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## Ranks and presentations for order-preserving transformations with one fixed point

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**Abstract:** In the present paper, we consider the semigroup  $O_{n,p}$  of all order-preserving full transformations  $\alpha$  on an  $n$ -elements chain  $X_n$ , where  $p \in X_n$  is the only fixed point of  $\alpha$ . The nilpotent semigroup  $O_{n,p}$  was first studied by Ayik et al. in 2011. Moreover,  $O_{n,1}$  is the maximal nilpotent subsemigroup of the Catalan Monoid  $C_n$ . Its rank is the difference of the  $(n-1)$ th and the  $(n-2)$ th Catalan number. The aim of the present paper is to provide further fundamental information about the nilpotent semigroup  $O_{n,p}$ . We will calculate the rank of  $O_{n,p}$  for  $p > 1$  and provide a semigroup presentation for  $O_{n,1}$ .

**Key words:** Order-preserving transformations, fixed point transformations, presentations, ranks

### 1. Introduction

The full transformation semigroup  $T_n$  on an  $n$ -elements set  $X_n = \{1, 2, \dots, n\}$  is the set of all transformations (i.e. self maps)  $X_n \rightarrow X_n$  of  $X_n$  with composition of transformations as multiplication. It is well known that every finite semigroup is isomorphic to a subsemigroup of a suitable finite transformation semigroup (the analogue of the Cayley's theorem for finite groups). Hence, the transformation semigroups have an important role in semigroup theory, as the symmetric groups in group theory. Various properties of  $T_n$  are known and many subsemigroups of  $T_n$  are studied. Among the most studied subsemigroups of  $T_n$  is the monoid  $O_n$  of all order-preserving transformations on the chain  $X_n = \{1 < 2 < \dots < n\}$ . A transformation  $\alpha \in T_n$  is called *order-preserving* if  $x < y$  implies  $x\alpha \leq y\alpha$ . The monoid  $O_n$  has long been considered in the literature. In particular, the rank of monoid  $O_n$  is  $n$  (see [7]). The rank of a semigroup  $S$  is the minimal size of a generating set for  $S$ , i.e.  $\text{rank}(S) = \min\{|G| : \langle G \rangle = S\}$ . Aizenštat (1962) exhibited presentations for  $O_n$  (see [1]). If  $Y$  is a set then we denote by  $Y^+$  the free semigroup on  $Y$ . If  $\mathcal{R} \subseteq Y^+ \times Y^+$ , then we denote by  $\mathcal{R}^\#$  the congruence on  $Y^+$  generated by  $\mathcal{R}$ . To say that a semigroup  $S$  has semigroup presentation  $\langle Y \mid \mathcal{R} \rangle$  is to say that  $S \cong Y^+ / \mathcal{R}^\#$  or, equivalently, that there is a semigroup epimorphism  $\varphi : Y^+ \rightarrow S$  with kernel  $\mathcal{R}^\#$  (i.e.  $\ker \varphi = \{(x, y) \in Y^+ \times Y^+ : x\varphi = y\varphi\} = \mathcal{R}^\#$ ). If  $\varphi$  is such an epimorphism, then we say that  $S$  has presentation  $\langle Y \mid \mathcal{R} \rangle$  via  $\varphi$ . The elements of  $Y$  are called *generators* and the elements of  $\mathcal{R}$  are called *relations*. In practice, a relation  $(x, y) \in \mathcal{R}$  will be written as  $x \approx y$ .

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Besides the already mentioned fundamental paper by Aizenštat [1], in 2010, East has given a presentation for the singular part of  $T_n$ , i.e. for  $T_n \setminus S_n$ , where  $S_n$  is the symmetric group on  $X_n$  [4]. In the literature, one can find presentations for several semigroups of partial transformations (see, e.g., [5, 6]).

A subset  $A \subseteq X_n$  is called invariant for a transformation  $\alpha \in T_n$  if  $A\alpha = \{x\alpha : x \in A\} \subseteq A$ . In 1966, Magill Jr. has considered the monoid  $T_n(A) = \{\alpha \in T_n : A\alpha \subseteq A\}$  [12]. However, the elements in  $A$  do not have to be included in the set  $Fix(\alpha) = \{x \in X_n : x\alpha = x\}$  of all fixed points of  $\alpha$ . The number  $|\{\alpha \in O_n : |Fix(\alpha)| = r\}|$  of all transformations on  $O_n$  with  $r$  fixed points, for a fixed positive integer  $r$ , was given by Higgins in 1993 in [9]. Honyam and Sanwong have studied the semigroup  $Fix(X_n, a)$  of all  $\alpha \in T_n$  such that  $A \subseteq Fix(\alpha)$ , for a fixed set  $A \subseteq X_n$  [10]. They have shown that  $Fix(X_n, a)$  is a regular monoid. Chinram and Yonthanthun have given a necessary and sufficient condition that  $Fix(X_n, a)$  is left-regular and right-regular, respectively [3]. If we restrict ourselves to the case that  $Fix(\alpha) = A$ , we miss that the corresponding semigroup is a monoid, whence  $A \neq X_n$ . In [2], the cardinality of the semigroup

$$O_{n,A} = \{\alpha \in O_n : Fix(\alpha) = A\}$$

was determined by Ayik et al.. This paper extends the considerations about the cardinalities of semigroups of order-preserving transformations on a finite set in [9]. We will write  $O_{n,p}$  instead of  $O_{n,A}$ , whenever  $A$  is the singleton set  $\{p\}$  ( $A = \{p\}$ ) for some  $p \in X_n$ . However, in [9], algebraic properties of  $O_{n,A}$  are not studied. The study of the nilpotent semigroup  $O_{n,p}$  gives more information about the subsemigroups of the well studied monoid  $O_n$ . In the present paper, we will study algebraic properties of  $O_{n,p}$ . We will calculate the ranks and give a presentation for  $O_{n,1}$ . The transformations in  $O_{n,p}$  can be described as follows:

**Proposition 1.1** ([2], Proposition 4) *Let  $p \in X_n$  and let  $\alpha \in O_{n,p}$ .*

- i) If  $1 \leq x < p$ , then  $x + 1 \leq x\alpha$ .*
- ii) If  $p < x \leq n$ , then  $x\alpha \leq x - 1$ .*

By Proposition 1.1, it is easy to verify that  $(p - 1)\alpha = p$  (whenever  $p > 1$ ) and  $(p + 1)\alpha = p$  (whenever  $p < n$ ).

An element  $a$  of a finite semigroup  $S$  with 0 is called nilpotent if  $a^m = 0$  for some positive integer  $m$ . The set of all nilpotent elements of  $S$  is denoted by  $N(S)$ . The semigroup  $S$  with 0 is called nilpotent if  $S^m = \{0\}$  for some positive integer  $m$ . It is a well known fact that  $S$  is nilpotent if and only if  $N(S) = S$  (see [8]). In the present paper, we deal with nilpotent semigroups. Let  $c_x$  be the constant map to  $x$ , for  $x \in X_n$ .

In particular, the semigroup  $O_{n,1}$  is already well studied because it is the maximal nilpotent subsemigroup of the Catalan Monoid  $C_n$ . A transformation  $\alpha \in T_n$  is called order-decreasing if  $x\alpha \leq x$  for all  $x \in X_n$ . The set of all order-decreasing transformations in  $T_n$  is denoted by  $D_n$ , which is a subsemigroup of  $T_n$ . The subsemigroup  $C_n = O_n \cap D_n$  is called Catalan Monoid since its cardinality is the  $n$ th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n} \binom{2n}{n-1}, \text{ where } C_0 = 1 \text{ (see [14]).}$$

We observe that  $c_1$  is the zero element in  $C_n$ . Moreover, an element  $\alpha \in C_n$  is nilpotent if and only if  $Fix(\alpha) = \{1\}$  (see [11], Lemma 2.2). From [2], Proposition 4, we have the fact  $O_{n,1} = N(C_n)$ .

**Lemma 1.2**  *$O_{n,p}$  is a nilpotent semigroup.*

**Proof** Since  $O_{n,1}$  is equal to the nilpotent semigroup  $N(C_n)$ , it is enough to consider the case  $p \geq 2$ . Let  $\alpha \in O_{n,p}$ . By Proposition 1.1, using  $1\alpha \geq 2$  and  $(p-1)\alpha = (p+1)\alpha = p$ , we can conclude that  $x\alpha^{p-1} = p$  for all  $x \in \{1, \dots, p-2\}$  and  $x\alpha^{n-p} = p$  for all  $x \in \{p+2, \dots, n\}$ . This shows that  $x\alpha^m = p$  for  $m = \max\{p-1, n-p\}$ . Consequently, we have  $\alpha^m = c_p$ . We obtain  $N(O_{n,p}) = O_{n,p}$ , i.e.  $O_{n,p}$  is a nilpotent semigroup with zero element  $c_p$ .  $\square$

It is easy to verify that  $O_{n,r}$  and  $O_{n,n-r+1}$  are isomorphic for  $r \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ . In particular,  $O_{n,1}$  and  $O_{n,n}$  are isomorphic. Thus, we get

**Lemma 1.3** ([2])  $|O_{n,1}| = |O_{n,n}| = C_{n-1}$ .

In general, it holds:

**Lemma 1.4** ([2], Lemma 6)  $|O_{n,p}| = C_{p-1}C_{n-p}$ .

Using the fact that  $O_{n,1} = N(C_n)$ , where  $rank(N(C_n)) = C_{n-1} - C_{n-2}$  ([15], Theorem 2), we obtain

**Proposition 1.5** ([15])  $rank(O_{n,1}) = rank(O_{n,n}) = C_{n-1} - C_{n-2}$ .

As  $O_{n,1}$  and  $O_{n,n}$ , we can show that  $O_{n,2}$  and  $O_{n-1,1}$  are isomorphic.

**Lemma 1.6**  $O_{n,2}$  and  $O_{n-1,1}$  are isomorphic.

**Proof** Let  $\varphi : O_{n,2} \rightarrow O_{n-1,1}$  with  $x(\alpha\varphi) = (x+1)\alpha - 1$  for all  $x \in \{1, \dots, n-1\} = X_{n-1}$  and  $\alpha \in O_{n,2}$ . Firstly, we show that  $\varphi$  maps into  $O_{n-1,1}$ . Let  $\alpha \in O_{n,2}$ . Then  $1(\alpha\varphi) = (1+1)\alpha - 1 = 2 - 1 = 1$ . Moreover, for  $x < y \in X_{n-1}$ , we have  $x(\alpha\varphi) = (x+1)\alpha - 1 \leq (y+1)\alpha - 1 = y(\alpha\varphi) \leq n-1$ . Further, we have  $x(\alpha\varphi) = (x+1)\alpha - 1 < x+1 - 1 = x$  for all  $x \in \{2, 3, 4, \dots, n-1\}$ . This shows that  $\alpha\varphi \in O_{n-1,1}$ .

On the other hand, for  $\alpha \in O_{n-1,1}$ , let  $\bar{\alpha} \in T_n$  with  $1\bar{\alpha} = 2$  and  $x\bar{\alpha} = (x-1)\alpha + 1 (\leq n)$  for all  $x \in \{2, \dots, n\}$ . We have  $2\bar{\alpha} = (2-1)\alpha + 1 = 1 + 1 = 2$ . It is routine to show that  $\bar{\alpha}$  is order-preserving and  $x\bar{\alpha} < x$  for  $x \in \{2, \dots, n\}$ , so  $\bar{\alpha} \in O_{n,2}$ . Further, we have  $x(\bar{\alpha}\varphi) = (x+1)\bar{\alpha} - 1 = (x+1-1)\alpha + 1 - 1 = x\alpha$  for  $x \in X_{n-1}$ , i.e.  $\bar{\alpha}\varphi = \alpha$ , which provides that  $\varphi$  is surjective. Next, we verify that  $\varphi$  is injective. Let  $\alpha, \beta \in O_{n,2}$  with  $\alpha\varphi = \beta\varphi$ . For  $x \in X_{n-1}$ , the assumption  $x\alpha\varphi = x\beta\varphi$  implies  $(x+1)\alpha = (x+1)\beta$ . Moreover, we have  $1\alpha = 2 = 1\beta$ , which completes the argumentation that  $\alpha = \beta$ .

Finally, let  $\alpha, \beta \in O_{n,2}$ . Then for  $x \in X_{n-1}$ , we have  $x((\alpha\beta)\varphi) = (x+1)\alpha\beta - 1 = (((x+1)\alpha - 1) + 1)\beta - 1 = ((x+1)\alpha - 1)(\beta\varphi) = (x(\alpha\varphi))(\beta\varphi)$ . This shows that  $(\alpha\beta)\varphi = (\alpha\varphi)(\beta\varphi)$ , which completes the proof.  $\square$

## 2. Rank of $O_{n,p}$

As already mentioned, the rank of  $O_{n,1}$  as well as the rank of  $O_{n,n}$  is  $C_{n-1} - C_{n-2}$ . Since  $O_{n,2}$  is isomorphic to both  $O_{n-1,1}$  and  $O_{n,n-1}$ , we can conclude that  $rank(O_{n,2}) = rank(O_{n,n-1}) = C_{n-2} - C_{n-3}$ . Since  $O_{n,p}$  is a nilpotent semigroup, we have  $rank(O_{n,p}) = |O_{n,p} \setminus O_{n,p}^2|$  (see [13], Lemma 2.1). Let

$$G_{n,p} = \{\alpha \in O_{n,p} : |x\alpha - x| = 1 \text{ for at least one } x \in X_n \setminus \{p-1, p+1\}\}$$

and we will show that  $O_{n,p} \setminus O_{n,p}^2 = G_{n,p}$ . For this, we show first that  $O_{n,p}^2 \cap G_{n,p} = \emptyset$ .

**Lemma 2.1** *If  $\alpha, \beta \in O_{n,p}$ , then  $\alpha\beta \notin G_{n,p}$ .*

**Proof** Assume that there are  $\alpha, \beta \in O_{n,p}$  such that  $\alpha\beta \in G_{n,p}$ . Then there is  $x \in X_n \setminus \{p-1, p, p+1\}$  such that  $|x\alpha\beta - x| = 1$ .

If  $x < p-1$ , then  $x\alpha \geq x+1$  by Proposition 1.1, where  $x+1 < p$ . This implies  $x\alpha\beta \geq (x+1)\beta \geq x+2$ . Thus,  $x\alpha\beta - x \geq x+2 - x = 2$ , a contradiction.

If  $x > p+1$ , then  $x\alpha \leq x-1$  by Proposition 1.1, where  $x-1 > p$ . Thus,  $x\alpha\beta \leq (x-1)\beta \leq x-2$ , i.e.  $x\alpha\beta - x \leq x-2 - x = -2$ , a contradiction.  $\square$

**Proposition 2.2**  $O_{n,p} \setminus O_{n,p}^2 = G_{n,p}$ .

**Proof** By Lemma 2.1, we can conclude that  $O_{n,p} \setminus O_{n,p}^2 \supseteq G_{n,p}$ . In order to show the converse inclusion, we verify that  $\alpha \in O_{n,p}^2$  for all  $\alpha \in O_{n,p} \setminus G_{n,p}$ . Let  $\alpha \in O_{n,p} \setminus G_{n,p}$ . We define  $\alpha^* : X_n \rightarrow X_n$  by

$$x\alpha^* = \begin{cases} x\alpha - 1 & \text{for } x \leq p-2; \\ x\alpha + 1 & \text{for } x \geq p+2; \\ p & \text{otherwise.} \end{cases}$$

Since  $\alpha \notin G_{n,p}$  implies  $|x - x\alpha| \geq 2$  for all  $x \in X_n \setminus \{p-1, p, p+1\}$ , it is easy to see that  $\alpha^* \in O_{n,p}$ . We show that  $\alpha = \alpha^*\gamma$ , where

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & \cdots & p-2 & p-1 & p & p+1 & p+2 & \cdots & n \\ 2 & 3 & 4 & \cdots & p-1 & p & p & p & p+1 & \cdots & n-1 \end{pmatrix}.$$

It is clear that  $\gamma \in O_{n,p}$ . Let  $x \in X_n$ . Suppose that  $x \leq p-2$ . We have  $x\alpha^*\gamma = (x\alpha - 1)\gamma$ , where  $x\alpha \leq p$  since  $\alpha$  is order-preserving. If  $x\alpha = p$ , then  $(x\alpha - 1)\gamma = (p-1)\gamma = p = x\alpha$ . If  $x\alpha < p$ , then  $(x\alpha - 1)\gamma = (x\alpha - 1) + 1 = x\alpha$ . Thus,  $x\alpha^*\gamma = x\alpha$ . If  $x \in \{p-1, p, p+1\}$ , then  $x\alpha = p = p\gamma = x\alpha^*\gamma$ . Suppose that  $x \geq p+2$ . Then  $x\alpha^*\gamma = (x\alpha + 1)\gamma$ . Since  $\alpha$  is order-preserving, we have  $x\alpha \geq p$ . If  $x\alpha = p$ , then  $x\alpha^*\gamma = (x\alpha + 1)\gamma = (p+1)\gamma = p = x\alpha$ . If  $x\alpha > p$ , then  $x\alpha + 1 > p+1$  and  $x\alpha^*\gamma = (x\alpha + 1)\gamma = (x\alpha + 1 - 1) = x\alpha$ .  $\square$

Proposition 2.2 shows that  $\alpha \in O_{n,p}^2$  if and only if  $|x\alpha - x| \geq 2$  for all  $x \in X_n \setminus \{p-1, p, p+1\}$ . We will use this fact for the calculation of the size of  $O_{n,p}^2$ .

**Lemma 2.3**  $|O_{n,p}^2| = |O_{n-2,p-1}|$ .

**Proof** Let  $\alpha \in O_{n,p}^2$ . We define  $\alpha^* \in T_{n-2}$  by

$$x\alpha^* = \begin{cases} x\alpha - 1 & \text{for } x \leq p-1; \\ (x+2)\alpha - 1 & \text{for } p \leq x \leq n-2. \end{cases}$$

Clearly,  $x\alpha^* \leq n-1$  for all  $x \in X_{n-2}$ . Since  $|x\alpha - x| \geq 2$  for all  $x \in X_n \setminus \{p-1, p, p+1\}$ , we have  $|x\alpha^* - x| \geq 1$  for all  $x \in \{1, \dots, p-2, p, \dots, n-2\}$ . Moreover,  $(p-1)\alpha = p$  implies  $(p-1)\alpha^* = (p-1)\alpha - 1 = p-1$ . Obviously,  $\alpha^*$  is order-preserving. Hence, we can conclude that  $\alpha^* \in O_{n-2,p-1}$ . We define a mapping  $\varphi : O_{n,p}^2 \rightarrow O_{n-2,p-1}$  by  $\alpha\varphi = \alpha^*$ . It is easy to verify that  $\varphi$  is a bijection, i.e.  $|O_{n,p}^2| = |O_{n-2,p-1}|$ .  $\square$

Now we are able to calculate the rank of the nilpotent semigroup  $O_{n,p}$ .

**Theorem 2.4**  $rank(O_{n,p}) = \mathcal{C}_{p-1}\mathcal{C}_{n-p} - \mathcal{C}_{p-2}\mathcal{C}_{n-p-1}$ .

**Proof** Note that  $rank(O_{n,p}) = |O_{n,p} \setminus O_{n,p}^2| = |O_{n,p}| - |O_{n,p}^2| = |O_{n,p}| - |O_{n-2,p-1}|$  by Lemma 2.3. Because of  $|O_{n,p}| = \mathcal{C}_{p-1}\mathcal{C}_{n-p}$  and  $|O_{n-2,p-1}| = \mathcal{C}_{p-2}\mathcal{C}_{n-p-1}$  by Lemma 1.4, we get  $rank(O_{n,p}) = |O_{n,p} \setminus O_{n,p}^2| = \mathcal{C}_{p-1}\mathcal{C}_{n-p} - \mathcal{C}_{p-2}\mathcal{C}_{n-p-1}$ .  $\square$

### 3. Presentations for $O_{n,1}$

The goal of this section is to give a presentation for  $O_{n,1}$ . Let  $A_n$  be set of all mappings  $g \in \{0, 1, \dots, n-1\}^{X_n}$  with  $1g = 0$ ,  $xg \geq 1$  for  $x \in \{2, \dots, n\}$ , where  $lg = 1$  for some  $l \in \{3, \dots, n\}$ , and either  $(x+1)g \leq xg$  or  $(x+1)g = xg + 1$  for all  $x \in \{1, \dots, n-1\}$ .

**Lemma 3.1** *Let  $g \in A_n$ . Then  $x - xg \geq 1$  for all  $x \in X_n$ .*

**Proof** We have  $1 - 1g = 1 - 0 = 1$ . Suppose that  $x - xg \geq 1$  for some  $x \in X_n$  and we will show that  $(x+1) - (x+1)g \geq 1$ . We have  $(x+1)g \leq xg$  or  $(x+1)g = xg + 1$ . If  $(x+1)g \leq xg$ , then  $1 \leq x - xg < x + 1 - xg \leq (x+1) - (x+1)g$ . If  $(x+1)g = xg + 1$ , then  $(x+1) - xg = (x+1) - (x+1)g + 1$ ; thus,  $1 \leq x - xg = (x+1) - (x+1)g$ .  $\square$

For each  $g \in A_n$ , let  $\alpha_g \in T_n$  with

$$x\alpha_g = x - xg \text{ for all } x \in X_n.$$

**Lemma 3.2**  $G_{n,1} = \{\alpha_g : g \in A_n\}$ .

**Proof** Let  $g \in A_n$  and we will show that  $\alpha_g \in G_{n,1}$ . Clearly,  $1\alpha_g = 1 - 1g = 1 - 0 = 1$  and  $|x - x\alpha_g| = xg \geq 1$  for all  $x \in \{2, 3, \dots, n\}$ . Note that there is  $l \in \{3, \dots, n\}$  such that  $lg = 1$ . This provides  $l\alpha_g = l - lg = l - 1$ ; thus,  $|l - l\alpha_g| = 1$ .

Next, we show that  $\alpha_g \in O_{n,1}$ , i.e.  $\alpha_g$  is order-preserving. Let  $x < y \in X_n$ . Then  $y = x + k$  for some  $k \in X_n$ . Firstly, we verify that  $x\alpha_g \leq (x+1)\alpha_g$ . We have  $xg \geq (x+1)g$  or  $(x+1)g = xg + 1$ . If  $xg \geq (x+1)g$ , then  $x\alpha_g = x - xg \leq x - (x+1)g < (x+1) - (x+1)g = (x+1)\alpha_g$ . If  $(x+1)g = xg + 1$ , then  $x\alpha_g = x - xg = x - (x+1)g + 1 = (x+1) - (x+1)g = (x+1)\alpha_g$ . By the same arguments, we obtain  $x\alpha_g \leq (x+1)\alpha_g \leq (x+2)\alpha_g \leq \dots \leq (x+k)\alpha_g$ , i.e.  $x\alpha_g \leq y\alpha_g$ .

Therefore,  $\alpha_g \in G_{n,1}$ . Altogether, we have shown that  $\{\alpha_g : g \in A_n\} \subseteq G_{n,1}$ .

For the converse inclusion, let  $\beta \in G_{n,1}$  and we define  $g_\beta \in \{0, 1, \dots, n-1\}^{X_n}$  by

$$xg_\beta = x - x\beta \text{ for all } x \in X_n.$$

We will show that  $g_\beta \in A_n$ . Clearly,  $1g_\beta = 1 - 1\beta = 1 - 1 = 0$  and  $xg_\beta = x - x\beta \geq 1$  for all  $x \in \{2, \dots, n\}$  by Proposition 1.1. Further, there is  $p \in \{3, \dots, n\}$  such that  $1 = p - p\beta = pg_\beta$ . Let  $x \in \{1, \dots, n-1\}$  with  $(x+1)g_\beta > xg_\beta$ . This implies  $(x+1) - (x+1)\beta > x - x\beta$  and so  $1 - (x+1)\beta > -x\beta$ , i.e.  $(x+1)\beta - 1 < x\beta$ . Since  $\beta$  is order-preserving,  $(x+1)\beta - 1 < x\beta$  is only possible if  $(x+1)\beta = x\beta$ , i.e.  $(x+1) - (x+1)\beta = x - x\beta + 1$ . This shows  $(x+1)g_\beta = xg_\beta + 1$ . Consequently,  $g_\beta \in A_n$ .

Finally, since  $x\alpha_{g\beta} = x - xg\beta = x - (x - x\beta) = x\beta$ , for all  $x \in X_n$ , we can conclude that  $\beta = \alpha_{g\beta} \in \{\alpha_g : g \in A_n\}$ . Consequently, we have shown the converse inclusion  $G_{n,1} \subseteq \{\alpha_g : g \in A_n\}$ , which completes the proof.  $\square$

Together with Proposition 2.2 we obtain:

**Corollary 3.3**  $\{\alpha_g : g \in A_n\}$  is a generating set of  $O_{n,1}$ .

For  $g, h \in A_n$ , let  $g \oplus h \in \{0, 1, 2, \dots, n - 1\}^{X_n}$  with  $1(g \oplus h) = 0, 2(g \oplus h) = 1$  and  $x(g \oplus h) = (x - xg)h + xg - 1$  for  $x \in \{3, 4, \dots, n\}$ . Note that  $2g = 1$  for all  $g \in A_n$ .

**Lemma 3.4** Let  $g, h \in A_n$ . Then  $g \oplus h \in A_n$ .

**Proof** For  $x \in \{3, 4, \dots, n\}$ , we have  $xg \geq 1$ . If  $xg = 1$ , then  $x - xg \geq 2$ ; thus,  $(x - xg)h \geq 1$ . This provides  $(x - xg)h + xg - 1 \geq 1$ . If  $xg > 1$ , then  $xg - 1 \geq 1$ ; thus  $(x - xg)h + xg - 1 \geq 1$ . We take  $x = 3$ . Then  $3g \in \{1, 2\}$  by Lemma 3.1. If  $3g = 1$ , then  $3(g \oplus h) = (3 - 1)h + 1 - 1 = 2h = 1$ . If  $3g = 2$ , then  $3(g \oplus h) = (3 - 2)h + 2 - 1 = 1h + 2 - 1 = 1$ . Let  $x \in \{2, 3, 4, \dots, n - 1\}$  such that  $(x + 1)(g \oplus h) > x(g \oplus h)$ . We show that  $(x + 1)(g \oplus h) = x(g \oplus h) + 1$ .

If  $xg > (x + 1)g$ , then  $xg = r + (x + 1)g$  for some  $r \in \{1, 2, \dots, n\}$ . Then  $(x + 1)(g \oplus h) = (x + 1 - (x + 1)g)h + (x + 1)g - 1$  and  $x(g \oplus h) = (x - xg)h + xg - 1 = (x - (r + (x + 1)g))h + r + (x + 1)g - 1$ , that is  $(x + 1 - (x + 1)g)h > (x - (r + (x + 1)g))h + r$ . Thus,  $(x + 1 - (x + 1)g)h \geq (x - (r + (x + 1)g))h + r + 1$ . On the other hand, we can calculate that  $(x + 1 - (x + 1)g)h \leq (x + 1 - (x + 1)g - (r + 1))h + (r + 1) = (x - (x + 1)g - r)h + (r + 1)$ . This implies  $(x + 1 - (x + 1)g)h = (x - r - (x + 1)g)h + r + 1$ . We get  $(x + 1 - (x + 1)g)h + (x + 1)g - 1 = (x - r - (x + 1)g)h + r + (x + 1)g = (x - r - (xg - r))h + r + xg - r = (x - xg)h + xg - 1 + 1 = x(g \oplus h) + 1$ . This implies  $(x + 1)(g \oplus h) = x(g \oplus h) + 1$ .

If  $xg = (x + 1)g$ , then  $(x + 1)(g \oplus h) = (x + 1 - xg)h + xg - 1 > (x - xg)h + xg - 1 = x(g \oplus h)$ . This gives  $(x + 1 - xg)h > (x - xg)h$ , i.e.  $(x + 1 - xg)h = (x - xg)h + 1$ . This provides  $(x + 1)(g \oplus h) = x(g \oplus h) + 1$ .

If  $xg < (x + 1)g$ , i.e.  $(x + 1)g = xg + 1$ , then we can calculate  $(x + 1)(g \oplus h) = (x + 1 - (x + 1)g)h + (x + 1)g - 1 = (x + 1 - xg - 1)h + xg + 1 - 1 = ((x - xg)h + xg - 1) + 1 = x(g \oplus h) + 1$ .  $\square$

Let us put  $\omega \in \{0, 1, 2, \dots, n - 1\}^{X_n}$  with  $1\omega = 0$  and  $x\omega = 1$  for  $x \in \{2, 3, 4, \dots, n\}$ . Clearly,  $\omega \in A_n$ .

It is useful to determine the product of any transformation in  $G_{n,1}$  with  $\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 2 & \dots & n - 1 \end{pmatrix}$ .

Let us define a mapping  $f : X_n \cup \{0\} \rightarrow X_n$  by  $f(0) = 1$  and  $f(x) = x$  for  $x \in X_n$ . In this case, we write the argument right of the mapping for convenient.

**Lemma 3.5** Let  $g \in A_n$ . Then

$$\alpha_g \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & f(2 - 3g) & f(3 - 4g) & \dots & f((n - 1) - ng) \end{pmatrix}.$$

**Proof** We have  $1\alpha_g \alpha_\omega = 1\alpha_\omega = 1$ ,  $2\alpha_g \alpha_\omega = 1\alpha_\omega = 1$ , and

$$x\alpha_g \alpha_\omega = \begin{cases} x - xg - 1 & \text{if } x - xg > 1; \\ 1 & \text{if } x - xg = 1, \end{cases}$$

i.e.  $x\alpha_g\alpha_\omega = f(x - xg - 1)$ , for all  $x \in \{3, 4, 5, \dots, n\}$ . □

For  $g \in A_n$  let  $\hat{g} \in \{0, 1, 2, \dots, n - 1\}^{X_n}$  with  $1\hat{g} = 0$ ,  $2\hat{g} = 1$ , and  $x\hat{g} = x - 1 - f(f(x - 1 - xg) - 1)$  for  $x \in \{3, 4, \dots, n\}$ .

**Lemma 3.6**  $\hat{g} \in A_n$  for all  $g \in A_n$ .

**Proof** Let  $g \in A_n$ . We have  $1\hat{g} = 0$  and  $2\hat{g} = 1$ . If  $x \geq 3$ , then we obtain  $x\hat{g} = x - 1 - f(f(x - 1 - xg) - 1) = x - 1 - x + 1 + xg + 1 = xg + 1 \geq 1$ , whenever  $f(x - 1 - xg) \geq 2$  and  $x\hat{g} = x - 1 - 1 \geq 3 - 2 = 1$ , whenever  $f(x - 1 - xg) = 1$ . Further,  $3g \in \{1, 2\}$  implies  $f(2 - 3g) = 1$  and  $3\hat{g} = 2 - f(f(2 - 3g) - 1) = 2 - f(1 - 1) = 2 - 1 = 1$ , i.e.  $2\hat{g} = 3\hat{g}$ . Let  $x \in \{3, \dots, n - 1\}$  with  $(x + 1)\hat{g} > x\hat{g}$ . Since  $\alpha_{\hat{g}}\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & & 4 & & \cdots & n \\ 1 & 1 & f(f(2 - 3g) - 1) & & f(f(3 - 4g) - 1) & & \cdots & f(f((n - 1) - ng) - 1) \end{pmatrix} \in O_{n,1}$ , we obtain  $f(f(x - 1 - xg) - 1) \leq f(f(x - (x + 1)g) - 1)$ ,  $-f(f(x - (x + 1)g) - 1) \leq -f(f(x - 1 - xg) - 1)$ ; thus,  $(x + 1)\hat{g} = x - f(f(x - (x + 1)g) - 1) \leq x - f(f(x - 1 - xg) - 1) = x - 1 - f(f(x - 1 - xg) - 1) + 1 = x\hat{g} + 1$ . On the other hand,  $x\hat{g} < (x + 1)\hat{g}$  implies  $x\hat{g} + 1 \leq (x + 1)\hat{g}$ . Therefore, we have  $x\hat{g} + 1 = (x + 1)\hat{g}$ . □

Now we define an alphabet set  $Y_n = \{x_g : g \in A_n\}$  and define two sets of relations on  $Y_n^+$ . Let

$$R_1 = \{x_g x_\omega^2 \approx x_{\hat{g}} x_\omega : g \in A_n\} \text{ and}$$

$$R_2 = \{x_g x_h \approx x_{g \oplus h} x_\omega : g, h \in A_n\}.$$

It is easy to see that  $|R_1| = |A_n|$  and  $|R_2| = |A_n|^2$ . Since  $R_1 \cap R_2 = \emptyset$ , we have

$$|R_1 \cup R_2| = |A_n| + |A_n|^2 = (1 + |A_n|) \cdot |A_n| = (1 + |G_{n,1}|)|G_{n,1}| = (1 + C_{n-1} - C_{n-2})(C_{n-1} - C_{n-2})$$

using Lemma 3.2 and Proposition 2.2.

We write  $\sim$  for the congruence on  $Y_n^+$  generated by  $R_1 \cup R_2$ . Note that  $\{\alpha_g : g \in A_n\}$  is a generating set of  $O_{n,1}$  (see Corollary 3.3). We define an epimorphism  $\varphi : Y_n \rightarrow O_{n,1}$  by  $x_g \varphi = \alpha_g$  for  $g \in A_n$ . We aim to show that  $\ker \varphi = \sim$ , so that  $O_{n,1}$  has presentation  $\langle Y_n | R_1 \cup R_2 \rangle$  via  $\varphi$ .

**Lemma 3.7** We have the inclusion  $\sim \subseteq \ker \varphi$ .

**Proof** It is sufficient to show that the relations in  $R_1 \cup R_2$  hold as equations in  $O_{n,1}$  when the variables are replaced by their images under  $\varphi$ . Note that  $x_\omega \varphi = \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 2 & 3 & \cdots & n - 1 \end{pmatrix}$ .

Let  $g \in A_n$ , where  $\hat{g} \in A_n$  by Lemma 3.6. Then  $\alpha_g \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & & \cdots & n \\ 1 & 1 & f(2 - 3g) & & \cdots & f((n - 1) - ng) \end{pmatrix}$  by Lemma 3.5 and it is easy to verify that  $\alpha_g \alpha_\omega = \alpha_h$  with  $1h = 0$ ,  $2h = 1$  and  $xh = x - f(x - 1 - xg)$  for  $x \in \{3, 4, \dots, n\}$ . Now we have

$$(\alpha_g \alpha_\omega) \alpha_\omega = \alpha_h \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & & \cdots & n \\ 1 & 1 & f(2 - 3 + f(2 - 3g)) & & \cdots & f(n - 1 - n + f((n - 1) - ng)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & & \cdots & n \\ 1 & 1 & f(f(2 - 3g) - 1) & f(f(3 - 4g) - 1) & \cdots & f(f((n - 1) - ng) - 1) \end{pmatrix}. \text{ On the other hand, Lemma}$$



3.5 gives

$$\begin{aligned} \alpha_{\hat{g}}\alpha_{\omega} &= \begin{pmatrix} 1 & 2 & 3 & & \cdots & n \\ 1 & 1 & f(2-2+f(f(2-3g)-1)) & & \cdots & f((n-1)-(n-1)+f(f((n-1)-ng)-1)) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(f(2-3g)-1) & f(f(3-4g)-1) & \cdots & f(f((n-1)-ng)-1) \end{pmatrix} \text{ using } f(f(x)) = f(x) \text{ for } x \in X_n. \end{aligned}$$

Hence,  $\alpha_g\alpha_{\omega}\alpha_{\omega} = \alpha_{\hat{g}}\alpha_{\omega}$  and we obtain  $x_gx_{\omega}^2\varphi = x_g\varphi x_{\omega}\varphi x_{\omega}\varphi = \alpha_g\alpha_{\omega}\alpha_{\omega} = \alpha_{\hat{g}}\alpha_{\omega} = x_{\hat{g}}\varphi x_{\omega}\varphi = x_{\hat{g}}x_{\omega}\varphi$ .

Let  $g, h \in A_n$ , where  $g \oplus h \in A_n$  by Lemma 3.4. Then we can easily calculate

$$\alpha_g\alpha_h = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3-3g-(3-3g)h & \cdots & n-ng-(n-ng)h \end{pmatrix}.$$

For  $x \in \{3, 4, 5, \dots, n\}$ , we have  $x - xg - (x - xg)h \geq 1$  since  $x - xg - (x - xg)h$  is in the image of  $\alpha_g\alpha_h \in O_{n,1}$ ; thus,  $f(x - xg - (x - xg)h) = x - xg - (x - xg)h$ . Hence,  $x_{g \oplus h}x_{\omega}\varphi = x_{g \oplus h}\varphi x_{\omega}\varphi = \alpha_{g \oplus h}\alpha_{\omega}$

$$= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & f(3-3g-(3-3g)h+1-1) & \cdots & f(n-ng-((n-ng)h)+1-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 3-3g-(3-3g)h & 4-4g-(4-4g)h & \cdots & n-ng-(n-ng)h \end{pmatrix} \quad \square$$

$$= \alpha_g\alpha_h = x_g\varphi x_h\varphi = (x_gx_h)\varphi.$$

Now, we want to show the converse inclusion  $\ker\varphi \subseteq \sim$ .

**Lemma 3.8** *Let  $v_1, v_2, \dots, v_r \in \{x_g : g \in A_n\}$ ,  $r \geq 2$ . Then there is  $h \in A_n$  such that  $v_1v_2 \cdots v_r \sim x_hx_{\omega}$ .*

**Proof** We prove by induction on  $r$ . If  $r = 2$ , then the statement is satisfied by  $R_2$ . Suppose that for  $v_1, v_2, \dots, v_r \in \{x_g : g \in A_n\}$ , for some integer  $r \geq 2$ , there is  $h \in A_n$  such that  $v_1v_2 \cdots v_r \sim x_hx_{\omega}$ . Let  $v_1, v_2, \dots, v_{r+1} \in \{x_g : g \in A_n\}$ . Then  $v_1(v_2v_2 \cdots v_rv_{r+1}) \sim v_1(x_hx_{\omega})$  for some  $h \in A_n$ . Further,  $(v_1x_h)x_{\omega} \sim (x_{\hat{h}}x_{\omega})x_{\omega}$  by  $R_2$ , for some  $\hat{h} \in A_n$ . By  $R_1$ , there is  $\tilde{h} \in A_n$  such that  $x_{\hat{h}}x_{\omega}x_{\omega} \sim x_{\tilde{h}}x_{\omega}$ , i.e.  $v_1v_2 \cdots v_rv_{r+1} \sim x_{\tilde{h}}x_{\omega}$ .  $\square$

We put now  $\hat{A}_n = \{g \in A_n : |x - xg| > 1 \text{ for all } x \in \{3, 4, 5, \dots, n\}\}$ .

**Lemma 3.9** *For all  $h \in A_n$ , there is  $g \in \hat{A}_n$  such that  $x_gx_{\omega} \sim x_hx_{\omega}$ .*

**Proof** Let  $h \in A_n$ . Suppose that  $i - ih = 1$  for some  $i \in \{3, 4, 5, \dots, n\}$ . Let  $Q = \{x \in \{3, 4, 5, \dots, n\} : x - xh = 1\} \neq \emptyset$ . Then  $xh \geq 2$  for all  $x \in Q$ . We put

$$xg = \begin{cases} xh & \text{if } x \notin Q; \\ xh - 1 & \text{if } x \in Q. \end{cases}$$

We have to show that  $g \in A_n$ .

We have  $1g = 1h = 0$  and for  $x \in \{2, \dots, n\}$ ,

$$xg = \begin{cases} xh \neq 0 & \text{if } x \notin Q; \\ xh - 1 \geq 2 - 1 = 1 \neq 0 & \text{if } x \in Q. \end{cases}$$

There is  $l \in \{3, 4, 5, \dots, n\}$  such that  $lh = 1$ , i.e.  $|l - lh| \neq 1$  and  $l \notin Q$ . Thus,  $lg = lh = 1$ . Let  $x \in \{2, 3, 4, \dots, n - 1\}$  such that  $(x + 1)g > xg$ . Then we have the following cases:

Case a:  $xg = xh$ . Assume that  $(x + 1)g = (x + 1)h - 1$ . Then  $(x + 1)h - 1 = (x + 1)g > xg = xh$ . This gives  $(x + 1)h > (x + 1)h - 1 > xh$ , i.e.  $(x + 1)h = xh + 1$ . However,  $(x + 1)h = (x + 1)h - 1 + 1 > xh + 1 = (x + 1)h$ . This is a contradiction. Hence,  $(x + 1)g = (x + 1)h$ . Thus,  $(x + 1)h > xh$ , i.e.  $(x + 1)h = xh + 1$ ; therefore,  $(x + 1)g = (x + 1)h = xh + 1 = xg + 1$ .

Case b:  $(x + 1)g = (x + 1)h$  and  $xg = xh - 1$ , i.e.  $(x + 1)h \geq xh$ . This implies that  $(x + 1)h = xh$  or  $(x + 1)h = xh + 1$ . If  $(x + 1)h = xh + 1$ , then  $1 = x + 1 - (x + 1)h = x + 1 - xh - 1 = x - xh \geq 2$ , a contradiction. Thus, we have  $(x + 1)h = xh$ . This means that  $(x + 1)g = (x + 1)h = xh$  and then  $(x + 1)g = xg + 1$ .

Case c:  $xg = xh - 1$  and  $(x + 1)g = (x + 1)h - 1$ . Then  $(x + 1)g > xg$  implies  $(x + 1)h > xh$ , i.e.  $(x + 1)h = xh + 1$ ,  $(x + 1)h - 1 = xh - 1 + 1$ ; thus,  $(x + 1)g = xg + 1$ .

Altogether, this shows that  $g \in A_n$ .

Let  $x \in \{3, 4, 5, \dots, n\}$ . If  $x \notin Q$ , then  $x - xg = x - xh > 1$ . If  $x \in Q$ , then  $x - xg = x - xh + 1 = 2 > 1$ . This means that  $x - xg > 1$  for all  $x \in \{3, 4, 5, \dots, n\}$ . Therefore,  $g \in \widehat{A}_n$ .

Next, we show that  $x_g x_\omega \sim x_h x_\omega$ . We have  $x_h x_\omega \approx x_{h \oplus \omega} x_\omega \in R_2$  and  $x_g x_\omega \approx x_{g \oplus \omega} x_\omega \in R_2$ . If  $x \in Q$ , then  $x - xh = 1$  and  $x - xg = x - xh + 1 = 2$ . This gives  $(x - xh)\omega + xh - 1 = 0 + xh - 1 = xh - 1 = 1 + xh - 1 - 1 = 1 + xg - 1 = (x - xg)\omega + xg - 1$ . If  $x \notin Q$ , then  $xh = xg$  and  $(x - xh)\omega + xh - 1 = (x - xg)\omega + xg - 1$ . This provides  $x_{h \oplus \omega} x_\omega = x_{g \oplus \omega} x_\omega$ . Then we obtain  $x_h x_\omega \sim x_g x_\omega$  by transitivity.  $\square$

We define a function  $\mu : \widehat{A}_n \rightarrow O_{n,1}$  by  $g \mapsto \alpha_g \alpha_\omega$  for all  $g \in \widehat{A}_n$ .

**Lemma 3.10**  $\mu$  is an injection from  $\widehat{A}_n$  into  $O_{n,1} \setminus G_{n,1}$ .

**Proof** We show that  $\mu$  is a function from  $\widehat{A}_n$  to  $O_{n,1} \setminus G_{n,1}$ . For this, we take  $g \in \widehat{A}_n$  such that  $g\mu = \alpha_g \alpha_\omega \in O_{n,1}$ . This shows that  $\alpha_g \alpha_\omega \notin G_{n,1}$  by Lemma 2.1.

We show  $\mu$  is injective. For this, let  $g_1, g_2 \in \widehat{A}_n$  such that  $g_1\mu = g_2\mu$ . By Lemma 3.5, we have  $g_1\mu = \alpha_{g_1} \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & f(3 - 3g_1 - 1) & f(4 - 4g_1 - 1) & \dots & f(n - ng_1 - 1) \end{pmatrix}$  and  $g_2\mu = \alpha_{g_2} \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & f(3 - 3g_2 - 1) & f(4 - 4g_2 - 1) & \dots & f(n - ng_2 - 1) \end{pmatrix}$ . For  $x \in \{3, 4, 5, \dots, n\}$ , we have  $x - xg_1 > 1$  and  $x - xg_2 > 1$ , this means  $x - xg_1 - 1 = f(x - xg_1 - 1) = f(x - xg_2 - 1) = x - xg_2 - 1$ . Hence  $xg_1 = xg_2$ . Altogether, we have  $xg_1 = xg_2$  for all  $x \in X_n$ , that means  $g_1 = g_2$ .  $\square$

**Theorem 3.11** The semigroup  $O_{n,1}$  has presentation  $\langle Y_n | R_1 \cup R_2 \rangle$  via  $\varphi$ .

**Proof** By Lemma 3.7, we have  $\sim \subseteq \ker \varphi$ . It remains to show that  $\ker \varphi \subseteq \sim$ . For this, let  $(w_1, w_2) \in \ker \varphi$ . By Lemmas 3.8 and 3.9, there are  $g_1, g_2 \in \widehat{A}_n$  such that  $w_1 \sim x_{g_1} x_\omega$  and  $w_2 \sim x_{g_2} x_\omega$ . Since  $\sim \subseteq \ker \varphi$ , we can calculate that  $x_{g_1} x_\omega \varphi = w_1 \varphi = w_2 \varphi = x_{g_2} x_\omega \varphi$ . In particular,  $g_1\mu = \alpha_{g_1} \alpha_\omega = x_{g_1} x_\omega \varphi$  and

$g_2\mu = \alpha_{g_2}\alpha_\omega = x_{g_2}x_\omega\varphi$ . This provides  $g_1\mu = g_2\mu$ . Since  $\mu$  is injective by Lemma 3.10, we obtain  $g_1 = g_2$ . This gives  $w_1 \sim x_{g_1}x_\omega = x_{g_2}x_\omega \sim w_2$ , i.e.  $w_1 \sim w_2$ .  $\square$

We want to illustrate Theorem 3.11 for  $n = 4$ . It is easy to verify that  $O_{4,1}$  consists of the five transformations

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}, b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix},$$

$$d = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \text{ and } e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let  $g_1, g_2 \in \{0, 1, 2, 3\}^{X_4}$  with  $1g_1 = 1g_2 = 0$ ,  $2g_1 = 2g_2 = 3g_2 = 4g_1 = 1$ , and  $3g_1 = 4g_2 = 2$ . It is easy to see that  $A_4 = \{g_1, g_2, w\}$ , i.e.  $Y_4 = \{x_{g_1}, x_{g_2}, w\}$ . For convenience, we use  $a, b$ , and  $c$  for  $x_{g_1}, x_{g_2}$ , and  $\omega$ , respectively. The relations in  $R_1$  provide

$$ac^2 \approx bc, bc^2 \approx bc, \text{ and } cc^2 \approx bc.$$

The relations in  $R_2$  provide

$$\begin{matrix} aa \approx bc, & ba \approx bc, & ca \approx bc, \\ ab \approx ac, & bb \approx bc, & cb \approx ac, \\ ac \approx ac, & bc \approx bc, & cc \approx ac. \end{matrix}$$

Then we obtain the following presentation for  $O_{4,1}$ :

$$\langle \{a, b, c\} \mid \{ac^2 \approx bc, bc^2 \approx bc, cc^2 \approx bc, aa \approx bc, ab \approx ac, ba \approx bc, bb \approx bc, ca \approx bc, cb \approx ac, cc \approx ac\} \rangle.$$

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