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A new approach to matrix isomorphisms of complex Clifford algebras via Cantor set

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Abstract: We give a new way to obtain the standard isomorphisms of complex Clifford algebras, known as the tensor product of Pauli matrices, by representing the complex Clifford algebras on the space of complex valued functions defined over a finite subset of the Cantor set.

Key words: Complex Clifford algebra, representation, Cantor set, Pauli matrices

1. Introduction

Clifford algebras (also known as "geometric algebras") are introduced (1878) by W. K. Clifford as a generalization of Grassmann algebras, complex numbers, and quaternions. In the area of mathematical physics, the representations of Clifford algebras are important for determining the topological and geometric structures of manifolds [8].

The idea that the Clifford algebras could be represented on fractals is discussed in the paper [7], where the envisaged representation of Clifford algebras is undertaken via Cuntz algebras (For representations of Cuntz algebras on fractals, see also [9]). In [2, 3], the authors give a direct realization for this pretty idea of representing Clifford algebras on fractals, without any use of Cuntz algebras. They represent the infinite dimensional complex Clifford algebra $\mathbb{C}l_\infty$ on $L^2\mathcal{K}$ which is the complex Hilbert space of square integrable, complex valued functions on $\mathcal{K}$, where $\mathcal{K}$ is the Cantor set.

In this note, we first present a representation for even complex Clifford algebra $\mathbb{C}l_{2n}$ using a special 2^n-element subset of the Cantor set, by the analogue of the representation for infinite-dimensional case [3]. Next, we show that the matrix for any image of the standard Clifford generator under this representation emerges as the tensor product of the standard Pauli matrices with respect to a suitable base of the representation space. In the case of the odd dimension, we can see easily from [7].

We will consider a special finite subset of $\mathcal{K}$, which is the attractor of the iterated functions system on $\mathbb{R}$ consisting of the functions φ_0 and φ_1 such that $\varphi_0(x) = \frac{1}{3}x$, $\varphi_1(x) = \frac{1}{3}x + \frac{2}{3}$, with 2^n elements. Let V_n denote the set of left endpoints of the nth stage of $\mathcal{K}$. The first three sets of endpoints illustrated in Figure 1 are as follows:

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$$V_1 = \left\{0, \frac{2}{3}\right\}, \; V_2 = \left\{0, \frac{2}{9}, \frac{2}{3}, \frac{8}{9}\right\}, \; V_3 = \left\{0, \frac{2}{27}, \frac{2}{9}, \frac{8}{27}, \frac{2}{3}, \frac{20}{27}, \frac{8}{9}, \frac{26}{27}\right\}$$

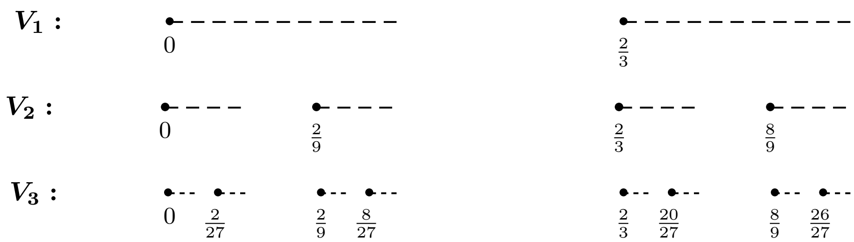

Figure 1. The finite subsets of the Cantor set V_1, V_2, and V_3.

Note that these endpoints are obtained by applying the transformations φ_0 and φ_1 to the point $x = 0$, successively. Thus, the first two sets V_1 and V_2 are also written as follows:

$$\begin{aligned} V_1 &= \{\varphi_0(0), \varphi_1(0)\} \\ V_2 &= \{\varphi_0(\varphi_0(0)), \varphi_0(\varphi_1(0)), \varphi_1(\varphi_0(0)), \varphi_1(\varphi_1(0))\} \end{aligned}$$

We denote the set of complex-valued functions on V_n by F_n and algebra of bounded linear operators on F_n by $B(F_n)$. In our representation, we will construct an algebra homomorphism from $\mathbb{C}l_{2n}$ to $B(F_n)$.

The rest of this paper is organized as follows. In Section 2, we will introduce special transformations that will be used in the construction of our representation, which we will call the tilt and switch operators here and illustrate their geometric behaviour with some examples. In Section 3, we will present our representation for $\mathbb{C}l_{2n}, n \in \mathbb{N}^+$. In that section, we will also construct a base for the representation space F_n by using symbolic notations of the elements of V_n and determine the matrix of any Clifford generator's image under the representation with respect to this base constructed.

2. Tilt and switch operators on F_n

In [3], tilt and switch operators on $L^2\mathcal{K}$, which are used to represent infinite dimensional complex Clifford algebra, were defined. We will define similar operators on F_n which will be used to construct the representation of $\mathbb{C}l_{2n}$ and call them tilt and switch operators too. We use the symbolic dynamics of these endpoints in V_n to describe these transformations. For any element x in V_n, it has unique address which are finite words $\omega_1\omega_2\ldots\omega_n$ such that

$$x = \varphi_{\omega_1\omega_2\ldots\omega_n}(0) = (\varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n})(0),$$

where each of the letters ω_i belongs to $\{0, 1\}$ (See [1] and [6] for symbolic dynamics of the points of an attractor.)

We identify a point $x \in V_n$ with its address $\omega_1\omega_2\ldots\omega_n$ and write $x = \omega_1\omega_2\ldots\omega_n$. For some fixed n and for all $1 \leq j \leq n, j \in \mathbb{N}$, one can decompose V_n with respect to the address-letter at a specific slot j:

$$V_n^{j,0} := \{x \in V_n \mid x = \omega_1\omega_2\cdots\omega_{j-1}0\omega_{j+1}\cdots\omega_n\}$$

and

$$V_n^{j,1} := \{x \in V_n \mid x = \omega_1\omega_2\cdots\omega_{j-1}1\omega_{j+1}\cdots\omega_n\}$$

with

$$V_n = V_n^{j,0} \cup V_n^{j,1}.$$

Now we define the operators T_j and $S_j, j = 1, 2, \ldots, n$ on F_n for $n \in \mathbb{N}^+$. For a given $f \in F_n$, $T_j f$ and $S_j f$ are defined as follows:

$$(T_j f)(x) = \begin{cases} f(x) & , \quad x \in V_n^{j,0} \\ -f(x) & , \quad x \in V_n^{j,1} \end{cases}$$

and

$$(S_j f)(x) = f(\tilde{x}^j) \text{ for } x = \omega_1\omega_2\ldots\omega_n \in V_n,$$

where

$$\tilde{x}^j = \begin{cases} \omega_1\omega_2\ldots\omega_{j-1}0\omega_{j+1}\ldots\omega_n & , \quad \text{for } \omega_j = 1 \\ \omega_1\omega_2\ldots\omega_{j-1}1\omega_{j+1}\ldots\omega_n & , \quad \text{for } \omega_j = 0. \end{cases}$$

T_j's are the "tilt" operators as they tilt the portion of the graph on $V_n^{j,1}$, and S_j's are the "switch" operators as they switch the portions of graphs on $V_n^{j,0}$ and $V_n^{j,1}$ like the tilt and switch operators defined on $L^2\mathcal{K}$ in [3]. We note that as n changes, the tilt and stitch operators will also be different, as the domains will change. We write T_j and S_j without n in order not to cause indices confusion.

Example 2.1 *Let a function f on V_2 be given as in Figure 2. We illustrate $T_1(f)$, $S_1(f)$, $T_2(f)$, and $S_2(f)$ as in Figures 3–6, respectively. Note that the elements in V_2 have been shown with their address representations.*

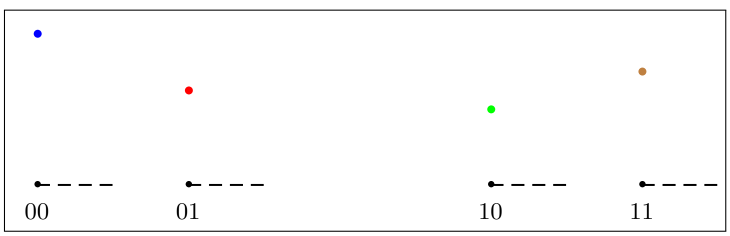

Figure 2. The graph of f on V_2.

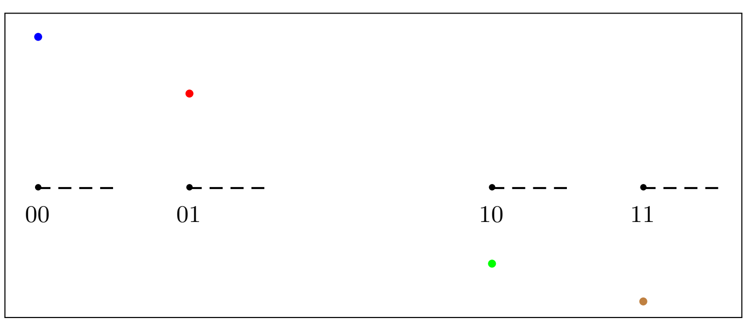

Figure 3. The graph of $T_1 f$.

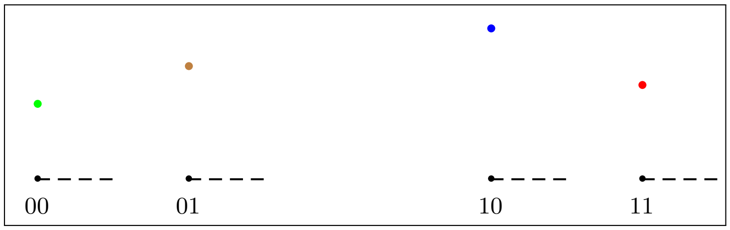

Figure 4. The graph of $S_1 f$.

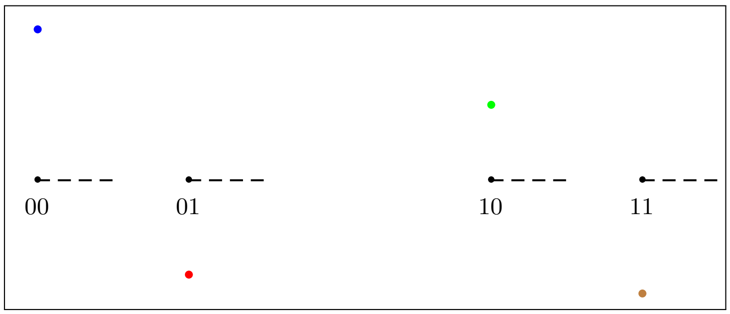

Figure 5. The graph of $T_2 f$.

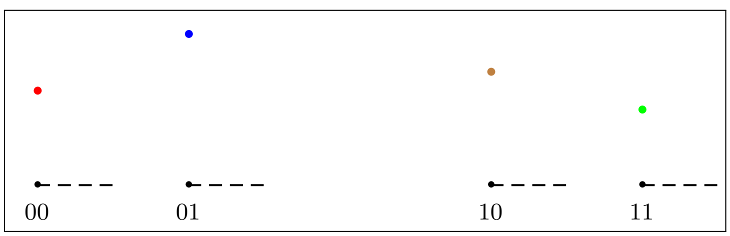

Figure 6. The graph of $S_2 f$.

We now give a lemma about commutation properties of tilt and switch operators on F_n.

Lemma 2.2 *For a fixed integer $n \geq 1$ and $p, q \in \mathbb{N}, 1 \leq p, q \leq n$, the following equalities hold:*

i) $T_p T_q = T_q T_p$
ii) $S_p S_q = S_q S_p$
iii) $T_p S_q = S_q T_p \ (p \neq q)$
iv) $T_p S_p = -S_p T_p$

Proof Let $f \in F_n$ be given. Then, i)

$$(T_pT_q)(f)(x) = \begin{cases} (T_qf)(x) & , \quad x \in V_n^{p,0} \\ -(T_qf)(x) & , \quad x \in V_n^{p,1} \end{cases} \tag{2.1}$$

$$= \begin{cases} f(x) & , \quad x \in V_n^{p,0} \text{ and } V_n^{q,0} \\ -f(x) & , \quad x \in V_n^{p,0} \text{ and } V_n^{q,1} \\ -f(x) & , \quad x \in V_n^{p,1} \text{ and } V_n^{q,0} \\ f(x) & , \quad x \in V_n^{p,1} \text{ and } V_n^{q,1} \end{cases} \tag{2.2}$$

$(T_pT_q)(f)(x)$ has the same explicit expression.

ii) $(S_pS_q)(f)(x) = (S_qf)(\tilde{x}^p) = f(\widetilde{(\tilde{x}^p)}^q) = f(\widetilde{(\tilde{x}^q)}^p) = (S_pf)(\tilde{x}^q) = (S_qS_p)(f)(x)$.

iii)

$$(T_pS_q)(f)(x) = \begin{cases} (S_qf)(x) & , \quad x \in V_n^{p,0} \\ -(S_qf)(x) & , \quad x \in V_n^{p,1} \end{cases} \tag{2.3}$$

$$= \begin{cases} f(\tilde{x}^q) & , \quad x \in V_n^{p,0} \\ -f(\tilde{x}^q) & , \quad x \in V_n^{p,1} \end{cases} \tag{2.4}$$

For $p \neq q$, we have $x \in V_n^{p,0} \Leftrightarrow \tilde{x}^q \in V_n^{p,0}$ and $x \in V_n^{p,1} \Leftrightarrow \tilde{x}^q \in V_n^{p,1}$. Hence, we can write:

$$(T_pS_q)(f)(x) = \begin{cases} f(\tilde{x}^q) & , \quad \tilde{x}^q \in V_n^{p,0} \\ -f(\tilde{x}^q) & , \quad \tilde{x}^q \in V_n^{p,1} \end{cases} \tag{2.5}$$

$$= (T_pf)(\tilde{x}^q) \tag{2.6}$$

$$= (S_qT_p)(f)(x). \tag{2.7}$$

iv)

$$(S_pT_p)(f)(x) = (T_pf)(\tilde{x}^p) \tag{2.8}$$

$$= \begin{cases} f(\tilde{x}^p) & , \quad \tilde{x}^p \in V_n^{p,0} \\ -f(\tilde{x}^p) & , \quad \tilde{x}^p \in V_n^{p,1} \end{cases} \tag{2.9}$$

$$= \begin{cases} f(\tilde{x}_p) & , \quad x \in V_n^{p,1} \\ -f(\tilde{x}_p) & , \quad x \in V_n^{p,0} \end{cases} \tag{2.10}$$

$$= \begin{cases} (S_pf)(x) & , \quad x \in V_n^{p,1} \\ -(S_pf)(x) & , \quad x \in V_n^{p,0} \end{cases} \tag{2.11}$$

$$= -(T_pS_p)(f)(x). \tag{2.12}$$

□

3. Representations of complex Clifford algebras on F_n

It is well known that the structures of finite-dimensional real and complex Clifford algebras for a nondegenerate quadratic form have been completely classified [4]. In this section, we construct representation for complex Clifford algebra in even dimension via tilt and switch operators and show that the matrix representation of every Clifford generator is the form of the tensor product of the Pauli matrices.

Let us denote the generators of the complex Clifford algebra $\mathbb{C}l_{2n}$ by e_j $(j = 1, 2, ..., 2n)$ with ${e_j}^2 = 1$ and $e_je_k = -e_ke_j$ for $j \neq k$. We map these generators in the following way into $B(F_n)$:

$$\begin{array}{rlcl} \psi_{2n}: & e_1 & \mapsto & T_1 \\ & e_2 & \mapsto & S_1 \\ & & \vdots & \\ (1 < k \leq n) & e_{2k-1} & \mapsto & i^{(k+1)^2}(T_kT_{k-1}S_{k-1}T_{k-2}S_{k-2}\cdots T_1S_1) \\ & e_{2k} & \mapsto & i^{(k+1)^2}(S_kT_{k-1}S_{k-1}T_{k-2}S_{k-2}\cdots T_1S_1), \end{array} \tag{3.1}$$

where i is the imaginary unit.

Theorem 3.1 *ψ_{2n}, defined above, induces an algebra homomorphism from $\mathbb{C}l_{2n}$ to $B(F_n)$, i.e. a representation of $\mathbb{C}l_{2n}$ on F_n.*

Proof

For all $1 \leq p, q \leq 2n,\ p \neq q$, we have to check that both

$$(\psi_{2n}(e_p))^2 = I$$

and

$$\psi_{2n}(e_p)\psi_{2n}(e_q) = -\psi_{2n}(e_q)\psi_{2n}(e_p).$$

We first show that $(\psi_{2n}(e_p))^2 = I$. For $p = 1$ and $p = 2$, it can be easily verified from the following equalities:

$$(\psi_{2n}(e_1))^2 = T_1T_1 = I$$

and

$$(\psi_{2n}(e_2))^2 = S_1S_1 = I.$$

Now let $p = 2k - 1$ $(k > 1)$:

$$\begin{aligned} (\psi_{2n}(e_{2k-1}))^2 &= (i^{(k+1)^2}T_kT_{k-1}S_{k-1}\cdots T_1S_1)(i^{(k+1)^2}T_kT_{k-1}S_{k-1}\cdots T_1S_1) \\ (\text{by Lemma } 2.2) &= i^{2(k+1)^2}(-1)^{k-1}(T_kT_kT_{k-1}T_{k-1}S_{k-1}S_{k-1}\cdots T_1T_1S_1S_1) \\ &= (-1)^{(k+1)^2}(-1)^{k-1}I = (-1)^{k(k+3)}I = I. \end{aligned}$$

For $p = 2k$, we obtain

$$\begin{aligned} (\psi_{2n}(e_{2k}))^2 &= (i^{(k+1)^2}S_kT_{k-1}S_{k-1}\cdots T_1S_1)(i^{(k+1)^2}S_kT_{k-1}S_{k-1}\cdots T_1S_1) \\ &= i^{2(k+1)^2}(-1)^{k-1}(S_kS_kT_{k-1}T_{k-1}S_{k-1}S_{k-1}\cdots T_1T_1S_1S_1) \\ &= I. \end{aligned}$$

Let us now check the anticommutativity relations. By Lemma 2.2,

$$\psi_{2n}(e_1)\psi_{2n}(e_2) = -\psi_{2n}(e_2)\psi_{2n}(e_1).$$

Likewise, $\psi_{2n}(e_1)$ and $\psi_{2n}(e_2)$ anticommute with all $\psi_{2n}(e_j)$ for $j > 2$ by Lemma 2.2.

Now we consider various cases:

i) Let $p = 2k-1$, $q = 2l-1$ for $k > 1, l > 1$ and $k < l$.

$$\begin{aligned}
\psi_{2n}(e_p)\psi_{2n}(e_q) &= (i^{(k+1)^2} T_k T_{k-1} S_{k-1} \cdots T_1 S_1)(i^{(l+1)^2} T_l T_{l-1} S_{l-1} \cdots T_1 S_1) \\
&= i^{(k+1)^2+(l+1)^2} (T_l T_{l-1} S_{l-1} \cdots T_{k+1} S_{k+1})(T_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&\qquad (T_k S_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&= i^{(k+1)^2+(l+1)^2} (-1)^{2k-1} (T_l T_{l-1} S_{l-1} \cdots T_1 S_1 (T_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&= -\psi_{2n}(e_q)\psi_{2n}(e_p).
\end{aligned}$$

ii) Let $p = 2k$, $q = 2l$ for $k > 1, l > 1$ and $k < l$.

$$\begin{aligned}
\psi_{2n}(e_p)\psi_{2n}(e_q) &= (i^{(k+1)^2} S_k T_{k-1} S_{k-1} \cdots T_1 S_1)(i^{(l+1)^2} S_l T_{l-1} S_{l-1} \cdots T_1 S_1) \\
&= i^{(k+1)^2+(l+1)^2} (S_l T_{l-1} S_{l-1} \cdots T_{k+1} S_{k+1})(S_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&\qquad (T_k S_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&= i^{(k+1)^2+(l+1)^2} (-1)^{2k-1} (S_l T_{l-1} S_{l-1} \cdots T_1 S_1)(T_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&= -\psi_{2n}(e_q)\psi_{2n}(e_p).
\end{aligned}$$

iii) Let $p = 2k-1 < q = 2l$, $k > 1, l > 1$.

$$\begin{aligned}
\psi_{2n}(e_p)\psi_{2n}(e_q) &= (i^{(k+1)^2} T_k T_{k-1} S_{k-1} \cdots T_1 S_1)(i^{(l+1)^2} S_l T_{l-1} S_{l-1} \cdots T_1 S_1) \\
&= i^{(k+1)^2+(l+1)^2} (S_l T_{l-1} S_{l-1} \cdots T_{k+1} S_{k+1})(T_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&\qquad (T_k S_k \cdots T_1 S_1) \\
&= i^{(k+1)^2+(l+1)^2} (-1)^{2k-1} (S_l T_{l-1} S_{l-1} \cdots T_1 S_1)(T_k T_{k-1} S_{k-1} \cdots T_1 S_1) \\
&= -\psi_{2n}(e_q)\psi_{2n}(e_p).
\end{aligned}$$

It can be shown similarly in the remaining cases with the help of Lemma 2.2. □

Our current aim is to determine the corresponding matrix for all $\psi_{2n}(e_i), i = 1, 2, \ldots, 2n$. We present the base for F_n denoted by

$$E_n = \{f_j \mid j = 0, 1, \ldots, 2^n - 1\}$$

that we use to determine the matrices such that

$$f_j(x) = \begin{cases} 1 & , \quad x = \varphi_{i_1 i_2 \ldots i_n}(0), j = (i_n \ldots i_1)_2 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{3.2}$$

As an example for $n = 1$ each of the base functions f_0 and f_1 can be thought of an element of $\mathbb{C}^2$ such as $f_0 = (1, 0), f_1 = (0, 1)$ and the base functions for F_2 will be the following elements of $\mathbb{C}^4$:

$$f_0 = (1, 0, 0, 0), \quad f_1 = (0, 0, 1, 0), \quad f_2 = (0, 1, 0, 0), \quad f_3 = (0, 0, 0, 1). \tag{3.3}$$

With these definitions, we can now state our main theorem.

Theorem 3.2 *For each* e_i, $1 \leq i \leq 2n$, *the matrix of* $\psi_{2n}(e_i)$ *with respect to* E_n *is obtained by the tensor product of Pauli matrices such as*

$$
\begin{array}{rlcl}
 & \psi_{2n}(e_1) & : & \underbrace{I_2 \otimes I_2 \otimes \cdots \otimes I_2}_{n-1} \otimes U, \\
 & \psi_{2n}(e_2) & : & \underbrace{I_2 \otimes I_2 \otimes \cdots \otimes I_2}_{n-1} \otimes V, \\
(1 < k \leq n) & \psi_{2n}(e_{2k-1}) & : & -\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \otimes U \otimes \underbrace{J \otimes \cdots \otimes J}_{k-1}, \\
 & \psi_{2n}(e_{2k}) & : & -\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \otimes V \otimes \underbrace{J \otimes \cdots \otimes J}_{k-1},
\end{array}
\tag{3.4}
$$

where

$$
U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
$$

Proof We prove this theorem by the method of induction. In the first step, we show the result is true for $n = 1$; in the second, we suppose that the result is true for n and prove it for $n+1$.

It can be easily verified that the matrices of $\psi_2(e_1)$ and $\psi_2(e_2)$ with respect to E_1 are as follows:

$$
U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Let us assume the claim is true for n and determine the matrix corresponding to $\psi_{2n+2}(e_i)$ with respect to E_{n+1} for all $i = 1, \ldots, 2n+2$. By definition, ψ_{2n+2} is as follows:

$$
\begin{array}{rrcl}
\psi_{2n+2}: & \mathbb{C}l_{2n+2} & \to & B(F_{n+1}) \\
 & e_1 & \mapsto & T_1 \\
 & e_2 & \mapsto & S_1 \\
 & & \vdots & \\
(1 < k \leq n+1), & e_{2k-1} & \mapsto & i^{(k+1)^2}(T_k T_{k-1} S_{k-1} T_{k-2} S_{k-2} \cdots T_1 S_1) \\
 & e_{2k} & \mapsto & i^{(k+1)^2}(S_k T_{k-1} S_{k-1} T_{k-2} S_{k-2} \cdots T_1 S_1).
\end{array}
$$

At this point, we need the transformation that gives the identification between the algebras $B(F_n)$ and $B(F_{n+1})$ defined in the following way:

$$
\sigma_n : B(F_n) \to B(F_{n+1}), (\sigma_n T)(f)(x) = (Tf|_{V_n})(x')
$$

for $T \in B(F_n), f \in F_{n+1}$ and $x' = \omega_1\omega_2\ldots\omega_n$ where $x = \omega_1\omega_2\ldots\omega_n\omega_{n+1}$. One can check that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{C}l_{2n} & \xrightarrow{\psi_{2n}} & B(F_n) \\
\iota\downarrow & & \downarrow\sigma_n \\
\mathbb{C}l_{2n+2} & \xrightarrow{\psi_{2n+2}} & B(F_{n+1}),
\end{array}
$$

where ι is the inclusion map. With the help of this diagram, we now identify the restriction of ψ_{2n+2} to $\mathbb{C}l_{2n}$ with ψ_{2n}. We know from the assumption that for all $j = 1, \ldots, 2n$ the matrix of $\psi_{2n}(e_j)$ relative to $E_n = \{f_0, f_1, \ldots, f_{2^n-1}\}$ is given as in 3.4.

We will use the following property given in [5] and apply this to ψ_{2n} and ψ_2:

Property: Let $f : \mathcal{A} \to End(V)$ and $g : \mathcal{B} \to End(W)$ be representations and $a \in \mathcal{A}$, $b \in \mathcal{B}$ be given. The matrix of $(f \otimes g)(a \otimes b)$ with respect to

$$\{v_1 \otimes w_1, \ldots, v_1 \otimes w_n, v_2 \otimes w_1, \ldots, v_2 \otimes w_n, v_m \otimes w_1, \ldots, v_m \otimes w_n\}$$

is $C \otimes D$ such that C is the matrix of $f(a)$, D is the matrix of $g(b)$ with respect to $\{v_1, v_2, \ldots, v_n\}$, $\{w_1, w_2, \ldots, w_m\}$ basis for V and W, respectively [5].

If we consider the following isomorphism with $F_n \otimes F_1$ and F_{n+1} as

$$\begin{aligned} \varphi : \quad & F_n \otimes F_1 \to F_{n+1} \\ & a \otimes b \mapsto b \otimes a \end{aligned}$$

then the base E_{n+1} of F_{n+1} emerges as the image of the ordered base of $F_n \otimes F_1$

$$\{f_0 \otimes f_0, f_0 \otimes f_1, f_1 \otimes f_0, f_1 \otimes f_1, \ldots, f_{2^n-1} \otimes f_0, f_{2^n-1} \otimes f_1\}.$$

Remark 3.3 *Note that the functions f_0 and f_1 in both basis are not the same. We will use the same notation to avoid indices confusion and distinguish these functions by looking at the spaces to which they belong.*

We now consider the isomorphism ρ between $\mathbb{C}l_{2n+2}$ and $\mathbb{C}l_n \otimes \mathbb{C}l_2$ given in [4].

$$\begin{aligned} \rho : \mathbb{C}l_{n+2} &\to \mathbb{C}l_n \otimes \mathbb{C}l_2 \\ e_1 &\mapsto 1 \otimes e_1 \\ e_2 &\mapsto 1 \otimes e_2 \\ e_3 &\mapsto ie_1 \otimes e_1e_2 \\ e_4 &\mapsto ie_2 \otimes e_1e_2 \\ k \geq 5,\ e_k &\mapsto -ie_{k-2} \otimes e_1e_2. \end{aligned}$$

Using our assumption for n and the isomorphism ρ, if we apply the above property mentioned in [5] to ψ_{2n}

and ψ_2, then we obtain the matrices from the following equalities:

$$\begin{aligned}
(\psi_{2n} \otimes \psi_2)(1 \otimes e_1) &= \psi_{2n}(1) \otimes \psi_2(e_1) \\
&= \underbrace{I_2 \otimes I_2 \otimes \cdots \otimes I_2}_{n} \otimes U \\
(\psi_{2n} \otimes \psi_2)(1 \otimes e_2) &= \psi_{2n}(1) \otimes \psi_2(e_2) \\
&= \underbrace{I_2 \otimes I_2 \otimes \cdots \otimes I_2}_{n} \otimes V \\
(\psi_{2n} \otimes \psi_2)(ie_1 \otimes e_1e_2) &= i\psi_{2n}(e_1) \otimes \psi_2(e_1e_2) \\
&= (\underbrace{I_2 \otimes I_2 \otimes \cdots \otimes I_2}_{n-1} \otimes U) \otimes (-J) \\
&= -I_2 \otimes \cdots \otimes I_2 \otimes U \otimes J \\
(\psi_{2n} \otimes \psi_2)(ie_2 \otimes e_1e_2) &= i\psi_{2n}(e_2) \otimes \psi_2(e_1e_2) \\
&= -I_2 \otimes I_2 \otimes \cdots \otimes I_2 \otimes V \otimes J \\
(\psi_{2n} \otimes \psi_2)(-ie_{2k-1} \otimes e_1e_2) &= -i\psi_{2n}(e_{2k-1}) \otimes \psi_2(e_1e_2) \\
&= (-\underbrace{I_2 \otimes \cdots I_2}_{n-k} \otimes U \otimes \underbrace{J \otimes \cdots \otimes J}_{k-1}) \otimes J \\
&= -\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \otimes U \otimes \underbrace{J \otimes \cdots \otimes J}_{k} \\
(\psi_{2n} \otimes \psi_2)(-ie_{2k} \otimes e_1e_2) &= -i\psi_{2n}(e_{2k}) \otimes \psi_2(e_1e_2) \\
&= (-\underbrace{I_2 \otimes \cdots I_2}_{n-k} \otimes V \otimes \underbrace{J \otimes \cdots \otimes J}_{k-1}) \otimes J \\
&= -\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \otimes V \otimes \underbrace{J \otimes \cdots \otimes J}_{k}
\end{aligned}$$

which completes the proof. We note that the identification between the spaces $B(F_n \otimes F_1)$ and $B(F_{n+1})$ is given as follows:

$$h : B(F_n \otimes F_1) \to B(F_{n+1}), h(T)(g) = (\varphi T \varphi^{-1})(g).$$

□

To understand the dynamics of the generators' image better, we present the case $n = 2$ in Example 3.4.

Example 3.4 *Let us consider the transformation ψ_4 and the corresponding base $\{f_0, f_1, f_2, f_3\}$ given in 3.3 such that*

$$\begin{aligned}
\psi_4 : \mathbb{C}l_4 &\to B(F_2) \\
e_1 &\mapsto \ T_1 \\
e_2 &\mapsto \ S_1 \\
e_3 &\mapsto \ iT_2T_1S_1 \\
e_4 &\mapsto \ iS_2T_1S_1.
\end{aligned}$$

Since

$$T_1(1,0,0,0) = (1,0,0,0)$$
$$T_1(0,0,1,0) = (0,0,-1,0)$$
$$T_1(0,1,0,0) = (0,1,0,0)$$
$$T_1(0,0,0,1) = (0,0,0,-1),$$

the matrix of $\psi_4(e_1)$ *is obtained as follows which is equal to* $I_2 \otimes U$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Similarly since

$$S_1(1,0,0,0) = (0,0,1,0)$$
$$S_1(0,0,1,0) = (1,0,0,0)$$
$$S_1(0,1,0,0) = (0,0,0,1)$$
$$S_1(0,0,0,1) = (0,1,0,0),$$

we obtain the following matrix which is equal to $I_2 \otimes V$ *as the matrix of* $\psi_4(e_2)$*:*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since

$$iT_2T_1S_1(1,0,0,0) = iT_2T_1(0,0,1,0) = iT_2(0,0,-1,0) = (0,0,-i,0)$$
$$iT_2T_1S_1(0,0,1,0) = iT_2T_1(1,0,0,0) = iT_2(1,0,0,0) = (i,0,0,0)$$
$$iT_2T_1S_1(0,1,0,0) = iT_2T_1(0,0,0,1) = iT_2(0,0,0,-1) = (0,0,0,i)$$
$$iT_2T_1S_1(0,0,0,1) = iT_2T_1(0,1,0,0) = iT_2(0,1,0,0) = (0,-i,0,0),$$

we obtain the following matrix which is equal to $-U \otimes J$ *as the matrix of* $\psi_4(e_3)$*:*

$$\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

And using the following equalities

$$iS_2T_1S_1(1,0,0,0) = iS_2T_1(0,0,1,0) = iS_2(0,0,-1,0) = (0,0,0,-i)$$
$$iS_2T_1S_1(0,0,1,0) = iS_2T_1(1,0,0,0) = iS_2(1,0,0,0) = (0,i,0,0)$$
$$iS_2T_1S_1(0,1,0,0) = iS_2T_1(0,0,0,1) = iS_2(0,0,0,-1) = (0,0,-i,0)$$
$$iS_2T_1S_1(0,0,0,1) = iS_2T_1(0,1,0,0) = iS_2(0,1,0,0) = (i,0,0,0),$$

we obtain the following matrix which is equal to $-V \otimes J$ *as the matrix of* $\psi_4(e_4)$*:*

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

which is equal to $-V \otimes J$*.*

Remark 3.5 *So far, we have verified our claim in every even dimensional case. The odd case of the theorem follows immediately from the results of [7].*

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