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Oscillation of second order mixed functional differential equations with sublinear and superlinear neutral terms

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Abstract: In this paper, we shall establish some new oscillation theorems for the functional differential equations with sublinear and superlinear neutral terms of the form

$$(r(t)(z'(t))^{\alpha})' = q(t)x^{\alpha}(\tau(t)),$$

where $z(t) = x(t) + p_1(t)x^{\beta}(\sigma(t)) - p_2(t)x^{\gamma}(\sigma(t))$ with $0 < \beta < 1$ and $\gamma > 1$. Moreover, $\sigma(t) \le t$ and $\tau(t)$ is a mixed type deviating argument specially. Finally, some relevant examples are indicated to illustrate the applicability of our results.

Key words: Sublinear and superlinear neutral terms, oscillation, delay, advanced

1. Introduction

The present paper is meant to investigate the oscillation for a class of second-order mixed-type differential equations with sublinear and superlinear neutral terms of the form

$$(r(t)(z'(t))^{\alpha})' = q(t)x^{\alpha}(\tau(t)),\ t \ge t_0, \tag{1.1}$$

where $z(t) = x(t) + p_1(t)x^{\beta}(\sigma(t)) - p_2(t)x^{\gamma}(\sigma(t))$ and $r,\ p_1,\ p_2,\ q,\ \sigma,\ \tau$ are continuous real-valued functions on $[t_0, \infty)$. It is worth noting that $\tau(t)$ is of mixed type. The deviating argument $\tau(t)$ is said to be of mixed type if its delay part

$$\mathcal{D}_{\tau} = \{t \in [t_0, \infty) : \tau(t) < t\}$$

and its advanced part

$$\mathcal{A}_{\tau} = \{t \in [t_0, \infty) : \tau(t) > t\}$$

are both unbounded subsets of $[t_0, \infty)$. This has been mentioned by Kusano in [11].

The following assumptions will be also needed throughout the paper:

(H1) $\alpha,\ \beta,\ \gamma$ are ratios of odd positive integers with $0 < \beta < 1$ and $\gamma > 1$;

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(H2) $r \in C^1([t_0,\infty),\mathbb{R}^+)$ and

$$R(t) := \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)} ds \to \infty \text{ as } t \to \infty; \tag{1.2}$$

(H3) $p_1,\ p_2,\ q \in C[t_0,\infty)$ are nonnegative, and $0 < p_2 \le p < 1$;

(H4) $\tau \in C^1[t_0,\infty),\ \tau'(t) > 0,\ \sigma(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

By a solution of Eq. (1.1), we mean a function $x(t) \in C([t_x,\infty)),\ t_x \ge t_0$, with $z,\ r(t)(z'(t))^\alpha \in C^1([t_x,\infty))$ that satisfies the differential equation Eq. (1.1) on $[t_x,\infty)$. We only focus on the solutions that satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge t_x$. As usual, a solution $x(t)$ of Eq. (1.1) is called *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is called *nonoscillatory*. Eq. (1.1) is *oscillatory* if all its solutions oscillate.

Differential equations with neutral terms have a special meaning because they appear in many applications, including control system, electrodynamics, mixed liquids, neutron transmission and so on. In the qualitative analysis of this kind of system, it is indeed the oscillation for the solution of the differential equation that plays a vital role, the growth rate of which not only depends on the current and past or future but also the rate of change in the past or future. And it has been studied extensively during the past the few decades, see [2, 4, 5, 8, 14] and the references cited therein.

Observing the literature, one can see that many results are applicable to the oscillation for second order differential equation with linear neutral term. Moreover, there are fewer articles devoted to investigating differential equations with the sublinear neutral term, or the superlinear neutral term, of the form

$$z(t) := x(t) + p(t)x^{\beta}(\sigma(t)) \ ,$$

where $0 < \beta < 1$ or $\beta > 1$, see for instance, [1, 6, 7, 9, 12, 13]. However, compared with the mentioned above, few results are available on the equations with both sublinear and superlinear neutral terms. Motivated by the above, the purpose of this paper is to establish the oscillation criteria for a class of second order mixed functional differential equation with subliner and superlinear neutral terms.

Recently, B. Baculíková studied halflinear second order differential equations with mixed argument of the form

$$(r(t)(y'(t))^{\alpha})' = q(t)x^{\alpha}(\tau(t)),$$

where $\tau(t)$ is a mixed deviating argument. The known results in [3] are generalized to the neutral differential equations that we studied.

If we denote the set of all nonoscillatory solutions of Eq. (1.1) by $\mathcal{N}$, then the set $\mathcal{N}$ is the union

$$\mathcal{N} = \bigcup_{i=1}^{4} \mathcal{N}_i,$$

where

$$\mathcal{N}_1 : \ z(t) > 0 \ \text{ and } \ z'(t) < 0, \qquad \mathcal{N}_2 : \ z(t) > 0 \ \text{ and } \ z'(t) > 0,$$

$$\mathcal{N}_3 : \ z(t) < 0 \ \text{ and } \ z'(t) < 0, \qquad \mathcal{N}_4 : \ z(t) < 0 \ \text{ and } \ z'(t) > 0.$$

We consider the situation in which $\mathcal{N} = \varnothing$ for Eq. (1.1), that is, all nontrivial solutions of Eq. (1.1) are oscillatory.

The paper is organized as follows: First, we present the auxiliary results and our main oscillation criteria. Then, some examples are proposed to illustrate the applicability of our criteria.

2. Main results

Without loss of generality, we only need to consider the case of eventually positive ones of Eq. (1.1). Next, we state a preliminary lemma, which will be necessary to proof our main results.

Lemma 2.1 *[10] If X and Y are nonnegative, then*

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda XY^{\lambda-1} \geq 0, \ \textit{for } \lambda > 1, \tag{2.1}$$

and

$$X^{\lambda} - (1 - \lambda)Y^{\lambda} - \lambda XY^{\lambda-1} \leq 0, \ \textit{for } 0 < \lambda < 1, \tag{2.2}$$

where equalities hold if and only if $X = Y$.

Now, we present the first oscillation theorem for Eq. (1.1). For the sake of convenience, we adopt the following notations:

$$g_1(t) := (\gamma - 1)\gamma^{\frac{\gamma}{1-\gamma}} p_2^{\frac{1}{1-\gamma}}(t) p^{\frac{\gamma}{\gamma-1}}(t),$$

$$g_2(t) := (1 - \beta)\beta^{\frac{\beta}{1-\beta}} p^{\frac{\beta}{\beta-1}}(t) p_1^{\frac{1}{1-\beta}}(t),$$

and

$$G_1(t) := 1 - \frac{g_1(t) + g_2(t)}{c_1 R(t)}, \quad G_2(t) := 1 - \frac{g_1(t) + g_2(t)}{c_2},$$

where c_1, c_2 are two constants with $c_1 < 0$, $c_2 > 0$ and $p(t)$ is a positive continuous real-valued function.

Theorem 2.2 *Suppose that there exists a positive continuous function $p(t)$ such that*

$$\lim_{t\to\infty} (g_1(t) + g_2(t)) = 0. \tag{2.3}$$

Further, assume the following condition:

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{\infty} q(s)ds \right)^{1/\alpha} du = \infty. \tag{2.4}$$

If there exist two sequences $\{t_k\}$, $\{s_k\}$ with t_k, $s_k \to \infty$ as $k \to \infty$ such that $t_k \in \mathcal{D}_\tau$ and $s_k \in \mathcal{A}_\tau$,

$$\limsup_{k\to\infty} \int_{\tau(t_k)}^{t_k} m_1^{\alpha} q(s)[R(\tau(t_k)) - R(\tau(s))]^{\alpha} ds > 1, \tag{2.5}$$

and

$$\limsup_{k\to\infty} \int_{s_k}^{\tau(s_k)} m_2^{\alpha} q(s)[R(\tau(s)) - R(\tau(s_k))]^{\alpha} ds > 1, \tag{2.6}$$

for all $k = 1, 2, \ldots$, where m_1 and m_2 are constants. Then all solutions of Eq. (1.1) are oscillatory.

Proof Assume that $x(t)$ is an eventually positive solution of Eq. (1.1). Hence, we get that $x(t) > 0$, $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$, for large enough t. Then, we have the following cases:

Case 1. Assume that $z(t) \in \mathcal{N}_1$. From the definition of z, we get

$$\begin{aligned} x(t) &= z(t) - p_1(t)x^\beta(\sigma(t)) + p_2(t)x^\gamma(\sigma(t)) \\ &= z(t) - (p(t)x(\sigma(t)) - p_2(t)x^\gamma(\sigma(t))) - (p_1(t)x^\beta(\sigma(t)) - p(t)x(\sigma(t))). \end{aligned}$$

Applying (2.1) with $\lambda = \gamma > 1$, $X = p_2^{\frac{1}{\gamma}}(t)x(\sigma(t))$ and $Y = (\frac{1}{\gamma}p(t)p_2^{-\frac{1}{\gamma}}(t))^{\frac{1}{\gamma-1}}$, we have

$$p(t)x(\sigma(t)) - p_2(t)x^\gamma(\sigma(t)) \le (\gamma-1)\gamma^{\frac{\gamma}{1-\gamma}}p_2^{\frac{1}{1-\gamma}}(t)p^{\frac{\gamma}{\gamma-1}}(t) := g_1(t).$$

Applying (2.2) with $0 < \lambda = \beta < 1$, $X = p_1^{\frac{1}{\beta}}(t)x(\sigma(t))$ and $Y = (\frac{1}{\beta}p(t)p_1^{-\frac{1}{\beta}}(t))^{\frac{1}{\beta-1}}$, we have

$$p_1(t)x^\beta(\sigma(t)) - p(t)x(\sigma(t)) \le (1-\beta)\beta^{\frac{\beta}{1-\beta}}p^{\frac{\beta}{\beta-1}}(t)p_1^{\frac{1}{1-\beta}}(t) := g_2(t).$$

Thus, we arrive at

$$x(t) \ge \left[1 - \frac{g_1(t) + g_2(t)}{z(t)}\right] z(t). \tag{2.7}$$

Since $r(t)(z'(t))^\alpha$ is increasing, we obtain that

$$\begin{aligned} z(t) &\ge \int_{t_0}^{t} \frac{r^{1/\alpha}(s)z'(s)}{r^{1/\alpha}(s)}ds \\ &\ge r^{1/\alpha}(t_0)z'(t_0)\int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}ds \\ &= c_1R(t), \end{aligned}$$

where $c_1 < 0$ is a constant. So

$$x(t) \ge \left[1 - \frac{g_1(t) + g_2(t)}{c_1R(t)}\right] z(t) := G_1(t)z(t). \tag{2.8}$$

Now, there exists a constant $m_1 \ge 1$ such that

$$x(t) \ge m_1 z(t). \tag{2.9}$$

There exists a sequence $\{t_k\}$ such that $t_k \in \mathcal{D}_\tau$, in view of $\tau'(t) > 0$, which implies $(\tau(t_k), t_k) \subset \mathcal{D}_\tau$. Integrating Eq. (1.1) from $\tau(t_k)$ to t_k, we get

$$\begin{aligned} -r(\tau(t_k))(z'(\tau(t_k)))^\alpha &\ge \int_{\tau(t_k)}^{t_k} q(s)x^\alpha(\tau(s))ds \\ &\ge m_1^\alpha \int_{\tau(t_k)}^{t_k} q(s)z^\alpha(\tau(s))ds. \end{aligned} \tag{2.10}$$

For $s \in (\tau(t_k), t_k)$, one gets

$$\begin{aligned} z(\tau(s)) &\geq \int_{\tau(s)}^{\tau(t_k)} \frac{-r^{1/\alpha}(u)z'(u)}{r^{1/\alpha}(u)}du \\ &\geq -r^{1/\alpha}(\tau(t_k))z'(\tau(t_k)) \int_{\tau(s)}^{\tau(t_k)} \frac{1}{r^{1/\alpha}(u)}du \\ &= -r^{1/\alpha}(\tau(t_k))z'(\tau(t_k))[R(\tau(t_k)) - R(\tau(s))], \end{aligned}$$

which, combining with (2.10), gives

$$-r(\tau(t_k))(z'(\tau(t_k)))^\alpha \geq -m_1^\alpha r(\tau(t_k))(z'(\tau(t_k)))^\alpha \int_{\tau(t_k)}^{t_k} q(s)[R(\tau(t_k)) - R(\tau(s))]^\alpha ds,$$

that is

$$1 \geq \int_{\tau(t_k)}^{t_k} m_1^\alpha q(s)[R(\tau(t_k)) - R(\tau(s))]^\alpha ds.$$

This contradicts with (2.5), and thus, $\mathcal{N}_1 = \varnothing$.

Case 2. Assume that $z(t) \in \mathcal{N}_2$. Since $z(t)$ is increasing, there exists a constant $c_2 > 0$ such that $z(t) \geq c_2$ for large enough t. Hence, the inequality (2.7) becomes

$$x(t) \geq \left[1 - \frac{g_1(t) + g_2(t)}{c_2}\right] z(t) := G_2(t)z(t). \tag{2.11}$$

Now, there exists a constant $m_2 \in (0,1)$ such that

$$x(t) \geq m_2 z(t). \tag{2.12}$$

Taking into account the fact that $\tau(t)$ is increasing, it is easy to see that $s_k \in \mathcal{A}_\tau$ implies that $(s_k, \tau(s_k)) \subset \mathcal{A}_\tau$. By integrating Eq. (1.1) and using $(r(t)(z'(t))^\alpha)' > 0$ and (2.12), we get

$$\begin{aligned} -r(\tau(s_k))(z'(\tau(s_k)))^\alpha &\geq \int_{s_k}^{\tau(s_k)} q(s)x^\alpha(\tau(s))ds \\ &\geq m_2^\alpha \int_{s_k}^{\tau(s_k)} q(s)z^\alpha(\tau(s))ds. \end{aligned} \tag{2.13}$$

For $s \in (s_k, \tau(s_k))$,

$$\begin{aligned} z(\tau(s)) &\geq \int_{\tau(s_k)}^{\tau(s)} \frac{r^{1/\alpha}(u)z'(u)}{r^{1/\alpha}(u)}du \\ &\geq r^{1/\alpha}(\tau(t_k))z'(\tau(t_k)) \int_{\tau(s_k)}^{\tau(s)} \frac{1}{r^{1/\alpha}(u)}du \\ &= r^{1/\alpha}(\tau(t_k))z'(\tau(t_k))[R(\tau(s)) - R(\tau(s_k))]. \end{aligned}$$

Therefore, (2.13) can be written as

$$r(\tau(s_k))(z'(\tau(s_k)))^\alpha \geq m_2^\alpha r(\tau(t_k))z'(\tau(s_k))^\alpha \int_{s_k}^{\tau(s_k)} q(s)[R(\tau(s)) - R(\tau(s_k))]^\alpha ds,$$

or, equivalent to

$$1 \geq \int_{s_k}^{\tau(s_k)} m_2^\alpha q(s)[R(\tau(s)) - R(\tau(s_k))]^\alpha ds,$$

which contradicts (2.6). So $\mathcal{N}_2 = \varnothing$.

Case 3. Assume that $z(t) \in \mathcal{N}_3$. In this case $z(t)$ satisfies either

$$\lim_{t\to\infty} z(t) = -\infty \tag{2.14}$$

or

$$\lim_{t\to\infty} z(t) = l < 0. \tag{2.15}$$

We claim that (2.14) holds. Otherwise, by the definition of $z(t)$, we obtain

$$x(t) \geq \left(\frac{-z(\sigma^{-1}(t))}{p}\right)^{1/\gamma}, \quad t \geq t_1 \; .$$

It is easy to see that $x(t)$ is bounded and there exists a $M_1 > 0$ such that $x(t) \geq M_1 > 0$ for all $t \geq t_2 \geq t_1$. From Eq. (1.1), we have

$$(r(t)(z'(t))^\alpha)' \geq M_1^\alpha q(t), \quad t \geq t_2.$$

Integrating it from t to u and let $u \to \infty$, we get

$$M_1^\alpha \int_t^\infty q(s)ds \leq -r(t)(z'(t))^\alpha.$$

Taking integration from t_2 to t again, we obtain that as $t \to \infty$

$$z(t) \leq -M_1 \int_{t_2}^\infty \frac{1}{r^{1/\alpha}(u)} \left(\int_u^\infty q(s)ds\right)^{1/\alpha} du,$$

which is a contradiction with (2.15) from (2.4). So (2.14) holds and $\mathcal{N}_3 = \varnothing$.

Case 4. Assume that $z(t) \in \mathcal{N}_4$. Since $r(t)(z'(t))^\alpha$ is increasing and positive, there exists a $M_2 > 0$ such that $r(t)(z'(t))^\alpha \geq M_2$ for all $t \geq t_1$. Integrating it from t_1 to t and let $t \to \infty$, we get

$$z(t) \geq z(t_1) + M_2^{1/\alpha} \int_{t_1}^\infty \frac{1}{r^{1/\alpha}(s)} ds.$$

However, this is impossible due to (1.2). Hence, $\mathcal{N}_4 = \varnothing$. The proof is complete. □

Next, we will further improve Theorem 2.2 by establishing new monotonicity of nonoscillatory solutions of Eq. (1.1). To this end, let us state the following necessary conclusions.

Lemma 2.3 *Assume that there exists a sequence* $\{t_k\}$, $t_k \to \infty$ *as* $k \to \infty$, *such that* $t_k \in \mathcal{D}_\tau$. *Suppose that there exists a* $\eta > 0$ *such that*

$$m_1^\alpha (R(t) - R(\tau(t)))^\alpha R(t) r^{1/\alpha}(t) q(t) \geq \eta, \quad on \quad (\tau(\tau(t_k)), \tau(t_k)) \tag{2.16}$$

for all $k = 1, 2, \ldots$. *If* $x(t)$ *is a positive solution of Eq.* (1.1) *such that* $z(t) \in \mathcal{N}_1$, *then* $-R^\eta(t) r(t)(z'(t))^\alpha$ *is decreasing on* $(\tau(\tau(t_k)), \tau(t_k))$.

Proof Since $-r(t)(z'(t))^\alpha$ is decreasing, then

$$\begin{aligned} z(\tau(t)) &\geq \int_{\tau(t)}^{t} \frac{-r^{1/\alpha}(u) z'(u)}{r^{1/\alpha}(u)} du \\ &\geq -r^{1/\alpha}(t) z'(t)[R(t) - R(\tau(t))]. \end{aligned}$$

By virtue of Eqs. (1.1) and (2.9), we have

$$(r(t)(z'(t))^\alpha)' \geq m_1^\alpha q(t) r(t)(-z'(t))^\alpha [R(t) - R(\tau(t))]^\alpha.$$

The rest of the proof is similar to that of the Lemmas in [3], hence we omitted it. □

Lemma 2.4 *Assume that there exists a sequence* $\{s_k\}$, $s_k \to \infty$ *as* $k \to \infty$, *such that* $s_k \in \mathcal{A}_\tau$. *Suppose that there exists a* $\xi > 0$ *such that*

$$m_2^\alpha (R(\tau(t)) - R(t))^\alpha R(t) r^{1/\alpha}(t) q(t) \geq \xi, \quad on \quad (\tau(s_k), \tau(\tau(s_k))) \tag{2.17}$$

for all $k = 1, 2, \ldots$. *If* $x(t)$ *is a positive solution of Eq.* (1.1) *such that* $z(t) \in \mathcal{N}_2$, *then* $R^{-\xi}(t) r(t)(z'(t))^\alpha$ *is increasing on* $(\tau(s_k), \tau(\tau(t_k)))$.

A similar proof of this Lemma can be referred to Lemma 2.3, which will be not described in detail here.

Theorem 2.5 *Assume that* (2.4) *holds and there exists a positive continuous function* $p(t)$ *such that* (2.3) *holds. Further, assume that there exist two sequences* $\{t_k\}$, $\{s_k\}$ *with* t_k, $s_k \to \infty$, $k \to \infty$ *such that* $t_k \in \mathcal{D}_\tau$ *and* $s_k \in \mathcal{A}_\tau$. *If*

$$\limsup_{k\to\infty} m_1^\alpha R^\eta(\tau(t_k)) \int_{\tau(t_k)}^{t_k} q(s) \left[\frac{R^{1-\eta/\alpha}(\tau(t_k)) - R^{1-\eta/\alpha}(\tau(s))}{1 - \eta/\alpha}\right]^\alpha ds > 1 \tag{2.18}$$

and

$$\limsup_{k\to\infty} m_2^\alpha R^{-\xi}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} q(s) \left[\frac{R^{1+\xi/\alpha}(\tau(s)) - R^{1+\xi/\alpha}(\tau(s_k))}{1 + \xi/\alpha}\right]^\alpha ds > 1, \tag{2.19}$$

where η *and* ξ *are defined by* (2.16) *and* (2.17), m_1 *and* m_2 *are constants, then all solutions of Eq.*(1.1) *are oscillatory.*

Proof Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of Eq. (1.1). Then, we have the following cases:

Case 1. Assume that $z(t) \in \mathcal{N}_1$. In view of Lemma 2.3, $-R^{\eta}(t)r(t)(z'(t))^{\alpha}$ is decreasing on $(\tau(\tau(t_k)), \tau(t_k))$. Thus, for $x \in (\tau(t_k), t_k)$, we have

$$\begin{aligned} z(\tau(s)) &\geq \int_{\tau(s)}^{\tau(t_k)} \frac{-r^{1/\alpha}(u)R^{\eta/\alpha}(u)z'(u)}{r^{1/\alpha}(u)R^{\eta/\alpha}(u)}du \\ &\geq -r^{1/\alpha}(\tau(t_k))R^{\eta/\alpha}(\tau(t_k))z'(\tau(t_k)) \int_{\tau(s)}^{\tau(t_k)} \frac{1}{r^{1/\alpha}(u)R^{\eta/\alpha}(u)}du \\ &= -r^{1/\alpha}(\tau(t_k))R^{\eta/\alpha}(\tau(t_k))z'(\tau(t_k)) \left[\frac{R^{1-\eta/\alpha}(\tau(t_k)) - R^{1-\eta/\alpha}(\tau(s))}{1-\eta/\alpha}\right]. \end{aligned}$$

Using the last inequality in (2.10), we conclude that

$$\begin{aligned} &-r(\tau(t_k))(z'(\tau(t_k)))^{\alpha} \\ \geq& m_1^{\alpha}(-r^{1/\alpha}(\tau(t_k))R^{\eta/\alpha}(\tau(t_k))z'(\tau(t_k))) \int_{\tau(t_k)}^{t_k} q(s) \left[\frac{R^{1-\eta/\alpha}(\tau(t_k)) - R^{1-\eta/\alpha}(\tau(s))}{1-\eta/\alpha}\right]^{\alpha} ds, \end{aligned}$$

or

$$1 \geq m_1^{\alpha} R^{\eta}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} q(s) \left[\frac{R^{1-\eta/\alpha}(\tau(t_k)) - R^{1-\eta/\alpha}(\tau(s))}{1-\eta/\alpha}\right]^{\alpha} ds,$$

which contradicts (2.18). Hence, $\mathcal{N}_1 = \varnothing$.

Case 2. Assume that $z(t) \in \mathcal{N}_2$. Using the monotonicity of $R^{-\xi}(t)r(t)(z'(t))^{\alpha}$ in Lemma 2.4, we find

$$\begin{aligned} z(\tau(s)) &\geq \int_{\tau(s_k)}^{\tau(s)} \frac{r^{1/\alpha}(u)R^{-\xi/\alpha}(u)z'(u)}{r^{1/\alpha}(u)R^{-\xi/\alpha}(u)}du \\ &\geq r^{1/\alpha}(\tau(s_k))R^{-\xi/\alpha}(\tau(s_k))z'(\tau(s_k)) \int_{\tau(s_k)}^{\tau(s)} \frac{1}{r^{1/\alpha}(u)R^{-\xi/\alpha}(u)}du \\ &= r^{1/\alpha}(\tau(s_k))R^{-\xi/\alpha}(\tau(s_k))z'(\tau(s_k)) \left[\frac{R^{1+\xi/\alpha}(\tau(s)) - R^{1+\xi/\alpha}(\tau(s_k))}{1+\xi/\alpha}\right]. \end{aligned}$$

It follows from (2.13) that

$$\begin{aligned} &r(\tau(s_k))(z'(\tau(s_k)))^{\alpha} \\ \geq& m_2^{\alpha}(r^{1/\alpha}(\tau(s_k))R^{-\xi/\alpha}(\tau(s_k))z'(\tau(s_k))) \int_{s_k}^{\tau(s_k)} q(s) \left[\frac{R^{1+\xi/\alpha}(\tau(s)) - R^{1+\xi/\alpha}(\tau(s_k))}{1+\xi/\alpha}\right]^{\alpha} ds, \end{aligned}$$

that is,

$$1 \geq m_2^{\alpha} R^{-\xi}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} q(s) \left[\frac{R^{1+\xi/\alpha}(\tau(s)) - R^{1+\xi/\alpha}(\tau(s_k))}{1+\xi/\alpha}\right]^{\alpha} ds,$$

which contradicts (2.19) and then $\mathcal{N}_2 = \varnothing$.

The remaining parts (Cases 3 and 4) are the same as in Theorem 2.2. This completes the proof. □

3. Examples

In this section, we will illustrate the applicability of our main result via some examples.

Example 3.1 *Consider the second order mixed differential equation with sublinear and superlinear neutral terms*

$$z''(t) = \frac{q_0}{t^2} x\left(t\left(1 - \frac{2\cos(\ln t)}{3}\right)\right), \quad t > 0, \tag{3.1}$$

where $z(t) = x(t) + \sqrt{t}x^{\frac{1}{2}}(\frac{t}{3}) - \frac{1}{t}x^2(\frac{t}{3})$. *It is a special form of Eq.* (1.1) *when* $r(t) = 1$, $p_1(t) = \sqrt{t}$, $p_2(t) = \frac{1}{t}$, $q(t) = \frac{q_0}{t^2}(q_0 > 0)$, $\sigma(t) = \frac{t}{3}$, $\tau(t) = t\left(1 - \frac{2\cos(\ln t)}{3}\right)$. *Clearly, the deviating argument* $\tau(t) = t\left(1 - \frac{2\cos(\ln t)}{3}\right)$ *is of mixed type.*

If we take $t_k = e^{2k\pi}$, $k = 1, 2, ...$, then $\tau(t_k) = \frac{1}{3}e^{2k\pi} \in \mathcal{D}_\tau$. Thus

$$\begin{aligned}
&\limsup_{k\to\infty} m_1 \int_{\tau(t_k)}^{t_k} \frac{q_0}{s^2}\left[\frac{1}{3}e^{2k\pi} - s(1 - \frac{2\cos(\ln s)}{3})\right] ds \\
= &\limsup_{k\to\infty} q_0 m_1 \int_{\tau(t_k)}^{t_k} \left[\frac{1}{3}e^{2k\pi}\frac{1}{s^2} - \frac{1}{s} + \frac{2}{3}\frac{\cos(\ln s)}{s}\right] ds \\
= &\limsup_{k\to\infty} q_0 m_1 \left[-\frac{1}{3}e^{2k\pi}\frac{1}{s}\Big|_{\tau(t_k)}^{t_k} - \ln s\Big|_{\tau(t_k)}^{t_k} + \frac{2}{3}\cos(\ln s)\Big|_{\tau(t_k)}^{t_k}\right] \\
= &\; q_0 m_1 \left[\frac{2}{3} - \ln 3 + \frac{2}{3}\sin(\ln 3)\right] > 1\ ,
\end{aligned}$$

which by Theorem 2.2 implies that $\mathcal{N}_1 = \varnothing$ for $q_0 > \frac{1}{m_1}6.1805$.

Moreover, if we take $s_k = e^{\pi+2k\pi}$, then $\tau(s_k) = \frac{5}{3}e^{\pi+2k\pi} \in \mathcal{A}_\tau$. Condition (2.6) becomes

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{s_k}^{\tau(s_k)} m_2 \frac{q_0}{s^2}\left[s(1 - \frac{2\cos(\ln s)}{3} - \frac{5}{3}e^{\pi+2k\pi})\right] ds \\
= &\limsup_{k\to\infty} q_0 m_2 \int_{s_k}^{\tau(s_k)} \left[\frac{1}{s} - \frac{2}{3}\frac{\cos(\ln s)}{s} - \frac{5}{3}e^{\pi+2k\pi}\frac{1}{s^2}\right] ds \\
= &\limsup_{k\to\infty} q_0 m_2 \int_{s_k}^{\tau(s_k)} \left[\ln s\Big|_{s_k}^{\tau(s_k)} - \frac{2}{3}\sin(\ln s)\Big|_{s_k}^{\tau(s_k)} + \frac{5}{3}e^{\pi+2k\pi}\frac{1}{s}\Big|_{s_k}^{\tau(s_k)}\right] ds \\
= &\; q_0 m_2 \left[\ln\frac{5}{3} + \frac{2}{3}\sin\left(\ln\frac{5}{3}\right) - \frac{2}{3}\right] > 1,
\end{aligned}$$

which ensures that $\mathcal{N}_2 = \varnothing$ for $q_0 > \frac{1}{m_2}6.7476$.

Further, we can verify that

$$\int_0^\infty \int_u^\infty \frac{q_0}{s^2} ds du = \infty,$$

that is, (2.4) is also satisfied. Hence, we see that $q_0 > \frac{1}{m_2}6.7476$ implies the oscillation of (3.1).

Example 3.2 *We also consider the differential equation* (3.1).

Firstly, we will show that $\mathcal{N}_1 = \varnothing$ for $q_0 \geq \frac{1}{m_1}5.8067637$. We set $q_0 = \frac{1}{m_1}5.8067637$. Let $t_k = e^{2k\pi}$, $k = 1, 2, \ldots$, then $\tau(t_k) = \frac{1}{3}e^{2k\pi}$ and $\tau(\tau(t_k)) = (\frac{1}{3} - \frac{2}{9}\cos(\ln 3))e^{2k\pi}$. In view of Lemma 2.3, we have

$$\frac{2}{3}m_1 q_0 \cos(\ln t) \geq \eta \quad \text{on each} \quad (\tau(\tau(t_k)), \tau(t_k)), \quad k = 1, 2, \ldots.$$

Note that $\frac{2}{3}m_1 q_0 \cos(\ln t)$ is increasing on $(\tau(\tau(t_k)), \tau(t_k))$, so we have

$$\eta = \frac{2}{3}m_1 q_0 \cos(\ln(\tau(\tau(t_k)))) = 0.428419.$$

Now, we verify the condition (2.18).

$$\begin{aligned}
&\limsup_{k\to\infty} m_1 \tau^{\eta}(t_k) \int_{\tau(t_k)}^{t_k} q(s) \left[\frac{\tau^{1-\eta}(t_k) - \tau^{1-\eta}(s)}{1-\eta}\right] ds \\
&= \limsup_{k\to\infty} \frac{m_1 q_0}{1-\eta} \left(\frac{1}{3}e^{2k\pi}\right)^{\eta} \int_{\tau(t_k)}^{t_k} \left[\frac{(\frac{1}{3}e^{2k\pi})^{1-\eta}}{s^2} - \frac{s^{1-\eta}(1 - \frac{2\cos(\ln s)}{3})^{1-\eta}}{s^2}\right] ds \\
&= \limsup_{k\to\infty} \frac{m_1 q_0}{1-\eta} \left[\frac{2}{3} - \left(\frac{1}{3}e^{2k\pi}\right)^{\eta} \int_{\tau(t_k)}^{t_k} \frac{s^{1-\eta}(1 - \frac{2\cos(\ln s)}{3})^{1-\eta}}{s^2} ds\right].
\end{aligned}$$

Let $s = e^{2k\pi}t$, the above formula becomes

$$\begin{aligned}
&\frac{m_1 q_0}{1-\eta} \left[\frac{2}{3} - (\frac{1}{3})^{\eta} \int_{\frac{1}{3}}^{1} \frac{t^{1-\eta}(1 - \frac{2\cos(\ln e^{2k\pi}t)}{3})^{1-\eta}}{t^2} dt\right] \\
&= \frac{m_1 q_0}{1-\eta} \left[\frac{2}{3} - (\frac{1}{3})^{\eta} \int_{\frac{1}{3}}^{1} t^{-1-\eta} \left(1 - \frac{2\cos(\ln t)}{3}\right)^{1-\eta} dt\right].
\end{aligned}$$

Using Matlab, we get

$$\frac{m_1 q_0}{1-\eta} \left[\frac{2}{3} - (\frac{1}{3})^{\eta} \int_{\frac{1}{3}}^{1} t^{-1-\eta} \left(1 - \frac{2\cos(\ln t)}{3}\right)^{1-\eta} dt\right] = 1.0000000049 > 1$$

with $\eta = 0.428419$, which by Theorem 2.5 implies $\mathcal{N}_1 = \varnothing$.

Secondly, we will show that $\mathcal{N}_2 = \varnothing$ for $q_0 \geq \frac{1}{m_2}4.4183548$. We set $q_0 = \frac{1}{m_2}4.4183548$. Take $s_k = e^{\pi+2k\pi}$, then $\tau(s_k) = \frac{5}{3}e^{\pi+2k\pi}$ and $\tau(\tau(s_k)) = (\frac{5}{3} + \frac{10}{9}\cos(\ln\frac{5}{3}))e^{\pi+2k\pi}$. From Lemma 2.4, we see that

$$-\frac{2}{3}m_2 q_0 \cos(\ln t) \geq \xi \quad \text{on each} \quad (\tau(s_k), \tau(\tau(s_k))), \ k = 1, 2, \ldots.$$

Since $-\frac{2}{3}m_2 q_0 \cos(\ln t)$ is decreasing on $(\tau(s_k), \tau(\tau(s_k)))$, we have

$$\xi = -\frac{2}{3}m_2 q_0 \cos(\ln(\tau(\tau(s_k)))) = 1.6669803.$$

Condition (2.19) becomes

$$\limsup_{k\to\infty} m_2\tau^{-\xi}(s_k)\int_{s_k}^{\tau(s_k)} q(s)\left[\frac{\tau^{1+\xi}(s)-\tau^{1+\xi}(s_k)}{1+\xi}\right]ds$$
$$=\limsup_{k\to\infty}\frac{m_2q_0}{1+\xi}\left(\frac{5}{3}e^{\pi+2k\pi}\right)^{-\xi}\int_{s_k}^{\tau(s_k)}\left[\frac{s^{1+\xi}(1-\frac{2\cos(\ln s)}{3})^{1+\xi}}{s^2}-\frac{\left(\frac{5}{3}e^{\pi+2k\pi}\right)^{1+\xi}}{s^2}\right]ds$$
$$=\limsup_{k\to\infty}\frac{m_2q_0}{1+\xi}\left[\left(\frac{5}{3}e^{\pi+2k\pi}\right)^{-\xi}\int_{s_k}^{\tau(s_k)}\frac{s^{1+\xi}(1-\frac{2\cos(\ln s)}{3})^{1+\xi}}{s^2}ds-\frac{2}{3}\right].$$

Taking $s=e^{\pi+2k\pi}t$, we obtain that

$$\limsup_{k\to\infty} m_2\tau^{-\xi}(s_k)\int_{s_k}^{\tau(s_k)} q(s)\left[\frac{\tau^{1+\xi}(s)-\tau^{1+\xi}(s_k)}{1+\xi}\right]ds$$
$$=\frac{m_2q_0}{1+\xi}\left[\left(\frac{5}{3}\right)^{-\xi}\int_1^{\frac{5}{3}}t^{\xi-1}\left(1-\frac{2\cos(\ln e^{\pi+2k\pi}t)}{3}\right)^{1+\xi}dt-\frac{2}{3}\right]$$
$$=\frac{m_2q_0}{1+\xi}\left[-\frac{2}{3}+\left(\frac{5}{3}\right)^{-\xi}\int_1^{\frac{5}{3}}t^{\xi-1}\left(1+\frac{2\cos(\ln t)}{3}\right)^{1+\xi}dt\right].$$

By Matlab, we can conclude that

$$\frac{m_2q_0}{1+\xi}\left[-\frac{2}{3}+\left(\frac{5}{3}\right)^{-\xi}\int_1^{\frac{5}{3}}t^{\xi-1}\left(1+\frac{2\cos(\ln t)}{3}\right)^{1+\xi}dt\right]=1.000000027>1$$

with $\xi=1.6669803$, which implies that $\mathcal{N}_2=\varnothing$.

Hence, (3.1) is oscillatory for $q_0>\frac{1}{m_2}4.4183548$. This also shows that Theorem 2.5 is an improvement over Theorem 2.1.

The results of Examples 3.1 and 3.2 are graphically represented as shown in Figure. We can see intuitively that the condition corresponding to Example 3.2 is weaker, that is, the condition of Theorem 2.2 is an improvement of Theorem 2.1.

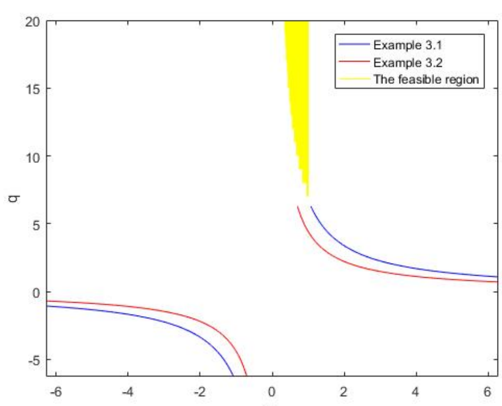

Figure. Comparison between Example 3.1 and Example 3.2.

4. Conclusion

In this paper, a class of second-order mixed functional differential equations with superlinear and sublinear neutral terms is studied and some oscillation criteria are established. The innovation of this paper is that the functional differential equation has both superlinear and sublinear terms, and it is a mixed type equation. Because the neutral term is indefinite, it is full of many uncertainties. In addition, we could consider whether the results presented in this paper apply to n order differential equations or to more general differential equations in the future work.

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References

[1] Agarwal RP, Bohner M, Li TX, Zhang CH. Oscillation of second-order differential equations with sublinear neutral term. Carpathian Journal of Mathematics 2014; 30 (1): 1-6.

[2] Agarwal RP, Zhang CH, Li TX. Some remarks on oscillation of second oreder neutral differential equatons. Applied Mathematics and Computation 2016; 274: 178-181.

[3] Baculíková B. Oscillation of second-order half-linear differential equations with deviating argument of mixed type. Applied Mathematics Letters 2019; 118, 107228.

[4] Baculíková B, Džurina J. Oscillation theorems for second-order nonlinear neutral differential equations. Computer & Mathematics with Applications 2011; 62 (12): 4472-4478.

[5] Bohner M, Grace SR, Jadlovská I. Oscillation criteria for second-order neutral delay differntial equations. Electronic Journal of Qualitative Theory of Differential Equations 2017; 60: 1-12.

[6] Bohner M, Sudha B, Tangavelu K, Thandapani E. Oscillation criteria for second-order differential equations with superlinear neutral term. Nonlinear Studies 2019; 26 (2): 425-434.

[7] Džurina J, Grace SR, Jadlovská I, Li TX. Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. Mathematische Nachrichten 2020; 295 (5): 910-922.

[8] Grace SR, Džurina J, Jadlovská I, Li TX. An improved approach for studying oscillation of second-order neutral delay differential equations. Journal of Inequality and Applications 2018; 2018(1): 193.

[9] Grace SR, Jadlovská I, Zafer A. On oscillation of second order delay differential equations with a sublinear term. Mediterranean Journal of Mathematics 2020; 17: 116.

[10] Hardy GH, Littlewood IE, Polya G. Inequalities. Cambridge University Press, Cambridge, 1959.

[11] Kusano T. Oscillation of even order linear functional differential equations with deviating arguments of mixed type. Journal of Mathmatical Analysis and Applications 1984; 98: 341–347.

[12] Lin XY, Tang XH. Oscillation of solutions of neutral differential equations with a superlinear neutral term. Applied Mathematics Letters 2007; 20: 1016-1022.

[13] Muhib A, Elabbasy EM, Moaaz O. New oscillation criteria for differential equations with subinear and superlinear neutral terms. Turkish Journal of Mathematics 2021; 45: 919-928.

[14] Sun SR, Li TX, Han ZL, Zhang CH. On oscillation of second-order nonlinear neutral functional differential equations. Bulletn of the Malaysiam Mathematical Sciences Society 2013; 36 (3): 541-554.