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Bessel equation and Bessel function on $\mathbb{T}_{(q,h)}$

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Abstract: This article is devoted to present nabla (q,h)-analogues of Bessel equation and Bessel function. In order to construct series solution of nabla (q,h)-Bessel equation, we present nabla (q,h)-analysis regarding nabla generalized quantum binomial, nabla (q,h)-analogues of Taylor's formula, Gauss's binomial formula, Taylor series, analytic functions, analytic exponential function with its fundamental properties, analytic trigonometric and hyperbolic functions. We emphasize that nabla (q,h)-Bessel equation recovers classical, h- and q-discrete Bessel equations. In addition, we establish nabla (q,h)-Bessel function which is expressed in terms of an absolutely convergent series in nabla generalized quantum binomials and intimately demonstrate its reductions. Finally, we develop modified nabla (q,h)-Bessel equation, modified nabla (q,h)-Bessel function and its relation with nabla (q,h)-Bessel function.

Key words: Nabla generalized quantum binomial, nabla (q,h)-Taylor series, nabla (q,h)-analytic functions, nabla (q,h)-Bessel equation, nabla (q,h)-Bessel function

1. Introduction

The concept of time scales $\mathbb{T}$ was discovered in [7] as an arbitrary nonempty closed subset of real numbers in order to unify and extend the discrete sets: h-lattice and q-numbers

$$h\mathbb{Z} := \{hx : x \in \mathbb{Z},\ h > 0\}, \quad \mathbb{K}_q := \{q^n : \quad n \in \mathbb{Z}, \quad q \in \mathbb{R}, \quad q \neq 1\} \cup \{0\},$$

as well as any type of discrete sets and combination of continuous and discrete sets. By the virtue of time scales, the theories of differential and difference equations have been extended to the dynamic equations [1] which have a vast variety of applications in economics, biomathematics and mathematical physics. A special discrete time scale $\mathbb{T}_{(q,h)}$ was introduced in [3] in order to unify and extend h- and q-analysis. The study of (q,h)-analysis is a fairly new subject and the research in this area is rapidly growing: (q,h)-analogue of fractional calculus [3], (q,h)-analogue of Laplace transform [13], and (q,h)-analogue of quantum splines [5].

The intrinsic feature of the special structure $\mathbb{T}_{(q,h)}$ is to get rid of the discrepancies and deficiencies on general time scales occurring even in some elementary subjects (e.g., the explicit form of the polynomials, Taylor series and exponential functions). In [14], the delta (q,h)-calculus was created. The essential aim in [14] was to introduce a precise, concrete and applicable form of the exponential function on $\mathbb{T}_{(q,h)}$. Parenthetically, the article also included (q,h)-analogues of Δ Cauchy-Euler and wave equations equipped with their solutions. In

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[15], delta (q,h)-integral was constructed as a series explicitly and its fundamental theorems were presented. The main purpose of [15] was to present delta (q,h)-analogue of Gauss's binomial formula expressed in terms of (q,h)-polynomials. As a consequence of (q,h)-Gauss's binomial formula, the additive property of exponential functions on $\mathbb{T}_{(q,h)}$ was stated. As applications, (q,h)-analogue of variation of parameter formula for a first order nonhomogeneous linear (q,h)-difference equation, (q,h)-analogues of dynamic generic diffusion equation and Burger's equation with its multi(q,h)-shock soliton solutions were derived.

The motivation of the current article comes from the absence of nabla (q,h)-analysis, nabla (q,h) Bessel equation and nabla (q,h) Bessel function. In Section 2, we first intensify on constructing proper polynomials on $\mathbb{T}_{(q,h)}$, namely nabla generalized quantum binomials, in a way that their role on $\mathbb{T}_{(q,h)}$ is similar to the role of the ordinary polynomials in $\mathbb{R}$. Nabla generalized quantum binomials allow us to evolve nabla (q,h)-Taylor's formula. We present related Leibniz rules and additive properties of nabla generalized quantum binomials. Furthermore, we state and prove nabla (q,h)-Gauss's binomial formula which is determined by the virtue of nabla generalized quantum binomials. We conclude this section by introducing nabla (q,h)-Taylor series expressed in terms of nabla generalized quantum binomials, analyzing its uniform, absolute convergence and defining nabla (q,h)-analytic functions.

Once we have convergent nabla (q,h)-Taylor series, it is possible to offer series solutions of nabla (q,h)-difference equations. For this purpose, we devote Section 3 to present nabla (q,h)-Bessel equation which reduces to classical Bessel equation as $(q,h)\to(1,0)$ and discrete Bessel equation as $q\to 1$. To the best of our knowledge, no research has been addressed to q-discrete Bessel equation. We emphasize that the presented nabla (q,h)-Bessel equation provides nabla q-Bessel equation as $h\to 0$. We construct an absolutely convergent nabla (q,h)-Taylor series solution which recovers the classical, discrete Bessel functions and produces a new nabla q-Bessel function. We finalize this section by presenting modified nabla (q,h)-Bessel equation, modified nabla (q,h)-Bessel function and its relation with nabla (q,h)-Bessel function. In order not to repeat, we do not add delta versions of (q,h)-Bessel equation or modified (q,h)-Bessel equation which can be derived by a similar fashion.

The significant contribution of Section 4 is to present nabla (q,h)-analogue of elementary functions. We introduce a nabla (q,h)-analytic exponential function which recovers ordinary exponential function, Jackson's q-exponential function [9], Euler's q-exponential function [11] and h-exponential function. Similar to the ordinary exponential function, we show that nabla (q,h)-exponential function acts as a translation operator. Furthermore, we prove that such an exponential function satisfies the additive property by using nabla (q,h)-Gauss's binomial formula. Finally, we present nabla (q,h)-analogues of trigonometric and hyperborlic functions with their properties.

2. Nabla (q,h)-analogues of polynomials, Taylor's formula, Gauss's binomial formula and Taylor series

Inspired by [3], for a given $x\in\mathbb{R}^+\cup\{0\}$, we define a two-parameter time scale, $\mathbb{T}_{(q,h)}$ by

$$\mathbb{T}_{(q,h)} := \{q^n x + [n]h : \quad n\in\mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\}, \quad h,q\in\mathbb{R}^+, \quad q\neq 1, \tag{2.1}$$

where $[n] := \dfrac{q^n - 1}{q - 1}$. On any time scale $\mathbb{T}$, the forward jump operator σ and the backward jump operator ρ are defined by

$$\sigma(x) := \inf\{s \in \mathbb{T} : s > x\}, \qquad \rho(x) := \sup\{s \in \mathbb{T} : s < x\}.$$

On $\mathbb{T}_{(q,h)}$, σ and ρ turn out to be

$$\sigma^n(x) = q^n x + [n]h, \qquad \rho^n(x) = q^{-n}(x - [n]h), \quad n \in \mathbb{N},$$

satisfying the relation $(\sigma \circ \rho)(x) = (\rho \circ \sigma)(x) = x$. Therefore, $\mathbb{T}_{(q,h)}$ is a regular discrete time scale. Note that, for $0 < q < 1$

$$\lim_{n\to\infty}(q^n x + [n]h) = \frac{h}{1-q}$$

while for $q > 1$

$$\lim_{n\to\infty} q^{-n}(x - [n]h) = \frac{h}{1-q}.$$

Thus, the point $\dfrac{h}{1-q}$ is an accumulation point.

Definition 2.1 *[3] Let $f(x)$ be any real valued function defined on $\mathbb{T}_{(q,h)}$ and $x \neq \frac{h}{1-q}$. The delta (q,h)-derivative and the nabla (q,h)-derivative of f are defined respectively by*

$$D_{(q,h)}f(x) := \frac{f(\sigma(x)) - f(x)}{\sigma(x) - x} = \frac{f(qx+h) - f(x)}{(q-1)x + h}, \tag{2.2}$$

and

$$\widetilde{D}_{(q,h)}f(x) := \frac{f(x) - f(\rho(x))}{x - \rho(x)} = \frac{f(x) - f(\frac{x-h}{q})}{x - (\frac{x-h}{q})}. \tag{2.3}$$

Furthermore, the delta and nabla (q,h)-derivatives of f at the right dense accumulation point $x = \frac{h}{1-q}$ are introduced as

$$D_{(q,h)}f(\frac{h}{1-q}) := \lim_{s\to(\frac{h}{1-q})^+} \frac{f(s) - f\left(\frac{h}{1-q}\right)}{s - \frac{h}{1-q}} = f'(\frac{h}{1-q}),$$

$$\widetilde{D}_{(q,h)}f(\frac{h}{1-q}) := \lim_{s\to(\frac{h}{1-q})^+} \frac{f(s) - f\left(\frac{h}{1-q}\right)}{s - \frac{h}{1-q}} = f'(\frac{h}{1-q}),$$

provided that the limits exist (see [1, Theorem 1.16 i]). We emphasize that the delta (q,h)-derivative recovers h, q and ordinary derivatives while nabla (q,h)-derivative reduces to nabla h, nabla q and ordinary derivatives in the proper limits of q and h.

Proposition 2.2 [14] *If* f, g *are any real valued functions defined on* $\mathbb{T}_{(q,h)}$*, then the product and quotient rules are given by*

$$\begin{aligned} D_{(q,h)}\left(f(x)g(x)\right) &= f(x)D_{(q,h)}g(x) + g(qx+h)D_{(q,h)}f(x) \\ &= g(x)D_{(q,h)}f(x) + f(qx+h)D_{(q,h)}g(x), \\ \widetilde{D}_{(q,h)}\left(f(x)g(x)\right) &= f(x)\widetilde{D}_{(q,h)}g(x) + g(\frac{x-h}{q})\widetilde{D}_{(q,h)}f(x) \\ &= g(x)\widetilde{D}_{(q,h)}f(x) + f(\frac{x-h}{q})\widetilde{D}_{(q,h)}g(x), \\ D_{(q,h)}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(x)D_{(q,h)}f(x) - f(x)D_{(q,h)}g(x)}{g(x)g(qx+h)} \\ &= \frac{g(qx+h)D_{(q,h)}f(x) - f(qx+h)D_{(q,h)}g(x)}{g(x)g(qx+h)}, \\ \widetilde{D}_{(q,h)}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(x)\widetilde{D}_{(q,h)}f(x) - f(x)\widetilde{D}_{(q,h)}g(x)}{g(x)g(\frac{x-h}{q})} \\ &= \frac{g(\frac{x-h}{q})\widetilde{D}_{(q,h)}f(x) - f(\frac{x-h}{q})\widetilde{D}_{(q,h)}g(x)}{g(x)g(\frac{x-h}{q})}. \end{aligned}$$

We introduce a nabla (q,h)-analogue of the polynomial $(x-x_0)^n$ whose features on $\mathbb{T}_{(q,h)}$ is the same with the characteristics of an ordinary polynomial in $\mathbb{R}$.

Definition 2.3 *Let* $x_0 \in \mathbb{R}$*. We define the nabla generalized quantum binomial as the polynomial*

$$(x-x_0)^n_{\widetilde{q,h}} := \begin{cases} 1 & \text{if} \quad n=0, \\ \prod_{i=1}^{n}\left(x - \frac{x_0-[i-1]h}{q^{i-1}}\right) & \text{if} \quad n \in \mathbb{N}. \end{cases} \tag{2.4}$$

The nabla generalized quantum binomial $(x-x_0)^n_{\widetilde{q,h}}$ recovers the ordinary polynomial $(x-x_0)^n$ as $(q,h) \to (1,0)$. Furthermore, similar to an ordinary polynomial, (2.4) satisfies the Leibniz rules on $\mathbb{T}_{(q,h)}$. To be more precise, the nabla generalized quantum binomial adopts exactly the same role that an ordinary polynomial performs in $\mathbb{R}$.

Proposition 2.4 *The following properties hold for the nabla generalized quantum binomial* (2.4)

(i) $\widetilde{D}_{(q,h)}(x-x_0)^n_{\widetilde{q,h}} = [n]_{\frac{1}{q}}(x-x_0)^{n-1}_{\widetilde{q,h}}, \quad n \in \mathbb{N},$

(ii) $\widetilde{D}^k_{(q,h)}(x-x_0)^n_{\widetilde{q,h}} = \dfrac{[n]_{\frac{1}{q}}!}{[n-k]_{\frac{1}{q}}!}(x-x_0)^{n-k}_{\widetilde{q,h}}, \quad 0 \le k \le n,$

where $\frac{1}{q}$*- numbers are defined by*

$$[n]_{\frac{1}{q}} := \frac{\left(\frac{1}{q}\right)^n - 1}{\frac{1}{q}-1} = 1 + \frac{1}{q} + \ldots + \frac{1}{q^{n-1}} = \frac{[n]}{q^{n-1}},$$

since $[n]_{\frac{1}{q}} = \frac{[n]}{q^{n-1}} \to n$ *as* $q \to 1$. *Consequently,*

$$[n]_{\frac{1}{q}}! := [n]_{\frac{1}{q}}.[n-1]_{\frac{1}{q}} \dots [2]_{\frac{1}{q}}.[1]_{\frac{1}{q}} = \frac{[n]!}{q^{\frac{n(n-1)}{2}}},$$

with convention $[0]_{\frac{1}{q}}! = 1$.

Proof Using the definition of nabla (q,h)-derivative (2.3) on nabla generalized quantum binomial (2.4), we obtain

$$\begin{aligned}
\widetilde{D}_{(q,h)}(x-x_0)^n_{\widetilde{q,h}} &= \frac{(x-x_0)^n_{\widetilde{q,h}} - (\frac{x-h}{q} - x_0)^n_{\widetilde{q,h}}}{(x - \frac{x-h}{q})} \\
&= \frac{(x-x_0)(x-\frac{x_0-h}{q})\cdots(x-\frac{x_0-[n-1]h}{q^{n-1}}) - (\frac{x-h}{q}-x_0)(\frac{x-h}{q}-\frac{x_0-h}{q})\cdots(\frac{x-h}{q}-\frac{x_0-[n-1]h}{q^{n-1}})}{\frac{(q-1)x+h}{q}} \\
&= (x-x_0)\left(x-\frac{x_0-h}{q}\right)\cdots\left(x-\frac{x_0-[n-2]h}{q^{n-2}}\right)\left(\frac{x-\frac{x_0-[n-1]h}{q^{n-1}} - \frac{1}{q^{n-1}}\left(\frac{x-h}{q}-x_0\right)}{\frac{(q-1)x+h}{q}}\right) \\
&= (x-x_0)^{n-1}_{\widetilde{q,h}} \frac{(q^n-1)x + (1+q+\dots q^{n-1})h}{q^{n-1}((q-1)x+h)} = \frac{[n]}{q^{n-1}}(x-x_0)^{n-1}_{\widetilde{q,h}} = [n]_{\frac{1}{q}}(x-x_0)^{n-1}_{\widetilde{q,h}}.
\end{aligned}$$

One can end up with the property (ii) by applying nabla (q,h)-derivative k-times successively on (2.4). □

Theorem 2.5 *If the sequences of polynomials are given by* $P_i(x) = \frac{(x-x_0)^i_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!}$, *for* $i \in \mathbb{N}_0$, *then the following statements hold.*

(a) *The polynomials* P_i *satisfy the criteria*

(i) $P_0(x_0) = 1$ *and* $P_i(x_0) = 0, \quad i \in \mathbb{N},$

(ii) $deg(P_i) = i, \quad i \in \mathbb{N}_0$

(iii) $\widetilde{D}_{(q,h)}(P_i) = P_{i-1}, \quad i \in \mathbb{N}.$

(b) *Any polynomial* $Q(x)$ *of degree* n *can be written as nabla* (q,h)*-Taylor's formula*

$$Q(x) = \sum_{i=0}^{n} \widetilde{D}^i_{(q,h)} Q(x_0) P_i(x) = \sum_{i=0}^{n} \widetilde{D}^i_{(q,h)} Q(x_0) \frac{(x-x_0)^i_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!}.$$

Proof (a) The polynomial $P_i(x)$ verifies the conditions (i) and (ii) straightforwardly. Now using Proposition 2.4, we obtain

$$\widetilde{D}_{(q,h)}P_i(x) = \widetilde{D}_{(q,h)}\left(\frac{(x-x_0)^i_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!}\right) = \frac{[i]_{\frac{1}{q}}(x-x_0)^{i-1}_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!} = \frac{(x-x_0)^{i-1}_{\widetilde{q,h}}}{[i-1]_{\frac{1}{q}}!} = P_{i-1}(x).$$

(b) Consider the set of polynomials $B = \{P_0(x), P_1(x), \dots, P_n(x)\}$. Since $deg(P_i) = i$ for each i, then B is a linearly independent set. Assume W is the vector space of polynomials of dimension $n+1$. Since $\mid B \mid = \mid W \mid = n+1$, then B spans W. Therefore, B is a basis for W and any polynomial $Q(x) \in W$ can be expressed as a linear combination of polynomials in B

$$Q(x) = \sum_{i=0}^{n} a_i P_i(x). \tag{2.5}$$

The condition (i) implies $Q(x_0) = \sum_{i=0}^{n} a_i P_i(x_0) = a_0 P_0(x_0) = a_0$. We employ the linearity of $\widetilde{D}_{(q,h)}$ and the condition (iii) to attain

$$\widetilde{D}_{(q,h)} Q(x) = \sum_{i=0}^{n} a_i \widetilde{D}_{(q,h)} P_i(x) = \sum_{i=1}^{n} a_i P_{i-1}(x),$$

which yields $a_1 = \widetilde{D}_{(q,h)} Q(x_0)$. We apply $\widetilde{D}_{(q,h)}$, k times successively to $Q(x)$ and obtain

$$\widetilde{D}^k_{(q,h)} Q(x) = \sum_{i=0}^{n} a_i \widetilde{D}^k_{(q,h)} P_i(x) = \sum_{i=k}^{n} a_i P_{i-k}(x),$$

which allows us to compute $a_k = \widetilde{D}^k_{(q,h)} Q(x_0),\ 0 \le k \le n$. Hence, by the use of (2.5), we conclude that

$$Q(x) = \sum_{i=0}^{n} \widetilde{D}^i_{(q,h)} Q(x_0) P_i(x) = \sum_{i=0}^{n} \widetilde{D}^i_{(q,h)} Q(x_0) \frac{(x-x_0)^{i}_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!},$$

which finishes the proof. □

Proposition 2.6 *The following additive identity holds for nabla generalized quantum binomial* (2.4)

$$(x-x_0)^{m+n}_{\widetilde{q,h}} = (x-x_0)^{m}_{\widetilde{q,h}} \cdot \left(x - \frac{x_0 - [m]h}{q^m}\right)^{n}_{\widetilde{q,h}}, \qquad m, n \in \mathbb{Z}.$$

Proof We investigate the proof in five cases.

(i) The proof is trivial if $m = 0$ or $n = 0$ or both.

(ii) Assume that both m and n are positive. Rewrite the nabla generalized quantum binomial (2.4) as

$$\begin{aligned}(x-x_0)^{m+n}_{\widetilde{q,h}} =& (x-x_0)\left(x - \frac{x_0 - h}{q}\right)\cdots\left(x - \frac{x_0-[m-1]h}{q^{m-1}}\right)\left(x - \frac{x_0-[m]h}{q^m}\right) \\ & \cdots\left(x - \frac{x_0-[m+n-1]h}{q^{m+n-1}}\right) \\ =& (x-x_0)^{m}_{\widetilde{q,h}} \cdot f(x, x_0),\end{aligned}$$

where $f(x,x_0) = \left(x - \frac{x_0-[m]h}{q^m}\right)\cdots\left(x - \frac{x_0-[m+n-1]h}{q^{m+n-1}}\right)$. The result follows by replacing x_0 by $\left(\frac{x_0-[m]h}{q^m}\right)$ in (2.4).

Inspired by the case (ii), we set $m = -n$ and introduce

$$(x - x_0)_{\widetilde{q,h}}^{-n} := \frac{1}{(x - q^n x_0 - [n]h)_{\widetilde{q,h}}^{n}}, \tag{2.6}$$

by the use of $[-n] = -q^{-n}[n]$.

(iii) Now we consider $m < 0$ and $n > 0$. Let $m = -m' < 0$

$$\begin{aligned}
(x - x_0)_{\widetilde{q,h}}^{-m'} \left(x - \frac{x_0 - [-m']h}{q^{-m'}} \right)_{\widetilde{q,h}}^{n} &= \frac{(x - q^{m'} x_0 - [m']h)_{\widetilde{q,h}}^{n}}{(x - q^{m'} x_0 - [m']h)_{\widetilde{q,h}}^{m'}} \\
&= \begin{cases} \prod_{i=0}^{n-m'-1} (x - \dfrac{q^{m'} x_0 + [m']h - [m'+i]h}{q^{m'+i}}), & n \geq m' > 0; \\ \dfrac{1}{\prod_{i=0}^{m'-n-1} (x - \frac{q^{m'} x_0 + [m']h - [n+i]h}{q^{n+i}})}, & m' > n > 0 \end{cases} \\
&= \begin{cases} \prod_{i=0}^{n-m'-1} (x - \dfrac{q^{m'} x_0 - q^{m'}[i]h}{q^{m'+i}}), & n \geq m' > 0; \\ \dfrac{1}{\prod_{i=0}^{m'-n-1} (x - \frac{q^{m'-n} x_0 + [-n+m']h - [i]h}{q^{1}})}, & m' > n > 0 \end{cases} \\
&= \frac{1}{(x - q^{m'-n} x_0 - [-n+m']h)_{\widetilde{q,h}}^{-n+m'}} = (x - x_0)_{\widetilde{q,h}}^{-m'+n} = (x - x_0)_{\widetilde{q,h}}^{m+n}.
\end{aligned}$$

The cases $m > 0$, $n < 0$ and $m < 0$, $n < 0$ can be proved by a similar fashion. □

Proposition 2.7 *The generalized nabla quantum binomial (2.4) satisfies the Leibniz rules for negative powers.*

(i) $\widetilde{D}_{(q,h)}(x - x_0)_{\widetilde{q,h}}^{-n} = [-n]_{\frac{1}{q}} (x - x_0)_{\widetilde{q,h}}^{-n-1}, \quad n \in \mathbb{N}$,

(ii) $\widetilde{D}_{(q,h)}^{k}(x - x_0)_{\widetilde{q,h}}^{-n} = (-1)^k q^{\frac{k(2n+k-1)}{2}} \dfrac{[n+k-1]_{\frac{1}{q}}!}{[n-1]_{\frac{1}{q}}!} (x - x_0)_{\widetilde{q,h}}^{-n-k}, \quad n, k \in \mathbb{N}$.

Proof (i) By the use of the equation (2.6), Propositions 2.2 and 2.4, we derive

$$\begin{aligned}\widetilde{D}_{(q,h)}(x-x_0)^{-n}_{\widetilde{q,h}} &= \widetilde{D}_{(q,h)}\left(\frac{1}{(x-q^nx_0-[n]h)^{n}_{\widetilde{q,h}}}\right)\\ &= \frac{-[n]_{\frac{1}{q}}(x-q^nx_0-[n]h)^{n-1}_{\widetilde{q,h}}}{(x-q^nx_0-[n]h)^{n}_{\widetilde{q,h}}\cdot(\frac{x-h}{q}-q^nx_0-[n]h)^{n}_{\widetilde{q,h}}}\\ &= \frac{-[n]_{\frac{1}{q}}}{\left(x-\frac{q^nx_0+[n]h-[n-1]h}{q^{n-1}}\right)\cdot\left(\frac{x-q^{n+1}x_0-[n+1]h}{q}\right)^{n}_{\widetilde{q,h}}}\\ &= \frac{-q^n[n]_{\frac{1}{q}}}{(x-qx_0-h)(x-q^{n+1}x_0-[n+1]h)^{n}_{\widetilde{q,h}}}\\ &= \frac{[-n]_{\frac{1}{q}}}{(x-q^{n+1}x_0-[n+1]h)^{n+1}_{\widetilde{q,h}}} = [-n]_{\frac{1}{q}}(x-x_0)^{-n-1}_{\widetilde{q,h}},\end{aligned}$$

where we used the relation $[-n]_{\frac{1}{q}} = -q^n[n]_{\frac{1}{q}}$. For the proof of (ii), one can apply $\widetilde{D}_{(q,h)}$, k times succesively on $(x-x_0)^{-n}_{\widetilde{q,h}}$. □

We intend to state and prove nabla (q,h)-analogue of Gauss's binomial formula which is determined by the nabla generalized quantum binomial (2.4).

Theorem 2.8 *We present the nabla (q,h)-Gauss's binomial formula as*

$$(x-x_0)^{n}_{\widetilde{q,h}} = \sum_{i=0}^{n}\left[\begin{array}{c} n\\ i\end{array}\right]_{\frac{1}{q}}(0-x_0)^{n-i}_{\widetilde{q,h}}\cdot(x-0)^{i}_{\widetilde{q,h}} = \sum_{i=0}^{n}\left[\begin{array}{c} n\\ i\end{array}\right]_{\frac{1}{q}}(0-x_0)^{i}_{\widetilde{q,h}}\cdot(x-0)^{n-i}_{\widetilde{q,h}}. \tag{2.7}$$

Proof In order to prove (2.7), we utilize Theorem 2.5 with the function $f(x)=(x-x_0)^{n}_{\widetilde{q,h}}$ about $x_0=0$. By Proposition 2.4, we have

$$\widetilde{D}^i_{(q,h)}f(x) = \widetilde{D}^i_{(q,h)}(x-x_0)^{n}_{\widetilde{q,h}} = \frac{[n]_{\frac{1}{q}}!}{[n-i]_{\frac{1}{q}}!}(x-x_0)^{n-i}_{\widetilde{q,h}}$$

which leads to

$$\widetilde{D}^i_{(q,h)}f(0) = \frac{[n]_{\frac{1}{q}}!}{[n-i]_{\frac{1}{q}}!}(0-x_0)^{n-i}_{\widetilde{q,h}}, \quad 0\le i\le n.$$

Armed with Theorem 2.5, we have

$$(x-x_0)^{n}_{\widetilde{q,h}} = \sum_{i=0}^{n}\frac{\widetilde{D}^i_{(q,h)}f(0)\ (x-0)^{i}_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!} = \sum_{i=0}^{n}\frac{[n]_{\frac{1}{q}}!}{[n-i]_{\frac{1}{q}}![i]_{\frac{1}{q}}!}(0-x_0)^{n-i}_{\widetilde{q,h}}\ (x-0)^{i}_{\widetilde{q,h}}. \tag{2.8}$$

We introduce the *nabla q-binomial coefficient* (or $\frac{1}{q}$*-binomial coefficient*) as

$$\left[\begin{array}{c} n\\ i\end{array}\right]_{\frac{1}{q}} := \frac{[n]_{\frac{1}{q}}!}{[n-i]_{\frac{1}{q}}![i]_{\frac{1}{q}}!} = q^{i(i-n)}\left[\begin{array}{c} n\\ i\end{array}\right]. \tag{2.9}$$

Using (2.9) on (2.8), the nabla (q,h)-Gauss's binomial formula results. Notice that,

$$\left[\begin{array}{c} n \\ n-i \end{array}\right]_{\frac{1}{q}} = q^{(n-i)(n-i-n)} \left[\begin{array}{c} n \\ n-i \end{array}\right] = q^{i(i-n)} \left[\begin{array}{c} n \\ i \end{array}\right] = \left[\begin{array}{c} n \\ i \end{array}\right]_{\frac{1}{q}}. \tag{2.10}$$

In order to obtain the second form in (2.7), one can rewrite (2.8) by the frame of (2.9) and (2.10)

$$\begin{aligned}(x-x_0)^{n}_{\widetilde{q,h}} &= \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array}\right]_{\frac{1}{q}} (0-x_0)^{n-i}_{\widetilde{q,h}} \, (x-0)^{i}_{\widetilde{q,h}} = \sum_{i=0}^{n} \left[\begin{array}{c} n \\ n-i \end{array}\right]_{\frac{1}{q}} (0-x_0)^{n-i}_{\widetilde{q,h}} \, (x-0)^{i}_{\widetilde{q,h}} \\ &= \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array}\right]_{\frac{1}{q}} (0-x_0)^{j}_{\widetilde{q,h}} \, (x-0)^{n-j}_{\widetilde{q,h}},\end{aligned}$$

what follows from the setting $j = n-i$. □

Let us investigate the reductions. When $\mathbb{T} = \mathbb{R}$, the nabla (q,h)-Gauss's binomial formula (2.7) reduces to the classical Newton's binomial formula

$$(x-x_0)^{n}_{\widetilde{1,0}} = (x-x_0)^n = \sum_{i=0}^{n} \binom{n}{i} (-x_0)^i x^{n-i},$$

since $(x-0)^{n-i}_{\widetilde{1,0}} = x^{n-i}$ and $(0-x_0)^{i}_{\widetilde{1,0}} = (-1)^i x_0^i$. When $\mathbb{T} = \mathbb{K}_q$, we have $(x-0)^{n-i}_{\widetilde{q,0}} = x^{n-i}$ and $(0-x_0)^{i}_{\widetilde{q,0}} = \frac{(-1)^i x_0^i}{q^{\frac{i(i-1)}{2}}}$. Then the nabla (q,h)-Gauss's binomial formula (2.7) reduces to

$$(x-x_0)^{n}_{\widetilde{q,0}} = \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array}\right]_{\frac{1}{q}} q^{\frac{i(1-i)}{2}} (-x_0)^i x^{n-i},$$

which can be regarded as nabla Gauss's binomial formula.

Furthermore, nabla generalized quantum binomial (2.4) produces the delta q-binomial

$$(x-a)^{n}_{q,0} = (x-a)(x-q^2 a)\dots(x-q^{n-1}a)$$

which is generalized in [14]. Indeed,

$$\begin{aligned}(x-x_0)^{n}_{\widetilde{q,0}} &= (x-x_0)(x-\frac{x_0}{q})(x-\frac{x_0}{q^2})\dots(x-\frac{x_0}{q^{n-1}}) \\ &= \frac{(x-x_0)(qx-x_0)(q^2x-x_0)\dots(q^{n-1}x-x_0)}{q^{\frac{n(n-1)}{2}}} = \frac{(-1)^n (x_0-x)^{n}_{q,0}}{q^{\frac{n(n-1)}{2}}},\end{aligned}$$

from which the nabla (q,h)-Gauss's binomial formula (2.7) becomes

$$\begin{aligned}(x-x_0)^{n}_{\widetilde{q,0}} = \frac{(-1)^n (x_0-x)^{n}_{q,0}}{q^{\frac{n(n-1)}{2}}} &= \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array}\right]_{\frac{1}{q}} q^{-\frac{(n-i)(n-i-1)}{2}} (-1)^{n-i} x_0^{n-i} x^i \\ &= \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array}\right] q^{i(i-n)} q^{-\frac{(n-i)(n-i-1)}{2}} (-1)^{n-i} x_0^{n-i} x^i,\end{aligned}$$

which implies the classical Gauss's binomial formula [11]

$$(x_0 - x)^n_{q,0} = \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array} \right] q^{\frac{i(i-1)}{2}} (-1)^i x^i x_0^{n-i}.$$

Definition 2.9 *We define the nabla (q,h)-Taylor series of f about x_0 as*

$$\sum_{n=0}^{\infty} \frac{\widetilde{D}^n_{(q,h)} f(x_0)(x - x_0)^n_{\widetilde{q,h}}}{[n]_{\frac{1}{q}}!}. \tag{2.11}$$

The convergence of nabla (q,h)-Taylor series (2.11) is proven by the following theorem.

Theorem 2.10 *If $q > 1$, $x, x_0 \in \mathbb{T}_{(q,h)}$ and $|\widetilde{D}^n_{(q,h)} f(x_0)| < M^n$ for $0 < M < \frac{q}{h}$, then the Taylor series (2.11) is absolutely and uniformly convergent on $\{x :| x |< \dfrac{q - Mh}{M(q-1)}\} \cap \mathbb{T}_{(q,h)}$.*

Proof Let $|\widetilde{D}^n_{(q,h)} f(x_0)| < M^n$ for some $0 < M < \frac{q}{h}$. Then we have

$$\left| \frac{\widetilde{D}^n_{(q,h)} f(x_0)(x - x_0)^n_{\widetilde{q,h}}}{[n]_{\frac{1}{q}}!} \right| \leq \frac{M^n}{[n]_{\frac{1}{q}}!} \left| (x - x_0)^n_{\widetilde{q,h}} \right|.$$

Applying the ratio test for the bounding series, we deduce

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{M^{n+1}(x - x_0)^{n+1}_{\widetilde{q,h}}}{[n+1]_{\frac{1}{q}}!} \cdot \frac{[n]_{\frac{1}{q}}!}{M^n (x - x_0)^n_{\widetilde{q,h}}} \right| &= M \lim_{n\to\infty} \frac{\left| (x - \frac{x_0 - [n]h}{q^n}) \right|}{[n+1]_{\frac{1}{q}}} \\
&= M \lim_{n\to\infty} \frac{\left| (x - \frac{x_0 - [n]h}{q^n}) q^n \right|}{[n+1]} \leq M \lim_{n\to\infty} \frac{q^n |x|}{[n+1]} + M \lim_{n\to\infty} \frac{x_0}{[n+1]} + \lim_{n\to\infty} \frac{[n]h}{[n+1]} \\
&= M\ |x|\ \frac{q-1}{q} + M\ \frac{h}{q} < 1
\end{aligned}$$

provided that $|x| < \dfrac{q - Mh}{M(q-1)}$ i.e. by Weierstrass M-test, the series (2.11) is absolutely and uniformly convergent on $\{x :| x |< \dfrac{q - Mh}{M(q-1)}\} \cap \mathbb{T}_{(q,h)}$. □

Definition 2.11 *A function $f : \mathbb{T}_{(q,h)} \to \mathbb{R}$ is said to be nabla (q,h)-analytic at x_0 if and only if there exists a power series centered at x_0 that converges to f in the neighborhood of x_0.*

To sum up, Theorem 2.10 provides sufficient conditions for a function f to be nabla (q,h)-analytic.

3. Nabla (q,h)-Bessel equation and nabla (q,h)-Bessel function

We propose the following second order delay nabla (q,h)-difference equation

$$(x-x_0)^2_{\widetilde{q,h}}\widetilde{D}^2_{(q,h)}u(q^2x+[2]h)+(x-x_0)^1_{\widetilde{q,h}}\widetilde{D}_{(q,h)}u(qx+h)+(x-x_0)^2_{\widetilde{q,h}}u(q^2x+[2]h)-\nu^2u(x)=0, \quad (3.1)$$

where ν is a real parameter. It is clear that when $x_0=0$ and as $(q,h)\to(1,0)$, Equation (3.1) reduces to the classical Bessel equation

$$x^2u^{''}(x)+xu^{'}(x)+(x^2-\nu^2)u(x)=0.$$

Therefore, Equation (3.1) can be regarded as nabla (q,h)-analogue of Bessel equation of order ν. As $q\to1$, Equation (3.1) reduces to

$$(x-x_0)(x-x_0+h)\nabla_h^2u(x+2h)+(x-x_0)\nabla_hu(x+h)+(x-x_0)(x-x_0+h)u(x+2h)-\nu^2u(x)=0, \quad (3.2)$$

whose Δ_h-version is given in [4]. If additionally $h=1$ and $x_0=0$, then (3.2) turns out to be

$$x(x+1)\nabla^2u(x+2)+x\nabla u(x+1)+x(x+1)u(x+2)-\nu^2u(x)=0,$$

whose Δ-analogue is presented in [2]. Furthermore, as $h\to0$, Equation (3.1) produces

$$(x-x_0)(x-\frac{x_0}{q})\ \nabla_q^2\ u(q^2x)+(x-x_0)\ \nabla_q\ u(qx)+(x-x_0)(x-\frac{x_0}{q})\ u(q^2x)-\nu^2u(x)=0,$$

and if additionally $x_0=0$, we encounter

$$x^2\nabla_q^2\ u(q^2x)+x\nabla_qu(qx)+x^2u(q^2x)-\nu^2u(x)=0. \quad (3.3)$$

In the literature q-Bessel function dates back to Jackson [10] and has been studied by many researchers [6, 8, 12]. To the best of our knowledge, q-discrete Bessel equation has not been stated elsewhere. We stress that the nabla (q,h)- Bessel equation (3.1) allows us to present a *nabla q-Bessel equation* (3.3). Furthermore, the nabla (q,h)-Bessel equation (3.1) which is proposed in the form of a delay nabla (q,h)-difference equation can also be written in the standard form.

Proposition 3.1 *Nabla (q,h)-Bessel equation (3.1) is equivalent to*

$$f_1(x;x_0,q,h,\nu)\widetilde{D}^2_{(q,h)}u(x)+f_2(x;x_0,q,h,\nu)\widetilde{D}_{(q,h)}u(x)+f_3(x;x_0,q,h,\nu)u(x)=0, \quad (3.4)$$

where

$$f_1=\left(\frac{x-[2]h}{q^2}-x_0\right)^2_{\widetilde{q,h}}-\frac{(x-[2]h-q^2x_0)((q-1)x+h)}{q^4}-\frac{\nu^2((q-1)x+h)^2}{q^3},$$

$$f_2=\frac{x-[2]h-q^2x_0}{q^3}+\frac{\nu^2((q-1)x+h)(1+q)}{q^2},$$

$$f_3=\left(\frac{x-[2]h}{q^2}-x_0\right)^2_{\widetilde{q,h}}-\nu^2.$$

Proof We replace x by $\dfrac{x-[2]h}{q^2}$ in (3.1) and obtain

$$\left(\frac{x-[2]h}{q^2}-x_0\right)^2_{\widetilde{q,h}}\widetilde{D}^2_{(q,h)}u(x)+\left(\frac{x-[2]h}{q^2}-x_0\right)^1_{\widetilde{q,h}}\widetilde{D}_{(q,h)}u\left(\frac{x-h}{q}\right)$$
$$+\left(\frac{x-[2]h}{q^2}-x_0\right)^2_{\widetilde{q,h}}u(x)-\nu^2u\left(\frac{x-[2]h}{q^2}\right)=0. \tag{3.5}$$

Using the definition of nabla (q,h)-derivative we encounter the following relations

$$\widetilde{D}_{(q,h)}u\left(\frac{x-h}{q}\right)=\frac{1}{q}\widetilde{D}_{(q,h)}u(x)-\frac{(q-1)x+h}{q^2}\widetilde{D}^2_{(q,h)}u(x),$$
$$u\left(\frac{x-[2]h}{q^2}\right)=\frac{((q-1)x+h)^2}{q^3}\widetilde{D}^2_{(q,h)}u(x)-\frac{((q-1)x+h)(1+q)}{q^2}\widetilde{D}_{(q,h)}u(x)+u(x).$$

Plugging the above relations in (3.5), we end up with (3.4). □

In this section, we focus on (q,h)-Bessel equation (3.1) and intend to present its solution. We also observe that the reductions of the solution recover many Bessel functions studied in the literature (see Remark 3.6). In order to present the source of the nabla (q,h)-Bessel function, we start with a formal power series of the form

$$u(x)=\sum_{k=0}^{\infty}a_k(x-x_0)^{k+m}_{\widetilde{q,h}},\quad m\in\mathbb{Z},$$

so that we can operate on this series formally. First, we calculate the terms of (3.1) as follows:

$$u(qx+h)=\sum_{k=0}^{\infty}a_k(qx+h-x_0)^{k+m}_{\widetilde{q,h}}=\sum_{k=0}^{\infty}a_kq^{k+m}\left(x-\frac{x_0-h}{q}\right)^{k+m}_{\widetilde{q,h}},$$

which implies that

$$\begin{aligned}(x-x_0)^1_{\widetilde{q,h}}\widetilde{D}_{(q,h)}u(qx+h)&=(x-x_0)^1_{\widetilde{q,h}}\widetilde{D}_{(q,h)}\sum_{k=0}^{\infty}a_kq^{k+m}\left(x-\frac{x_0-h}{q}\right)^{k+m}_{\widetilde{q,h}}\\&=(x-x_0)^1_{\widetilde{q,h}}\sum_{k=0}^{\infty}a_kq^{k+m}[k+m]_{\frac{1}{q}}\left(x-\frac{x_0-h}{q}\right)^{k+m-1}_{\widetilde{q,h}}\\&=\sum_{k=0}^{\infty}qa_k[k+m](x-x_0)^1_{\widetilde{q,h}}\left(x-\frac{x_0-h}{q}\right)^{k+m-1}_{\widetilde{q,h}}\\&=\sum_{k=0}^{\infty}qa_k[k+m](x-x_0)^{k+m}_{\widetilde{q,h}},\end{aligned} \tag{3.6}$$

where we used Proposition 2.4, Proposition 2.6 and the relation $[n]_{\frac{1}{q}}=\frac{[n]}{q^{n-1}}$ which holds for every $n\in\mathbb{Z}$. By a similar fashion, one can calculate other terms as

$$(x-x_0)^2_{\widetilde{q,h}}\widetilde{D}^2_{(q,h)}u(q^2x+[2]h)=\sum_{k=0}^{\infty}q^3a_k[k+m][k+m-1](x-x_0)^{k+m}_{\widetilde{q,h}}, \tag{3.7}$$

$$(x-x_0)^{2}_{\widetilde{q,h}}u(q^2x+[2]h)=\sum_{k=0}^{\infty}a_kq^{2k+2m}(x-x_0)^{2}_{\widetilde{q,h}}\left(x-\frac{x_0-[2]h}{q^2}\right)^{k+m}_{\widetilde{q,h}}$$

$$=\sum_{k=0}^{\infty}a_kq^{2k+2m}(x-x_0)^{k+m+2}_{\widetilde{q,h}}=\sum_{k=2}^{\infty}a_{k-2}q^{2k+2m-4}(x-x_0)^{k+m}_{\widetilde{q,h}}. \tag{3.8}$$

Therefore, plugging (3.6), (3.7) and (3.8) in Equation (3.1) we derive

$$\sum_{k=2}^{\infty}\left(q^3a_k[k+m][k+m-1]+qa_k[k+m]+q^{2m+2k-4}a_{k-2}-\nu^2a_k\right)(x-x_0)^{k+m}_{\widetilde{q,h}}$$
$$+a_0\left(q^3[m][m-1]+q[m]-\nu^2\right)(x-x_0)^{m}_{\widetilde{q,h}}$$
$$+a_1\left(q^3[m+1][m]+q[m+1]-\nu^2\right)(x-x_0)^{m+1}_{\widetilde{q,h}}=0. \tag{3.9}$$

As a consequence of Equation (3.9), we have

$$a_0\left(q^3[m][m-1]+q[m]-\nu^2\right)=0,\quad a_1\left(q^3[m+1][m]+q[m+1]-\nu^2\right)=0$$

and

$$a_k=\frac{-q^{2m+2k-4}a_{k-2}}{q^3[k+m][k+m-1]+q[k+m]-\nu^2}. \tag{3.10}$$

Assume $a_1=0$, by (3.10) all $a_{2k+1}=0$ for $k\in\mathbb{N}$. Assume also $a_0\neq 0$ which implies that

$$\nu^2=q^3[m][m-1]+q[m]. \tag{3.11}$$

After plugging (3.11) in (3.10) with simplifications, we derive

$$a_k=\frac{-q^{m+2k-5}a_{k-2}}{[k]([m+1]+q^2[m+k-1])}$$

which produces the general term of the series as

$$a_{2k}=\frac{(-1)^kq^{k(m+2k-3)}a_0}{[2]^k[k]_{q^2}!\{q\}_{m,k}},$$

where

$$[k]_{q^2}!:=\begin{cases}1 & \text{if } \ k=0,\\ (1+q^2)(1+q^2+q^4)\cdots(1+q^2+q^4+\ldots+q^{2k-2}) & \text{if } \ k\geq 1,\end{cases}$$

and

$$\{q\}_{m,k}:=\begin{cases}1 & \text{if } \ k=0,\\ \displaystyle\prod_{i=1}^{k}([m+1]+q^2[m+2i-1]) & \text{if } \ k\geq 1.\end{cases}$$

Therefore, we construct one solution for the nabla (q,h)-Bessel equation (3.1) as

$$u(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\ q^{k(m+2k-3)}\ a_0}{[2]^k\ [k]_{q^2}!\ \{q\}_{m,k}}\ (x-x_0)^{2k+m}_{\widetilde{q,h}}. \tag{3.12}$$

Theorem 3.2 *The series* (3.12) *solves the nabla* (q,h)*-Bessel equation* (3.1).

Proof We compute the terms as follows:

$$\nu^2 u(x) = a_0(q^3[m][m-1]+q[m])(x-x_0)^{\underline{m}}_{\widetilde{q,h}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} a_0(q^3[m][m-1]+q[m])}{[2]^k [k]_{q^2}!\{q\}_{m,k}} (x-x_0)^{\underline{2k+m}}_{\widetilde{q,h}}, \tag{3.13}$$

$$(x-x_0)^{\underline{1}}_{\widetilde{q,h}} \widetilde{D}_{(q,h)} u(qx+h) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} q^{2k+m} a_0 [2k+m]_{\frac{1}{q}}}{[2]^k [k]_{q^2}!\{q\}_{m,k}} (x-x_0)^{\underline{2k+m}}_{\widetilde{q,h}}$$
$$= a_0 q[m](x-x_0)^{\underline{m}}_{\widetilde{q,h}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} a_0(q[m]+q^{m+1}[2k])}{[2]^k [k]_{q^2}!\{q\}_{m,k}} (x-x_0)^{\underline{2k+m}}_{\widetilde{q,h}} \tag{3.14}$$

and

$$(x-x_0)^{\underline{2}}_{\widetilde{q,h}} \widetilde{D}^2_{(q,h)} u(q^2x+[2]h) = a_0 q^3[m][m-1](x-x_0)^{\underline{m}}_{\widetilde{q,h}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} a_0(q^3[m]^2 + q^{m+3}[m]([2k]+[2k-1]) + q^{2m+3}[2k][2k-1])}{[2]^k [k]_{q^2}!\{q\}_{m,k}} (x-x_0)^{\underline{2k+m}}_{\widetilde{q,h}}, \tag{3.15}$$

where we used the relation $[2k+m]=[m]+q^m[2k]$ which holds for every $m \in \mathbb{Z}$, $k \in \mathbb{N}$. Similarly we compute

$$(x-x_0)^{\underline{2}}_{\widetilde{q,h}} u(q^2x+[2]h) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} q^{4k+2m} a_0}{[2]^k [k]_{q^2}!\{q\}_{m,k}} (x-x_0)^{\underline{2k+m+2}}_{\widetilde{q,h}}$$
$$= -\sum_{k=1}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} q^{m+1} a_0}{[2]^{k-1} [k-1]_{q^2}!\{q\}_{m,k-1}} (x-x_0)^{\underline{2k+m}}_{\widetilde{q,h}}. \tag{3.16}$$

Since $\{q\}_{m,k} = ([m+1]+q^2[m+2k-1])\ \{q\}_{m,k-1}$ and $[k]_{q^2}! = [k]_{q^2}\ [k-1]_{q^2}!$, we obtain a relation

$$[2]^k\ [k]_{q^2}!\ \{q\}_{m,k} = [2k]\ ([m+1]+q^2[m+2k-1])\ [2]^{k-1}\ [k-1]_{q^2}\ !\{q\}_{m,k-1}. \tag{3.17}$$

Substituting (3.17) in (3.16), we find

$$(x-x_0)^{\underline{2}}_{\widetilde{q,h}} u(q^2x+[2]h) = -\sum_{k=1}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} a_0 q^{m+1}[2k]([m+1]+q^2[m]+q^{m+2}[2k-1])}{[2]^k [k]_{q^2}!\{q\}_{m,k}} (x-x_0)^{\underline{2k+m}}_{\widetilde{q,h}}. \tag{3.18}$$

The proof follows by plugging the expressions (3.13), (3.14), (3.15) and (3.18) into the left hand side of (3.1).
□

Theorem 3.3 *The series* (3.12) *is absolutely convergent if* $x < \frac{q^3-h}{q-1}$.

Proof Let $u_k(x) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} a_0}{[2]^k [k]_{q^2}! \{q\}_{m,k}} (x - x_0)^{2k+m}_{\widetilde{q,h}}$ and apply the ratio test:

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} q^{(k+1)(m+2k-1)} (x - x_0)^{2k+m+2}_{\widetilde{q,h}}}{[2][4]\cdots[2k][2k+2]\{q\}_{m,k+1}} \cdot \frac{[2][4]\cdots[2k]\{q\}_{m,k}}{(-1)^k q^{k(m+2k-3)} (x - x_0)^{2k+m}_{\widetilde{q,h}}} \right|$$

$$= \lim_{k\to\infty} \frac{q^{4k+m-1} \left| \left(x - \frac{x_0 - [2k+m]h}{q^{2k+m}} \right)^2_{\widetilde{q,h}} \right|}{[2k+2]([m+1] + q^2[m+2k+1])}$$

$$= q^{m-1} \lim_{k\to\infty} \frac{q^{4k} \left| \left(x - \frac{x_0 - [2k+m]h}{q^{2k+m}} \right) \left(x - \frac{x_0 - [2k+m+1]h}{q^{2k+m+1}} \right) \right|}{[2k+2]([m+1] + q^2[m+2k+1])}$$

$$= \frac{1}{q^{m+2}} \lim_{k\to\infty} \frac{|q^{2k+m}x - x_0 + [2k+m]h|}{[m+1] + q^2[m+2k+1]} \cdot \lim_{k\to\infty} \frac{|q^{2k+m+1}x - x_0 + [2k+m+1]h|}{[2k+2]}$$

$$\leq \frac{1}{q^{m+2}} \left(\lim_{k\to\infty} \frac{q^{2k+m}|x|}{[m+1] + q^2[m+2k+1]} + \frac{|x_0|}{[m+1] + q^2[m+2k+1]} + \frac{[2k+m]h}{[m+1] + q^2[m+2k+1]} \right)$$
$$\cdot \left(\lim_{k\to\infty} \frac{q^{2k+m+1}|x|}{[2k+2]} + \frac{|x_0|}{[2k+2]} + \frac{[2k+m+1]h}{[2k+2]} \right)$$

$$= \frac{1}{q^{m+2}} \left(\frac{q-1}{q^3}|x| + \frac{h}{q^3} \right) q^{m-1}((q-1)|x| + h) = \left(\frac{(q-1)|x| + h}{q^3} \right)^2.$$

Hence, the series (3.12) converges absolutely provided that $|x| < \frac{q^3 - h}{q-1}$. □

We are ready to introduce the nabla (q,h)-Bessel function as follows.

Definition 3.4 *We define the nabla (q,h)-Bessel function of order ν, by the series*

$$J_m(x; x_0, h, q) := \sum_{k=0}^{\infty} \frac{(-1)^k \; q^{k(m+2k-3)} \; a_0}{[2]^k \; [k]_{q^2}! \; \{q\}_{m,k}} \; (x - x_0)^{2k+m}_{\widetilde{q,h}}, \tag{3.19}$$

provided that $\nu^2 = q^3[m][m-1] + q[m]$.

Remark 3.5 *Let us analyze the order ν. It is clear that*

$$\lim_{q\to 1} \nu^2 = \lim_{q\to 1} (q^3[m][m-1] + q[m]) = m^2$$

which recovers the classical and discrete case, i.e. $\nu = \pm m$. But in general, for instance when $q \neq 1$, solving the exponential equation

$$\nu^2 = q^3[m][m-1] + q[m] = q^3 \left(\frac{q^m - 1}{q-1} \right) \left(\frac{q^{m-1} - 1}{q-1} \right) + q \left(\frac{q^m - 1}{q-1} \right), \tag{3.20}$$

for m, we compute two roots (by MATLAB)

$$m_1 = \frac{\log \left(\frac{q\sqrt{q^2 - 2q + 4\nu^2 + 1} - \sqrt{q^2 - 2q + 4\nu^2 + 1} + q^2 + 1}{2q} \right)}{\log q}, \tag{3.21}$$

$$m_2 = \frac{\log\left(\frac{\sqrt{q^2-2q+4\nu^2+1}-q\sqrt{q^2-2q+4\nu^2+1}+q^2+1}{2q}\right)}{\log q}. \tag{3.22}$$

Conversely the roots (3.21) and (3.22) imply (3.20). Therefore, we obtain the following table for the different choices of ν^2 (or m_1 and m_2):

$\boldsymbol{\nu^2}$	*0*	q	$q[2](1+q^2)$	$\frac{[2](1+q^2)}{q^2}$	$q[3](1+q^2+q^3)$	$\frac{[3](1+q+q^3)}{q^4}$
$\mathbf{m_1}$	*0 (for $q<1$)*	1	$\frac{\log(\frac{-q^3+q^2+1}{q})}{\log q}$	-2	$\frac{\log(\frac{-q^4+q^2+1}{q})}{\log q}$	-3
$\mathbf{m_2}$	0 *(for $q>1$)*	-1	2	$\frac{\log(\frac{q^3+q-1}{q^2})}{\log q}$	3	$\frac{\log(\frac{q^4+q^2-1}{q^3})}{\log q}$

Analyzing the above table, we conclude that, in general for the roots of ν^2 , $m_1 \neq -m_2$, i.e. the roots having the relation $m_1 = -m_2$ do not belong to the same specific value of ν^2. For that reason, the roots of ν^2 do not have a symmetric property which violates to present additional properties of J_m (3.19) unlike the classical or discrete case.

Remark 3.6 *We investigate the reductions of the nabla (q,h)-Bessel function (3.19).*

(i) If we set $x_0 = 0$, $a_0 = \dfrac{1}{2^m\Gamma(m+1)}$ and consider the limit as $(q,h) \to (1,0)$, the terms in the denominator become

$$\lim_{(q,h)\to(1,0)} [2]^k = 2^k,$$

$$\lim_{(q,h)\to(1,0)} [k]_{q^2}! = k!,$$

$$\lim_{(q,h)\to(1,0)} \{q\}_{m,k} = \prod_{i=1}^{k}((m+1)+(m+2i-1)) = 2^k(m+1)(m+2)\cdots(m+k).$$

To be more precise, (3.19) reduces to the classical Bessel function

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+m}}{2^{2k+m}k!(m+1)\cdots(m+k)\Gamma(m+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+m}}{2^{2k+m}k!\Gamma(m+k+1)}. \tag{3.23}$$

As illustration: when $m=0$, the order $\nu = 0$ and the nabla (q,h)-Bessel function of order 0, can be presented as

$$J_0(x;x_0,h,q) := \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(2k-3)} a_0}{[2]^k [k]_{q^2}! \{q\}_{0,k}} (x-x_0)_{\widetilde{q,h}}^{2k},$$

which reduces to the classical Bessel function of order 0, under the settings $a_0 = 1$, $x_0 = 0$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}k!k!}.$$

(ii) If we set $a_0 = \dfrac{1}{2^m\Gamma(m+1)}$ *and consider the limit as* $q \to 1$, (3.19) *reduces to* h*-discrete Bessel function*

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-x_0)_{\widetilde{h}}^{2k+m}}{2^{2k+m} k!\Gamma(m+k+1)}.$$

whose Δ_h*-version is given in [4]. Moreover, if* $x_0 = 0$ *and* $h = 1$, (3.19) *produces discrete Bessel function*

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-0)_{\widetilde{1}}^{2k+m}}{2^{2k+m} k!(m+k)!}.$$

whose Δ*-version is presented in [2].*

(iii) As $h \to 0$, *the nabla* (q,h)*-Bessel function* (3.19) *leads to*

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(m+2k-3)} a_0}{[2]^k [k]_{q^2}! \{q\}_{m,k}} (x-x_0)_{\widetilde{q,0}}^{2k+m},$$

where $(x-x_0)_{\widetilde{q,0}}^{2k+m} = (x-x_0)(x-\frac{x_0}{q})\cdots(x-\frac{x_0}{q^{2k+m-1}})$ *which implies* $(x-0)_{\widetilde{q,0}}^{2k+m} = x^{2k+m}$ *for* $x_0 = 0$. *In this context, setting* $a_0 = \dfrac{1}{[2]^m[m]!}$, *we offer a new* q*-Bessel function*

$$J_m(x) = \left(\frac{x}{[2]}\right)^m \frac{1}{[m]!} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(m+2k-3)}}{[k]_{q^2}! \{q\}_{m,k}} \left(\frac{x^2}{[2]}\right)^k, \tag{3.24}$$

which reduces directly to the classical Bessel function (3.23) *in the limit* $q \to 1$ *while the celebrated Jackson's* q*-Bessel function [10]*

$$J_m^{(1)}(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(\frac{-x^2}{4})^k}{(q,q)_{m+k}(q,q)_k} = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{(1-q)^{2k+m}[k]![k+m]!} \left(\frac{x^2}{4}\right)^k$$

and Hahn's q*-Bessel function [6]*

$$J_m^{(2)}(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{q^{k(k+m)}(\frac{-x^2}{4})^k}{(q,q)_{m+k}(q,q)_k} = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+m)}}{(1-q)^{2k+m}[k]![k+m]!} \left(\frac{x^2}{4}\right)^k$$

recover (3.23) *under the transformation* $x \to (1-q)x$ *and as* $q \to 1$.

We finish this section by presenting nabla (q,h)-analogue of modified Bessel equation

$$(x-x_0)_{\widetilde{q,h}}^2 \widetilde{D}_{(q,h)}^2 u(q^2x+[2]h) + (x-x_0)_{\widetilde{q,h}}^1 \widetilde{D}_{(q,h)} u(qx+h) - (x-x_0)_{\widetilde{q,h}}^2 u(q^2x+[2]h) - \nu^2 u(x) = 0, \tag{3.25}$$

where ν is a real parameter. It is straightforward that as $(q,h) \to (1,0)$ and $x_0 = 0$ Equation (3.25) reduces to the classical modified Bessel equation

$$x^2 u^{''}(x) + xu^{'}(x) - (x^2+\nu^2)u(x) = 0.$$

Similar to the nabla (q,h)-Bessel equation, one can derive a solution of the form

$$I_m(x;x_0,h,q) := \sum_{k=0}^{\infty} \frac{q^{k(m+2k-3)}a_0}{[2]^k[k]_{q^2}!\{q\}_{m,k}}(x-x_0)^{2k+m}_{\widetilde{q,h}}, \tag{3.26}$$

provided that $\nu^2 = q^3[m][m-1]+q[m]$. We may call the convergent series solution (3.26) as modified nabla (q,h)-Bessel function of order ν. Furthermore, the convergence of (3.26) is a direct consequence of Theorem 3.3.

Proposition 3.7 *The modified nabla (q,h)-Bessel function and nabla (q,h)-Bessel function are interrelated as*

$$I_m(x;x_0,h,q) = (-i)^m J_m(ix;ix_0,ih,q).$$

Proof The nabla (q,h)-Bessel function (3.19) can be rewritten

$$J_m(ix;ix_0,ih,q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(m+2k-3)}a_0}{[2]^k[k]_{q^2}!\{q\}_{m,k}}(ix-ix_0)^{2k+m}_{\widetilde{q,h}},$$

where

$$(ix-ix_0)^n_{\widetilde{q,ih}} = (ix-ix_0)(ix-\frac{ix_0-ih}{q})\cdots(ix-\frac{ix_0-i[n-1]h}{q^{n-1}}) = i^n(x-x_0)^n_{\widetilde{q,h}},$$

which yields the relation

$$\begin{aligned}(-i)^m J_m(ix;ix_0,ih,q) &= \sum_{k=0}^{\infty} \frac{(-i)^m(-1)^k\ (i)^{2k+m}\ q^{k(m+2k-3)}a_0}{[2]^k[k]_{q^2}!\{q\}_{m,k}}(x-x_0)^{2k+m}_{\widetilde{q,h}} \\ &= I_m(x;x_0,h,q).\end{aligned}$$

□

4. Nabla (q,h)-elementary functions and properties

Definition 4.1 *A nabla exponential function on $\mathbb{T}_{(q,h)}$ is introduced by the series*

$$\widetilde{Exp}_{(q,h)}(c(x-x_0)) := \sum_{n=0}^{\infty} \frac{c^n(x-x_0)^n_{\widetilde{q,h}}}{[n]_{\frac{1}{q}}!}, \qquad c\in\mathbb{R}/\{0\}. \tag{4.1}$$

Notice that, $\widetilde{Exp}_{(q,h)}(0) = 1$ and Theorem 2.10 guarantees the absolute and uniform convergence of the nabla (q,h)-exponential function (4.1) on $\{x :| x |< \frac{q-ch}{c(q-1)}\}\cap\mathbb{T}_{(q,h)}$ and for every $x_0\in\mathbb{T}_{(q,h)}$.

Proposition 4.2 *The nabla exponential function (4.1) solves the linear, homogeneous nabla (q,h)-initial value problem*

$$\begin{aligned}&\widetilde{D}_{(q,h)}u(x) = cu(x), \qquad c\in\mathbb{R}/\{0\},\\ &u(x_0) = 1.\end{aligned}$$

Proof As a direct consequence of Proposition 2.4, we compute the nabla (q,h)-derivative of (4.1)

$$\widetilde{D}_{(q,h)}\widetilde{Exp}_{(q,h)}(c(x-x_0)) = \sum_{n=0}^{\infty}\frac{c^n}{[n]_{\frac{1}{q}}!}\widetilde{D}_{(q,h)}(x-x_0)^{n}_{\widetilde{q,h}}$$

$$= \sum_{n=1}^{\infty}\frac{c^n}{[n-1]_{\frac{1}{q}}!}(x-x_0)^{n-1}_{\widetilde{q,h}} = c\widetilde{Exp}_{(q,h)}(c(x-x_0))$$

and clearly $\widetilde{Exp}_{(q,h)}(c(x_0-x_0)=1$. □

One of the main advantages of the ordinary exponential function is that, it acts as a translation operator producing the binomial expansion

$$e^{cD}x^n = (x+c)^n, \quad c\in\mathbb{R}.$$

This fact can be also generated to an action on an arbitrary analytic function $f(x)=\sum_{i=0}^{\infty}a_i x^i$ producing its translation as

$$e^{cD}f(x) = \sum_{i=0}^{\infty}a_i(x+c)^i = f(x+c), \quad c\in\mathbb{R}.$$

In this context, we introduce *the nabla* (q,h)*-translation operator* as

$$\widetilde{Exp}_{(q,h)}((0-c)\widetilde{D}_{(q,h)}) := \sum_{i=0}^{\infty}\frac{(0-c)^{i}_{\widetilde{q,h}}\widetilde{D}^{i}_{(q,h)}}{[i]_{\frac{1}{q}}!}, \quad c\in\mathbb{R},$$

which produces nabla (q,h)-binomial and translation of nabla (q,h)-analytic functions as follows.

Theorem 4.3 *(i) The nabla* (q,h)*-translation operator provides nabla generalized quantum binomial*

$$\widetilde{Exp}_{(q,h)}((0-c)\widetilde{D}_{(q,h)})(x-0)^{n}_{\widetilde{q,h}} = (x-c)^{n}_{\widetilde{q,h}}, \quad n\in\mathbb{N}_0, \quad c\in\mathbb{R}.$$

(ii) Let $f(x)$ *be a nabla* (q,h)*-analytic function as* $f(x)=\sum_{n=0}^{\infty}a_n(x-0)^{n}_{\widetilde{q,h}}$. *Then nabla* (q,h)*-exponential function provides the translation of* f

$$\widetilde{Exp}_{(q,h)}((0-c)\widetilde{D}_{(q,h)})f(x) = f(x-c), \quad c\in\mathbb{R}.$$

Proof The proofs are based on the nabla (q,h)-Gauss's binomial formula (2.7).

(i) Consider

$$\widetilde{Exp}_{(q,h)}((0-c)\widetilde{D}_{(q,h)})(x-0)^{n}_{\widetilde{q,h}} = \sum_{i=0}^{\infty}\frac{(0-c)^{i}_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!}\left(\widetilde{D}^{i}_{(q,h)}(x-0)^{n}_{\widetilde{q,h}}\right)$$

$$= \sum_{i=0}^{n}\frac{(0-c)^{i}_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!}\frac{[n]_{\frac{1}{q}}!}{[n-i]_{\frac{1}{q}}!}(x-0)^{n-i}_{\widetilde{q,h}}$$

$$= \sum_{i=0}^{n}\left[\begin{array}{c} n \\ i \end{array}\right]_{\frac{1}{q}}(0-c)^{i}_{\widetilde{q,h}}(x-0)^{n-i}_{\widetilde{q,h}} = (x-c)^{n}_{\widetilde{q,h}}.$$

(ii) As a consequence of part (i), one can derive

$$\widetilde{Exp}_{(q,h)}((0-c)\widetilde{D}_{(q,h)})f(x) = \sum_{i=0}^{\infty} \frac{(0-c)^{i}_{\widetilde{q,h}} \widetilde{D}^{i}_{(q,h)}}{[i]_{\frac{1}{q}}!} \sum_{n=0}^{\infty} a_n (x-0)^{n}_{\widetilde{q,h}}$$

$$= \sum_{i=0}^{\infty} \frac{(0-c)^{i}_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!} \sum_{n=0}^{\infty} a_n \left(\widetilde{D}^{i}_{(q,h)}(x-0)^{n}_{\widetilde{q,h}}\right) = \sum_{i=0}^{\infty} \frac{(0-c)^{i}_{\widetilde{q,h}}}{[i]_{\frac{1}{q}}!} \sum_{n=0}^{\infty} a_n \frac{[n]_{\frac{1}{q}}!}{[n-i]_{\frac{1}{q}}!}(x-0)^{n-i}_{\widetilde{q,h}}$$

$$= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} a_n \begin{bmatrix} n \\ i \end{bmatrix}_{\frac{1}{q}} (0-c)^{i}_{\widetilde{q,h}}(x-0)^{n-i}_{\widetilde{q,h}} = \sum_{n=0}^{\infty} a_n \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_{\frac{1}{q}} (0-c)^{i}_{\widetilde{q,h}}(x-0)^{n-i}_{\widetilde{q,h}}\right)$$

$$= \sum_{n=0}^{\infty} a_n (x-c)^{n}_{\widetilde{q,h}} = f(x-c).$$

□

As an example we may suggest $f(x) = \widetilde{Exp}_{(q,h)}(x-0)$ and deduce that

$$\widetilde{Exp}_{(q,h)}((0-k)\widetilde{D}_{(q,h)})\widetilde{Exp}_{(q,h)}(x-0)) = \widetilde{Exp}_{(q,h)}(x-k).$$

Our next result is the additivity of nabla (q,h)-exponential function which is a very beneficial tool not only in the field of analysis but also in the field of difference equations.

Theorem 4.4 *The nabla (q,h)-exponential function (4.1) satisfies the additive property*

$$\widetilde{Exp}_{(q,h)}(c(x-y)) = \widetilde{Exp}_{(q,h)}(c(0-y)) \cdot \widetilde{Exp}_{(q,h)}(c(x-0)), \quad c \in \mathbb{R}. \tag{4.2}$$

Proof The proof is based on the nabla (q,h)-Gauss's binomial formula (2.7). Adopting (4.1), we compute

$$\widetilde{Exp}_{(q,h)}(c(0-y)) \cdot \widetilde{Exp}_{(q,h)}(c(x-0)) = \sum_{j=0}^{\infty} \frac{c^j (0-y)^{j}_{\widetilde{q,h}}}{[j]_{\frac{1}{q}}!} \sum_{k=0}^{\infty} \frac{c^k (x-0)^{k}_{\widetilde{q,h}}}{[k]_{\frac{1}{q}}!}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{c^{j+k}(0-y)^{j}_{\widetilde{q,h}}(x-0)^{k}_{\widetilde{q,h}}}{[j]_{\frac{1}{q}}![k]_{\frac{1}{q}}!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{c^{n}(0-y)^{n-k}_{\widetilde{q,h}}(x-0)^{k}_{\widetilde{q,h}}[n]_{\frac{1}{q}}!}{[n-k]_{\frac{1}{q}}![k]_{\frac{1}{q}}![n]_{\frac{1}{q}}!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} (0-y)^{n-k}_{\widetilde{q,h}}(x-0)^{k}_{\widetilde{q,h}}\right) \frac{c^n}{[n]_{\frac{1}{q}}!} = \sum_{n=0}^{\infty} \frac{c^n (x-y)^{n}_{\widetilde{q,h}}}{[n]_{\frac{1}{q}}!} = \widetilde{Exp}_{(q,h)}(c(x-y)),$$

where we multiplied and divided by $[j+k]!$ in the second line and then changed the index as $n = j+k$. □

Remark 4.5 *The nabla (q,h)-exponential function (4.1) produces the exponential functions widely investigated in the literature. For simplicity, set $c=1$.*

(i) $\mathbb{T} = \mathbb{R}$: As $(q,h) \to (1,0)$, the nabla (q,h)-exponential function (4.1) reduces to the ordinary exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for $x_0 = 0$.

(ii) $\mathbb{T}=\mathbb{K}_q$: *When* $x_0=0$ *and as* $h\to 0$, *the generalized nabla quantum binomial turns out to be* $(x-0)^{n}_{\widetilde{q,0}}=x^n$ *from which we encounter Euler's* q*-exponential function* E_q *[11]*

$$\widetilde{Exp}_{(q,0)}(x-0)=\sum_{n=0}^{\infty}\frac{(x-0)^{n}_{\widetilde{q,0}}}{[n]_{\frac{1}{q}}!}=\sum_{n=0}^{\infty}\frac{x^n}{[n]_{\frac{1}{q}}!}=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}x^n}{[n]!}=E_q^x.$$

Similarly when $x=0, x_0=y$, *the nabla* (q,h)*-exponential function* (4.1) *reduces to Jackson's* q*-exponential function* e_q *[9]*

$$\widetilde{Exp}_{(q,0)}(0-y)=\sum_{n=0}^{\infty}\frac{(-1)^n y^n q^{\frac{-n(n-1)}{2}}}{[n]_{\frac{1}{q}}!}=\sum_{n=0}^{\infty}\frac{(-1)^n y^n}{[n]!}=e_q^{-y}.$$

(iii) $\mathbb{T}=h\mathbb{Z}$: *When* $x_0=0$ *and as* $q\to 1$, *the nabla* (q,h)*-exponential function* (4.1) *recovers* h*-exponential function*

$$\widetilde{Exp}_{(1,h)}(x-0)=\sum_{n=0}^{\infty}\frac{(x-0)^{n}_{\widetilde{1,h}}}{n!}=\sum_{n=0}^{\infty}\binom{-\frac{x}{h}}{n}(-h)^n=(1-h)^{-\frac{x}{h}},$$

where $(x-0)^{n}_{\widetilde{1,h}}=x(x+h)(x+2h)\cdots(x+(n-1)h)=\frac{-x}{h}(\frac{-x}{h}-1)\cdots(\frac{-x}{h}-(n-1))(-h)^n$. *When* $x=0, x_0=y$, *we get*

$$(0-y)^{n}_{\widetilde{1,h}}=(-1)^n y(y-h)(y-2h)\cdots(y-(n-1)h),$$

then (4.1) *turns out to be*

$$\widetilde{Exp}_{(1,h)}(0-y)=\sum_{n=0}^{\infty}\frac{(0-y)^{n}_{\widetilde{1,h}}}{n!}=\sum_{n=0}^{\infty}\binom{\frac{y}{h}}{n}(-h)^n=(1-h)^{\frac{y}{h}}.$$

Let us analyze the reductions of Theorem 4.4. Note that $(x-y)^{n}_{\widetilde{q,h}}=0$ *when* $x=\frac{y-[j]h}{q^j}$ *for* $0\le j\le n-1$. *For simplicity, set* $j=0$ *and* $y=x$, *that implies*

$$\widetilde{Exp}_{(q,h)}(c(x-x))=\widetilde{Exp}_{(q,h)}(c(0-x))\widetilde{Exp}_{(q,h)}(c(x-0))=1. \tag{4.3}$$

Then Theorem 4.4 provides the following relations

$$\widetilde{Exp}_{(q,0)}(c(0-x))\widetilde{Exp}_{(q,0)}(c(x-0))=e_q^{-cx}E_q^{cx}=1,$$

as $h\to 0$,

$$\widetilde{Exp}_{(1,h)}(c(0-x))\widetilde{Exp}_{(1,h)}(c(x-0))=(1-ch)^{\frac{x}{h}}(1-ch)^{\frac{-x}{h}}=1,$$

as $q\to 1$ *and*

$$\widetilde{Exp}_{(1,0)}(c(0-x))\widetilde{Exp}_{(1,0)}(c(x-0))=e^{-cx}e^{cx}=1,$$

as $(q,h)\to(1,0)$. *Note that, other solutions of* $(x-y)^{n}_{\widetilde{q,h}}=0$ *give similar reductions of Theorem 4.4.*

We finish this section by introducing nabla (q,h)-analogues of trigonometric and hyperbolic functions with their properties. For this purpose, we adopt the special case (4.3) which allows us to define

$$\left(\widetilde{Exp}_{(q,h)}(c(x-0))\right)^{-1} := \widetilde{Exp}_{(q,h)}(c(0-x)).$$

Definition 4.6 *We define nabla (q,h)-analogue of trigonometric and hyperbolic functions as*

$$\widetilde{\sin}_{(q,h)}(x-x_0) := \frac{\widetilde{Exp}_{(q,h)}(i(x-x_0)) - \widetilde{Exp}_{(q,h)}(i(-x+x_0))}{2i}, \tag{4.4}$$

$$\widetilde{\cos}_{(q,h)}(x-x_0) := \frac{\widetilde{Exp}_{(q,h)}(i(x-x_0)) + \widetilde{Exp}_{(q,h)}(i(-x+x_0))}{2}, \tag{4.5}$$

$$\widetilde{\sinh}_{(q,h)}(x-x_0) := \frac{\widetilde{Exp}_{(q,h)}(x-x_0) - \widetilde{Exp}_{(q,h)}(-x+x_0)}{2}, \tag{4.6}$$

$$\widetilde{\cosh}_{(q,h)}(x-x_0) := \frac{\widetilde{Exp}_{(q,h)}(x-x_0) + \widetilde{Exp}_{(q,h)}(-x+x_0)}{2}. \tag{4.7}$$

Note that the functions introduced in Definition 4.6 are well-defined because of the fact that the linear, homogenous nabla (q,h)-IVP

$$\widetilde{D}^2_{(q,h)}u(x) + au(x) = 0,$$
$$u(x_0) = b, \quad \widetilde{D}_{(q,h)}u(x_0) = c,$$

has unique solution (4.4) if $a=1, b=0, c=1$; (4.5) if $a=1, b=1, c=0$; (4.6) if $a=-1, b=0, c=1$ and (4.7) if $a=-1, b=1, c=0$, respectively. Furthermore, it is possible to present uniform and absolute convergent series representations for nabla (q,h)-trigonometric and hyperbolic functions:

$$\widetilde{\sin}_{(q,h)}(x-x_0) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-x_0)^{\widetilde{2n+1}}_{q,h}}{[2n+1]_{\frac{1}{q}}!}, \quad \widetilde{\cos}_{(q,h)}(x-x_0) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-x_0)^{\widetilde{2n}}_{q,h}}{[2n]_{\frac{1}{q}}!},$$

$$\widetilde{\sinh}_{(q,h)}(x-x_0) = \sum_{n=0}^{\infty} \frac{(x-x_0)^{\widetilde{2n+1}}_{q,h}}{[2n+1]_{\frac{1}{q}}!}, \qquad \widetilde{\cosh}_{(q,h)}(x-x_0) = \sum_{n=0}^{\infty} \frac{(x-x_0)^{\widetilde{2n}}_{q,h}}{[2n]_{\frac{1}{q}}!}.$$

Proposition 4.7 *The nabla (q,h)-trigonometric and hyperbolic functions admit the following relations.*

(i) $\widetilde{D}_{(q,h)}\widetilde{\sin}_{(q,h)}(x-x_0) = \widetilde{\cos}_{(q,h)}(x-x_0), \quad \widetilde{D}_{(q,h)}\widetilde{\cos}_{(q,h)}(x-x_0) = -\widetilde{\sin}_{(q,h)}(x-x_0).$

(ii) $\widetilde{D}_{(q,h)}\widetilde{\sinh}_{(q,h)}(x-x_0) = \widetilde{\cosh}_{(q,h)}(x-x_0), \quad \widetilde{D}_{(q,h)}\widetilde{\cosh}_{(q,h)}(x-x_0) = \widetilde{\sinh}_{(q,h)}(x-x_0).$

The proof is a direct consequence of Definition 4.6.

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