{\kappa}_s = -q\tilde{\tau}$ and $\tilde{\tau}_s = \tilde{\kappa}q$. In that case, we find

$$\frac{d}{ds}\left(\tilde{\kappa}^2 + \tilde{\tau}^2\right) = 2\tilde{\kappa}\tilde{\kappa}_s + 2\tilde{\tau}\tilde{\tau}_s = 2\tilde{\kappa}\left(-q\tilde{\tau}\right) + 2\tilde{\tau}\left(\tilde{\kappa}q\right) = 0.$$

As a result, we get

$$\frac{d}{ds}\left(p^2 + q_y^2\right) = 0,$$

where $p = \frac{1}{\omega}\left(\frac{\mu\rho - \lambda u_y}{u_y(\lambda^2+\mu^2)}\right)$ and $q_y = \omega^{-1}\left(\frac{1}{cu_y(\lambda^2+\mu^2)}\right)$. Hence, we can easily see that the conserved quantity of the CD equations with the curve $\tilde{\gamma}$ is constant. □

Corollary 3.11 *Let γ and $\tilde{\gamma}$ be families of Bertrand partner curves, then the conserved quantity of the CD equations with the curve γ and the curve $\tilde{\gamma}$ is constant.*

4. Conclusion

Although the connections of the integrable models to the motion of space curves can be found in the literature in various ways, the perspective in this paper is focused on the conjugate of a Bertrand curve since the curve pairs are necessary for mechanics, kinematics, and physics. Based on the connection between the coupled dispersionless (CD) equations system with the motion of Bertrand curve pairs, the Lax equations have been obtained. Moreover, it has been proved that the conserved quantity of the corresponding coupled dispersionless equations of each of these curve pairs is constant.

Today, the CD-type equations and the SP-type equations are active study areas. In a recent paper [16], the modified types of these equations are considered and their links of the motions of space curves are expressed. Moreover, the integrability of these equations is verified by constructing their Lax pairs geometrically. Thus, CD-type equations and the SP-type equations of the motion of known curve pairs should be considered further in future research.

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