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MEIRKHAH BORIKHAHNOV

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## Globally unsolvability of integro-differential diffusion equation and system with exponential nonlinearities

Meiirkhan BORIKHANOV<sup>1,2,3,\*</sup> 

<sup>1</sup>Khoja Akhmet Yassawi International Kazakh-Turkish University, Turkistan, Kazakhstan

<sup>2</sup>Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

<sup>3</sup>Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Ghent, Belgium

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**Abstract:** In this paper, the Cauchy problem for an integro-differential diffusion equation and a system with nonlocal nonlinear sources are considered. The results on the existence of local integral solutions and the nonexistence of global weak solutions to the nonlinear integro-differential diffusion equation and system are presented.

**Key words:** Local existence, blow-up, global weak solution, integro-differential diffusion system

### 1. Introduction and statement of the problem

In this paper, we consider the integro-differential diffusion equation with the Cauchy data

$$\begin{cases} u_t(x, t) - \Delta D_{0|t}^{1-\alpha} u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} e^{u(x,s)} ds, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $N \geq 1$ ,  $0 < \alpha < 1$ ,  $D_{0|t}^{1-\alpha}$  is the left-handed Riemann–Liouville fractional derivative of order  $1 - \alpha$ .

Moreover, we consider local existence and global nonexistence of the solution for the integro-differential diffusion system

$$\begin{cases} u_t(x, t) - \Delta D_{0|t}^{1-\alpha} u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} e^{v(x,s)} ds, \\ v_t(x, t) - \Delta D_{0|t}^{1-\beta} v(x, t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} e^{u(x,s)} ds, \end{cases} \quad (1.2)$$

for  $(x, t) \in \mathbb{R}^N \times (0, T)$ , subject to the initial conditions

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N, \quad (1.3)$$

where  $\alpha, \beta, \gamma, \delta \in (0, 1)$ .

Before we present and prove our results, let us dwell on existing literature related to our problems. The Cauchy problem for the semilinear parabolic equation with a nonlinear memory (for  $0 < \gamma < 1$ )

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(x, s) ds, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.4)$$

has been considered by Cazenave et al. [6].

Then, Fino and Kirane in [7] and [8] studied the semilinear parabolic equation:

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(x, s) ds, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.5)$$

and system

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |v|^{p-1} v(x, s) ds, & x \in \mathbb{R}^N, t > 0, \\ v_t(x, t) - \Delta v(x, t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} |u|^{p-1} u(x, s) ds, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.6)$$

In addition, Ahmad et al. [1] considered the semilinear nonlocal in time and space reaction diffusion equation:

$$\begin{cases} u_t(x, t) + (-\Delta)^{\beta/2} u(x, t) = I_{0|t}^{1-\gamma} [e^{u(x,t)}], & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.7)$$

where  $I_{0|t}^{1-\gamma} [e^{u(x,t)}]$  is the Riemann–Liouville fractional integral of order  $1 - \gamma \in (0, 1)$  of  $e^{u(x,t)}$ .

Moreover, the system of (1.7) was investigated by Ahmad et al. [2] in the following form:

$$\begin{cases} u_t(x, t) + (-\Delta)^{\beta/2} u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} e^{v(x,s)} ds, & x \in \mathbb{R}^N, t > 0, \\ v_t(x, t) + (-\Delta)^{\beta/2} v(x, t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} e^{u(x,s)} ds, & x \in \mathbb{R}^N, t > 0, \end{cases} \quad (1.8)$$

supplemented with the initial conditions

$$u(x, 0) = u_0(x), v(x, 0) = v_0(x), x \in \mathbb{R}^N, \quad (1.9)$$

where  $u_0(x), v_0(x) \in C_0(\mathbb{R}^N)$ ,  $0 < \beta \leq 2$ ,  $\gamma, \delta \in (0, 1)$ .

Recently, Kirane et al. in [4] treated the time-space fractional evolution equation with a time nonlocal nonlinearity of exponential growth

$$\begin{cases} \mathcal{D}_{0|t}^\alpha u(x, t) + (-\Delta)^{\beta/2} u(x, t) = I_{0|t}^{1-\alpha} [e^{u(x,t)}], & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.10}$$

where  $N \geq 1, 0 < \alpha < 1, 0 < \beta \leq 2$ ,  $\mathcal{D}_{0|t}^\alpha$  is the left-handed Caputo fractional derivative operator of order  $\alpha$ .

Later on, Alsaedi et al. [3] studied the system of (1.10)

$$\begin{cases} \mathcal{D}_{0|t}^{\alpha_1} u_t(x, t) + (-\Delta)^{\beta/2} u(x, t) = I_{0|t}^{1-\alpha_1} (e^v), & x \in \mathbb{R}^N, t > 0, \\ \mathcal{D}_{0|t}^{\alpha_2} v_t(x, t) + (-\Delta)^{\beta/2} v(x, t) = I_{0|t}^{1-\alpha_2} (e^u), & x \in \mathbb{R}^N, t > 0, \end{cases} \tag{1.11}$$

subject to the initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^N, \tag{1.12}$$

where  $\alpha_1, \alpha_2 \in (0, 1)$ .

After that, in [5] the integro-differential diffusion system

$$\begin{cases} u_t(x, t) - \Delta D_{0+,t}^{1-\alpha} u(x, t) = |x|^{\rho_1} t^{\sigma_1} v^p(x, t), & x \in \mathbb{R}^N, t > 0, \\ v_t(x, t) - \Delta D_{0+,t}^{1-\beta} v(x, t) = |x|^{\rho_2} t^{\sigma_2} u^q(x, t), & x \in \mathbb{R}^N, t > 0, \end{cases} \tag{1.13}$$

with Cauchy data

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \text{ for } x \in \mathbb{R}^N, \tag{1.14}$$

where  $0 < \alpha, \beta < 1$  and  $\sigma_1, \sigma_2 > -1, \rho_1, \rho_2 \geq 0$  was considered.

The here above cited papers treated the local existence, the blowing-up solutions and the blow-up profile of exploding solutions.

## 2. Preliminaries

**Definition 2.1** *The left and right Riemann–Liouville fractional integrals  $I_{0|t}^\alpha f(t)$  and  $I_{t|T}^\alpha f(t)$  of order  $\alpha \in \mathbb{R} (\alpha > 0)$ , for all  $f(t) \in L^q(0, T) (1 \leq q \leq \infty)$ , we defined as [11, p. 69]*

$$I_{0|t}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad a.e. \quad t \in (0, T],$$

and

$$I_{t|T}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) ds, \quad a.e. \quad t \in [0, T),$$

respectively.

**Definition 2.2** If  $f(t) \in C^1([0, T])$  the left-handed and right-handed Riemann–Liouville fractional derivatives  $D_{0|t}^\alpha f(t)$  and  $D_{t|T}^\alpha f(t)$  of order  $\alpha \in (0, 1)$  are defined by [11, p.70]

$$D_{0|t}^\alpha f(t) = \frac{d}{dt} I_{0|t}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \quad t \in (0, T),$$

and

$$D_{t|T}^\alpha f(t) = -\frac{d}{dt} I_{t|T}^{1-\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (s-t)^{-\alpha} f(s) ds, \quad t \in [0, T).$$

**Definition 2.3** The left and right Caputo fractional derivatives of order  $\alpha \in \mathbb{R}$  ( $0 < \alpha < 1$ ), for  $f \in C^1([0, T])$  are defined, respectively, by [11, p. 91]

$$\mathcal{D}_{0|t}^\alpha f(t) = I_{0|t}^{1-\alpha} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad \forall t \in (0, T),$$

and

$$\mathcal{D}_{t|T}^\alpha f(t) = -I_{t|T}^{1-\alpha} \frac{d}{dt} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s) ds, \quad \forall t \in [0, T).$$

**Lemma 2.4** Let  $\alpha > 0$ ,  $p \geq 1$ ,  $q \geq 1$  and  $1/p + 1/q \leq 1 + \alpha$ . If  $f(t) \in L^p(0, T)$ ,  $g(t) \in L^q(0, T)$ , then we have the formula of integration by parts (see [11], Lemma 2.7. p. 76)

$$\int_0^T I_{0|t}^\alpha [f(t)] g(t) dt = \int_0^T f(t) I_{t|T}^\alpha [g(t)] dt.$$

**Lemma 2.5** [10] For any  $p \geq 0$  and any  $f \in C^2(\mathbb{R}^N)$  the pointwise inequality

$$(p+1)f^p(x)(-\Delta)f(x) \geq (-\Delta)f^{p+1}(x), \quad x \in \mathbb{R}^N,$$

holds.

Later on, we will use the following results (see [9]):

Suppose,  $\varphi_2(t) = \left(1 - \frac{t}{T}\right)^\eta$  and  $t \geq 0$ ,  $T > 0$ ,  $\eta > 1$ , then

$$I_{t|T}^{1-\gamma} \varphi_2(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} T^{1-\gamma} \left(1 - \frac{t}{T}\right)^{\eta-\gamma+1}, \tag{2.1}$$

$$\mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha)} T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1}, \tag{2.2}$$

for all  $\alpha, \gamma \in (0, 1)$ .

**Definition 2.6** The Mittag-Leffler function is defined by [11, p. 42]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \geq 0,$$

where  $\beta \in \mathbb{R}$  is an arbitrary constant.

### 3. Fractional diffusion equation with exponential nonlinearity

#### 3.1. Local existence of integral solution

**Definition 3.1 (Integral solution)** Let  $u_0 \in L^\infty(\mathbb{R}^N)$  and  $T > 0$ . We say that  $u \in L^\infty(\mathbb{R}^N; C[0, T])$  is an integral solution (see [13, p. 78]) of (1.1) if  $u$  satisfies the following integral equation

$$u(x, t) = \int_{\mathbb{R}^N} G(x - y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\gamma}(e^u) dy d\tau, \quad (3.1)$$

for  $t \in [0, T]$ ,  $x \in \mathbb{R}^N$ , where

$$G(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\langle x, \xi \rangle} E_{\alpha,1}(-\xi^2 t^\alpha) d\xi. \quad (3.2)$$

**Theorem 3.2 (Local existence)** For given  $u_0(x) \in L^\infty(\mathbb{R}^N)$ , there exists a maximal time  $T > 0$  such that Equation (1.1) has a unique integral solution  $u(x, t) \in L^\infty(\mathbb{R}^N; C[0, T])$ .

Furthermore, either  $T = \infty$  or  $T < \infty$  and  $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \rightarrow \infty$ , as  $t \rightarrow T$ .

Proof. For arbitrary  $T > 0$ , we define the ball

$$B_T = \{u \in L^\infty(\mathbb{R}^N; C[0, T]); \|u\|_1 \leq 2 \|u_0\|_{L^\infty(\mathbb{R}^N)}\}, \quad (3.3)$$

where  $\|\cdot\|_1 = \|\cdot\|_{L^\infty(\mathbb{R}^N, L^\infty(0, T))}$ .

Next, for every  $u(x, t) \in B_T$ , we define the map

$$\begin{aligned} \Psi(u) &= \int_{\mathbb{R}^N} G(x - y, t) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\gamma}(e^u) dy d\tau, \quad t \in [0, T]. \end{aligned}$$

The existence of a local solution will be proved as a fixed point of  $\Psi(u)$  using the Banach fixed point theorem.

•  $\Psi : B_T \rightarrow B_T$ . According to Lemma 2.2 in [5], it holds

$$\begin{aligned} \|\Psi(u)\|_{B_T} &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \|e^u\|_\infty d\tau ds \right\|_{L^\infty(0, T)} \\ &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \|e^u\|_\infty ds d\tau \right\|_{L^\infty(0, T)} \\ &\leq \|u_0\|_\infty + CT^{2-\gamma} e^{\|u\|_1} \\ &\leq \|u_0\|_\infty + CT^{2-\gamma} e^{2\|u_0\|_\infty}, \end{aligned}$$

where

$$C := \frac{1}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} = \frac{1}{\Gamma(3-\gamma)}.$$

Now, we choose  $T$  so that

$$CT^{2-\gamma}e^{2\|u_0\|_\infty} \leq \|u_0\|_\infty,$$

to conclude that  $\Psi(u) \in B_T$ .

- $\Psi$  is a contraction. For  $(u, v) \in B_T$ , we have

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_{B_T} &\leq \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} (e^{u(\tau)} - e^{v(\tau)}) d\tau ds \right\|_{L^\infty(0,T)} \\ &\leq \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \|e^{u(\tau)} - e^{v(\tau)}\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\ &\leq CT^{2-\gamma} \|e^{u(s)} - e^{v(s)}\|_1 \\ &\leq CT^{2-\gamma} e^{2\|u_0\|_\infty} \|u - v\|_1 \\ &\leq \frac{1}{2} \|u - v\|_1, \end{aligned}$$

thanks to

$$|e^{u(s)} - e^{v(s)}| = e^{\lambda u(s) + \mu v(s)} |u(s) - v(s)|, \quad 0 < \lambda, \mu < 1, \quad \lambda + \mu = 1,$$

whenever  $T$  is chosen such that

$$CT^{2-\gamma}e^{2\|u_0\|_\infty} \leq \frac{1}{2}.$$

Then,  $\Psi$  is contractive on  $B_T$ . According to the Banach fixed point theorem, Equation (1.1) admits a unique integral solution  $u(x, t) \in B_T$ .

### 3.2. Global nonexistence of weak solution

**Definition 3.3 (Weak solution)** Let  $u_0(x) \in L^\infty_{loc}(\mathbb{R}^N)$  and  $T > 0$ . We say that  $u(x, t) \in L^\infty_{loc}(\mathbb{R}^N; L^p(0, T))$  is a weak solution of Equation (1.1) if it satisfies

$$\begin{aligned} &\int_{\mathbb{R}^N} u_0(x) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} I_{0|t}^{1-\gamma} [e^u] \varphi(x, t) dx dt \\ &= - \int_0^T \int_{\mathbb{R}^N} u(x, t) \Delta \mathcal{D}_{t|T}^{1-\alpha} \varphi(x, t) dx dt - \int_0^T \int_{\mathbb{R}^N} u(x, t) \varphi_t(x, t) dx dt, \end{aligned} \tag{3.4}$$

for all compactly supported function  $\varphi(x, t) \in C^2(\mathbb{R}^N; C^1[0, T])$ , such that  $\varphi(x, T) = 0$ .

**Lemma 3.4 (Integral  $\rightarrow$  Weak solution)** Assume  $u_0(x) \in L^\infty(\mathbb{R}^N)$ , let  $u \in L^\infty(\mathbb{R}^N; C[0, T])$  be an integral solution of (1.1), then  $u$  is also a weak solution of (1.1).

For the proof of Lemma 3.4, see [8, Lemma 4.2].

**Theorem 3.5** *Let  $u_0(x) \in L^\infty(\mathbb{R}^N)$  with  $u_0(x) \geq 0$  and  $u_0(x) \not\equiv 0$ . Then problem (1.1) does not admit a global solution.*

**Proof** The proof is by contradiction. Assume that  $u(x, t)$  is a global integral solution of (1.1), then  $u$  is a solution of (1.1) where  $u(x, t) \in L^\infty(\mathbb{R}^N; C[0; T])$ .

Let us choose as in [4]

$$\varphi(x, t) := \varphi_1^l(x)\varphi_2(t),$$

with

$$\varphi_1(x) := \Phi\left(\frac{|x|}{T^{\alpha/2}}\right),$$

and

$$\varphi_2(t) := \begin{cases} \left(1 - \frac{t}{T}\right)^\eta, & t \leq T, \eta > 1, \\ 0, & t > T. \end{cases}$$

The function  $\Phi$  is the smooth nonnegative function

$$\Phi(x) = \begin{cases} 1 & \text{if } x \leq 1, \\ \searrow & \text{if } 1 < x \leq 2, \\ 0 & \text{if } x > 2. \end{cases}$$

According to Definition 3.3 and using Lemma 2.4 for set  $\{x \in \mathbb{R}^N; |x| \leq 2T^{\alpha/2}\} \times [0, T]$ , we get

$$\begin{aligned} & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} I_{t|T}^{1-\gamma} \varphi(x, t) dx dt \\ &= - \int_0^T \int_{\mathbb{R}^N} u \Delta \mathcal{D}_{t|T}^{1-\alpha} \varphi(x, t) dx dt - \int_0^T \int_{\mathbb{R}^N} u \varphi_t(x, t) dx dt. \end{aligned}$$

Moreover, a use of Lemma 2.5 allows us to obtain, as  $u > 0$ ,

$$\begin{aligned} & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} I_{t|T}^{1-\gamma} \varphi(x, t) dx dt \\ & \leq l \int_0^T \int_{\mathbb{R}^N} u |\varphi_1^{l-1}(x) (-\Delta) \varphi_1(x) \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) dx dt \\ & \quad - \int_0^T \int_{\mathbb{R}^N} u \varphi_1^l(x) \frac{d}{dt} \varphi_2(t) dx dt \tag{3.5} \\ & \leq l \int_0^T \int_{\mathbb{R}^N} u \left| (-\Delta) \varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) dx dt + \int_0^T \int_{\mathbb{R}^N} u \left| \frac{d}{dt} \varphi_2(t) \right| dx dt \\ & \leq lM + K, \end{aligned}$$

where

$$M := \int_0^T \int_{\mathbb{R}^N} u \left| (-\Delta) \varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) dx dt,$$



$$K := \int_0^T \int_{\mathbb{R}^N} u \left| \frac{d}{dt} \varphi_2(t) \right| dx dt.$$

Then, using Young's inequality

$$AB \leq \varepsilon e^A + B \ln \frac{B}{\varepsilon e}, \text{ for } A, B > 0, \varepsilon > 0,$$

with  $\varepsilon = \frac{1}{4l} \varphi(x, t)$ ,  $A = u(x, t)$  and  $B = |(-\Delta)\varphi_1(x)| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t)$  for  $M$ , we obtain

$$\begin{aligned} M &\leq \int_0^T \int_{\mathbb{R}^N} \left| (-\Delta)\varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) \ln \left( \frac{4l |(-\Delta)\varphi_1(x)| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t)}{e \varphi(x, t)} \right) dx dt \\ &\quad + \frac{1}{4l} \int_0^T \int_{\mathbb{R}^N} e^{u(x, t)} \varphi(x, t) dx dt. \end{aligned}$$

Similarly, for  $K$  with  $\varepsilon = \frac{1}{4} \varphi(x, t)$ ,  $A = u(x, t)$  and  $B = \left| \frac{d}{dt} \varphi_2(t) \right|$ , we have

$$K \leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \varphi(x, t)} \right) dx dt + \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{u(x, t)} \varphi(x, t) dx dt.$$

Furthermore, using (2.2) we may write

$$\begin{aligned} M &\leq l C_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left( 1 - \frac{t}{T} \right)^{\eta+\alpha-1} \\ &\quad \times \ln \left( \frac{4l C_1 |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left( 1 - \frac{t}{T} \right)^{\eta+\alpha-1}}{e \varphi_1^l(x) \left( 1 - \frac{t}{T} \right)^\eta} \right) dx dt \\ &\quad + \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{u(x, t)} \varphi(x, t) dx dt, \end{aligned}$$

and

$$\begin{aligned} K &\leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \varphi_1^l(x) \left( 1 - \frac{t}{T} \right)^\eta} \right) dx dt \\ &\quad + \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{u(x, t)} \varphi(x, t) dx dt, \end{aligned}$$

where  $C_1 := \frac{\Gamma(\eta + 1)}{\Gamma(\eta + \alpha)}$ .

Consequently, according to (2.1) we finally obtain the estimate

$$\begin{aligned}
 & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + C_2 \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \varphi_1^l(x) T^{1-\gamma} \left(1 - \frac{t}{T}\right)^{\eta-\gamma+1} dx dt \\
 & \leq l C_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1} \\
 & \quad \times \ln\left(\frac{4l C_1 |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1}}{e \varphi_1^l(x) \left(1 - \frac{t}{T}\right)^\eta}\right) dx dt \\
 & \quad + \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln\left(\frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \varphi_1^l(x) \left(1 - \frac{t}{T}\right)^\eta}\right) dx dt \\
 & \quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \varphi(x,t) dx dt,
 \end{aligned} \tag{3.6}$$

where  $C_2 := \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}$ .

At this stage, let us introduce the scaled variables  $\tau = \frac{t}{T}$  and  $y = \frac{x}{T^{\alpha/2}}$ , it follows that

$$dx dt = T^{\frac{\alpha N}{2}+1} dy d\tau,$$

$$\left| \frac{d}{dt} \varphi_2(t) \right| = \eta T^{-1} (1-\tau)^{\eta-1},$$

$$(-\Delta)\varphi_1(x) = T^{-\alpha} (-\Delta_y)\varphi_1(y).$$

Then, for  $\{y \in \mathbb{R}^N, |y| \leq 2\} \times [0, 1]$ , the inequality (3.6) can be rewritten as

$$\begin{aligned}
 & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \varphi_1^l(x,t) \left[ C_2 T^{1-\gamma} \left(1 - \frac{t}{T}\right)^{1-\gamma} - \frac{1}{2} \right] dx dt \\
 & \leq l C_1 T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} |(-\Delta_y)\varphi_1(y)| (1-\tau)^{\eta+\alpha-1} \\
 & \quad \times \ln\left(\frac{4l C_1 T^{\alpha-1} |(-\Delta)\varphi_1(y)| (1-\tau)^{\eta+\alpha-1}}{e \varphi_1^l(y) (1-\tau)^\eta}\right) dy d\tau \\
 & \quad + \eta T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} (1-\tau)^{\eta-1} \ln\left(\frac{4\eta T^{-1} (1-\tau)^{\eta-1}}{e \varphi_1^l(y) (1-\tau)^\eta}\right) dy d\tau.
 \end{aligned}$$

Thus, we have two bounded functions  $\varphi_1(y)$  and  $(-\Delta)\varphi_1(y)$ , also  $\varphi_1(y) \rightarrow 1$  as  $T \rightarrow \infty$ . Using Lebesgue's dominated convergence theorem, we deduce that the right-hand side diverges to  $-\infty$ , while the left-hand side is positive when  $T \rightarrow +\infty$ .

This leads to a contradiction. □

**4. Fractional diffusion system with nonlinearities of exponential growth**

In this section, we study the system:

$$\begin{cases} u_t(x, t) - D_{0|t}^{1-\alpha} \Delta u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} e^{v(x,s)} ds, \\ v_t(x, t) - D_{0|t}^{1-\beta} \Delta v(x, t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} e^{u(x,s)} ds, \end{cases} \tag{4.1}$$

for  $x \in \mathbb{R}^N, t > 0$ , subject to the initial conditions

$$u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, x \in \mathbb{R}^N, \tag{4.2}$$

where  $\alpha, \beta, \gamma, \delta \in (0, 1)$ .

**4.1. Local existence of integral solution**

**Definition 4.1 (Integral solution)** Let  $u_0(x), v_0(x) \in L^\infty(\mathbb{R}^N)$ . We say that  $(u, v) \in L^\infty(\mathbb{R}^N; C[0, T]) \times L^\infty(\mathbb{R}^N; C[0, T])$  is an integral solution of the system (4.1)–(4.2) if  $(u, v)$  satisfies:

$$\begin{cases} u(x, t) = \int_{\mathbb{R}^N} G(x-y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^N} G(x-y, t-\tau) I_{0|s}^{1-\gamma}(e^v) dy d\tau, \\ v(x, t) = \int_{\mathbb{R}^N} G(x-y, t) v_0(y) dy + \int_0^t \int_{\mathbb{R}^N} G(x-y, t-\tau) I_{0|s}^{1-\delta}(e^u) dy d\tau, \end{cases} \tag{4.3}$$

for  $x \in \mathbb{R}^N, t \in [0, T]$ , where

$$G(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\langle x, \xi \rangle} E_{\alpha, 1}(-\xi^2 t^\alpha) d\xi \tag{4.4}$$

is a heat kernel of the problem (4.1)–(4.2) [5].

**Theorem 4.2 (Local existence)** Given  $u_0(x), v_0(x) \in L^\infty(\mathbb{R}^N)$ . Then, there exists a maximal time  $T > 0$  such that the system (4.1)–(4.2) has a unique integral solution  $(u, v) \in L^\infty(\mathbb{R}^N; C[0, T]) \times L^\infty(\mathbb{R}^N; C[0, T])$ .

Proof. For arbitrary  $T > 0$ , we define the ball

$$B_T = \{(u, v) \in L^\infty(\mathbb{R}^N; C[0, T]) \times L^\infty(\mathbb{R}^N; C[0, T]); \\ \|(u, v)\|_{B_T} \leq 2(\|u_0\|_{L^\infty(\mathbb{R}^N)} + \|v_0\|_{L^\infty(\mathbb{R}^N)})\},$$

where  $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbb{R}^N)}$  and  $\|\cdot\|_{B_T}$  is the norm of  $B_T$  defined by

$$\|(u, v)\|_{B_T} = \|u\|_1 + \|v\|_1 = \|u\|_{L^\infty(\mathbb{R}^N, L^\infty(0, T))} + \|v\|_{L^\infty(\mathbb{R}^N, L^\infty(0, T))},$$

and

$$d(u, v) = \max_{t \in [0, T]} \|u(t) - v(t)\|_{L^\infty(\mathbb{R}^N)} \text{ for } u, v \in B_T.$$

Since  $L^\infty(\mathbb{R}^N; C[0, T])$  is a Banach space,  $(B_T; d)$  is a complete metric space.

Next, for every  $(u, v) \in B_T$ , we introduce the map  $\Psi(u, v)$  defined on  $B_T$  by

$$\Psi(u, v) := (\Psi_1(u, v), \Psi_2(u, v)),$$

where

$$\begin{aligned} \Psi_1(u, v) &= \int_{\mathbb{R}^N} G(x - y, t) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\gamma}[e^v] dy d\tau, \quad t \in [0, T], \end{aligned}$$

and

$$\begin{aligned} \Psi_2(u, v) &= \int_{\mathbb{R}^N} G(x - y, t) v_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\delta}[e^u] dy d\tau, \quad t \in [0, T]. \end{aligned}$$

We are going to prove the existence of the local solution as a fixed point of  $\Psi$  via the Banach fixed point theorem.

- $\Psi : B_T \rightarrow B_T$ .

If  $(u, v) \in B_T$ , using Lemma 2.2 in [5], we obtain

$$\begin{aligned} \|\Psi(u, v)\|_{B_T} &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \|e^v\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\ &+ \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_0^s (s-\tau)^{-\delta} \|e^u\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\ &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \|e^v\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\ &+ \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\delta} \|e^u\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\ &\leq \|u_0\|_\infty + \|v_0\|_\infty + \mathcal{C}_1 T^{2-\gamma} e^{\|v\|_1} + \mathcal{C}_2 T^{2-\delta} e^{\|u\|_1}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}_1 &:= \frac{1}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} = \frac{1}{\Gamma(3-\gamma)}, \\ \mathcal{C}_2 &:= \frac{1}{(1-\delta)(2-\delta)\Gamma(1-\delta)} = \frac{1}{\Gamma(3-\delta)}. \end{aligned}$$

Hence, using the fact that  $(u, v) \in B_T$ , we get

$$\begin{aligned} \|\Psi(u, v)\|_{B_T} &\leq \|u_0\|_\infty + \|v_0\|_\infty + \mathcal{C}_1 T^{2-\gamma} e^{\|v\|_1} + T^{2-\delta} e^{\|u\|_1} \\ &\leq (\|u_0\|_\infty + \|v_0\|_\infty) + \max\{\mathcal{C}_1 T^{2-\gamma} e^{\|v\|_1}; \mathcal{C}_2 T^{2-\delta} e^{\|u\|_1}\} \\ &\leq (\|u_0\|_\infty + \|v_0\|_\infty) + T_{\gamma, \delta} e^{2(\|u_0\|_\infty + \|v_0\|_\infty)}, \end{aligned}$$

where

$$T_{\gamma,\delta} = \max\{\mathcal{C}_1 T^{2-\gamma}; \mathcal{C}_2 T^{2-\delta}\}.$$

If we choose  $T_{\gamma,\delta}$  such that

$$2T_{\gamma,\delta} \leq 1,$$

we conclude that  $\|\Psi(u, v)\|_{B_T} \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)$ . Hence  $\Psi(u, v) \in B_T$ .

- $\Psi$  is a contraction.

For  $(u, v), (\tilde{u}, \tilde{v}) \in B_T$ , by the same computations as above, we have

$$\begin{aligned} & \|\Psi(u, v) - \Psi(\tilde{u}, \tilde{v})\|_{B_T} \\ & \leq \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \|e^v - e^{\tilde{v}}\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\ & + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_0^s (s-\tau)^{-\delta} \|e^u - e^{\tilde{u}}\|_\infty d\tau ds \right\|_{L^\infty(0,T)} \\ & = \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \|e^v - e^{\tilde{v}}\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\ & + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\delta} \|e^u - e^{\tilde{u}}\|_\infty ds d\tau \right\|_{L^\infty(0,T)} \\ & = \mathcal{C}_1 T^{2-\gamma} \|e^v - e^{\tilde{v}}\|_1 + \mathcal{C}_2 T^{2-\delta} \|e^u - e^{\tilde{u}}\|_1. \end{aligned}$$

Now, thanks to

$$|e^{u(s)} - e^{v(s)}| = e^{\lambda u(s) + \mu v(s)} |u(s) - v(s)|, \quad 0 < \lambda, \mu < 1, \quad \lambda + \mu = 1,$$

we have the estimate

$$\begin{aligned} \|\Psi(u, v) - \Psi(\tilde{u}, \tilde{v})\|_{B_T} & \leq \mathcal{C}_2 T^{2-\delta} \|e^u - e^{\tilde{u}}\|_1 + \mathcal{C}_1 T^{2-\gamma} \|e^v - e^{\tilde{v}}\|_1 \\ & \leq \mathcal{C}_1 T^{2-\gamma} e^{2(\|u_0\|_\infty + \|v_0\|_\infty)} \|v - \tilde{v}\|_1 \\ & + \mathcal{C}_2 T^{2-\delta} e^{2(\|u_0\|_\infty + \|v_0\|_\infty)} \|u - \tilde{u}\|_1 \\ & \leq T_{\gamma,\delta} \|(u, v) - (\tilde{u}, \tilde{v})\| \\ & \leq \frac{1}{2} \|(u, v) - (\tilde{u}, \tilde{v})\|, \end{aligned}$$

and the choice of  $T_{\gamma,\delta}$

$$T_{\gamma,\delta} \leq \frac{1}{2}.$$

Hence, by the Banach fixed point theorem, the system (4.1)–(4.2) has an integral solution  $(u, v) \in B_T$ .

**4.2. Global nonexistence of weak solution**

**Definition 4.3 (Weak solution)** Let  $u_0(x), v_0(x) \in L^\infty_{loc}(\mathbb{R}^N)$  and  $T > 0$ . Then, we say that  $(u, v) \in L^\infty_{loc}(\mathbb{R}^N; L^p(0, T)) \times L^\infty_{loc}(\mathbb{R}^N; L^p(0, T))$  is a weak solution of the system (4.1)-(4.2) if  $(u, v)$  satisfies

$$\begin{aligned} & \int_{\mathbb{R}^N} u_0 \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} I_{0|t}^{1-\gamma} [e^v] \varphi(x, t) dx dt \\ &= - \int_0^T \int_{\mathbb{R}^N} u \Delta \mathcal{D}_{t|T}^{1-\alpha} \varphi(x, t) dx dt - \int_0^T \int_{\mathbb{R}^N} u \varphi_t(x, t) dx dt, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^N} v_0 \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} I_{0|t}^{1-\delta} [e^u] \varphi(x, t) dx dt \\ &= - \int_0^T \int_{\mathbb{R}^N} v \Delta \mathcal{D}_{t|T}^{1-\beta} \varphi(x, t) dx dt - \int_0^T \int_{\mathbb{R}^N} v \varphi_t(x, t) dx dt, \end{aligned} \tag{4.6}$$

for all compactly supported function  $\varphi(x, t) \in C^2(\mathbb{R}^N; C^1[0, T])$ , such that  $\varphi(x, T) = 0$ .

**Theorem 4.4** Let  $u_0(x), v_0(x) \in L^\infty(\mathbb{R}^N)$  with  $u_0(x), v_0(x) \geq 0$  and  $u_0(x), v_0(x) \not\equiv 0$ . Then, the system (4.1)-(4.2) does not admit global solutions.

**Proof** The proof is by contradiction. Suppose that  $(u, v)$  is a global integral solution to (4.1)-(4.2), then  $(u, v)$  is a weak solution of (4.1)-(4.2) where  $(u, v) \in L^\infty(\mathbb{R}^N; C[0, T]) \times L^\infty(\mathbb{R}^N; C[0, T])$ .

Let us choose as in [3]

$$\varphi(x, t) := \varphi_1^l(x) \varphi_2(t),$$

with

$$\varphi_1(x) := \Phi \left( \frac{|x|}{T^{\theta/2}} \right), \quad \theta = \min\{\alpha, \beta\},$$

and

$$\varphi_2(t) := \begin{cases} \left(1 - \frac{t}{T}\right)^\eta, & t \leq T, \quad \eta > 1, \\ 0, & t > T. \end{cases}$$

The function  $\Phi$  is a smooth nonnegative function such that

$$\Phi(x) = \begin{cases} 1 & \text{if } x \leq 1, \\ \searrow & \text{if } 1 < x \leq 2, \\ 0 & \text{if } x > 2. \end{cases}$$

According to Definition 4.3 and using Lemma 2.4 we get

$$\begin{aligned} & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} I_{t|T}^{1-\gamma} \varphi(x, t) dx dt \\ &= - \int_0^T \int_{\mathbb{R}^N} u \Delta \mathcal{D}_{t|T}^{1-\alpha} \varphi(x, t) dx dt - \int_0^T \int_{\mathbb{R}^N} u \varphi_t(x, t) dx dt, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^N} v_0 \varphi_1^l(x) dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} I_{t|T}^{1-\delta} \varphi(x,t) dx dt \\ &= - \int_0^T \int_{\mathbb{R}^N} v \Delta \mathcal{D}_{t|T}^{1-\beta} \varphi(x,t) dx dt - \int_0^T \int_{\mathbb{R}^N} v \varphi_t(x,t) dx dt. \end{aligned} \tag{4.8}$$

Furthermore, from Lemma 2.5 we obtain, as  $u > 0$ ,

$$\begin{aligned} & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} I_{t|T}^{1-\gamma} \varphi(x,t) dx dt \\ & \leq l \int_0^T \int_{\mathbb{R}^N} u \varphi_1^{l-1}(x) (-\Delta) \varphi_1(x) \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) dx dt \\ & \quad - \int_0^T \int_{\mathbb{R}^N} u \varphi_1^l(x) \frac{d}{dt} \varphi_2(t) dx dt \\ & \leq l \int_0^T \int_{\mathbb{R}^N} u \left| (-\Delta) \varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) dx dt + \int_0^T \int_{\mathbb{R}^N} u \left| \frac{d}{dt} \varphi_2(t) \right| dx dt \\ & \leq l M_1 + K_1, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} I_{t|T}^{1-\delta} \varphi(x,t) dx dt \\ & \leq l \int_0^T \int_{\mathbb{R}^N} v \varphi_1^{l-1}(x) (-\Delta) \varphi_1(x) \mathcal{D}_{t|T}^{1-\beta} \varphi_2(t) dx dt \\ & \quad - \int_0^T \int_{\mathbb{R}^N} v \varphi_1^l(x) \frac{d}{dt} \varphi_2(t) dx dt \\ & \leq l \int_0^T \int_{\mathbb{R}^N} v \left| (-\Delta) \varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\beta} \varphi_2(t) dx dt + \int_0^T \int_{\mathbb{R}^N} v \left| \frac{d}{dt} \varphi_2(t) \right| dx dt \\ & \leq l M_2 + K_2, \end{aligned} \tag{4.10}$$

where

$$M_1 := \int_0^T \int_{\mathbb{R}^N} u \left| (-\Delta) \varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) dx dt,$$

$$K_1 := \int_0^T \int_{\mathbb{R}^N} u \left| \frac{d}{dt} \varphi_2(t) \right| dx dt,$$

$$M_2 := \int_0^T \int_{\mathbb{R}^N} v \left| (-\Delta) \varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\beta} \varphi_2(t) dx dt,$$

$$K_2 := \int_0^T \int_{\mathbb{R}^N} v \left| \frac{d}{dt} \varphi_2(t) \right| dx dt.$$

Hence, using Young's inequality

$$AB \leq \varepsilon e^A + B \ln \frac{B}{\varepsilon e}, \text{ for } A, B > 0, \varepsilon > 0,$$

with

$$\begin{aligned} \varepsilon &= \frac{1}{4l} \frac{\Gamma(\eta + 1)}{\Gamma(\eta - \delta + 2)} (T - t)^{1-\delta} \varphi(x, t), \\ A &= u(x, t), B = |(-\Delta)\varphi_1(x)| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) \text{ for } M_1, \end{aligned}$$

and

$$\begin{aligned} \varepsilon &= \frac{1}{4l} \frac{\Gamma(\eta + 1)}{\Gamma(\eta - \gamma + 2)} (T - t)^{1-\gamma} \varphi(x, t), \\ A &= v(x, t), B = |(-\Delta)\varphi_1(x)| \mathcal{D}_{t|T}^{1-\beta} \varphi_2(t) \text{ for } M_2, \end{aligned}$$

we obtain

$$\begin{aligned} M_1 &\leq \int_0^T \int_{\mathbb{R}^N} \left| (-\Delta)\varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t) \ln \left( \frac{4l |(-\Delta)\varphi_1(x)| \mathcal{D}_{t|T}^{1-\alpha} \varphi_2(t)}{e \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)} (T-t)^{1-\delta} \varphi(x,t)} \right) dx dt \\ &\quad + \frac{1}{4l} \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \frac{\Gamma(\eta + 1)}{\Gamma(\eta - \delta + 2)} (T - t)^{1-\delta} \varphi(x, t) dx dt, \\ M_2 &\leq \int_0^T \int_{\mathbb{R}^N} \left| (-\Delta)\varphi_1(x) \right| \mathcal{D}_{t|T}^{1-\beta} \varphi_2(t) \ln \left( \frac{4l |(-\Delta)\varphi_1(x)| \mathcal{D}_{t|T}^{1-\beta} \varphi_2(t)}{e \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} (T-t)^{1-\gamma} \varphi(x,t)} \right) dx dt \\ &\quad + \frac{1}{4l} \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \frac{\Gamma(\eta + 1)}{\Gamma(\eta - \gamma + 2)} (T - t)^{1-\gamma} \varphi(x, t) dx dt, \end{aligned}$$

respectively. Let

$$\begin{aligned} \varepsilon &= \frac{1}{4} \frac{\Gamma(\eta + 1)}{\Gamma(\eta - \delta + 2)} (T - t)^{1-\delta} \varphi(x, t), \\ A &= u(x, t), B = \left| \frac{d}{dt} \varphi_2(t) \right| \text{ for } K_1, \end{aligned}$$

and

$$\begin{aligned} \varepsilon &= \frac{1}{4} \frac{\Gamma(\eta + 1)}{\Gamma(\eta - \gamma + 2)} (T - t)^{1-\gamma} \varphi(x, t), \\ A &= v(x, t), B = \left| \frac{d}{dt} \varphi_2(t) \right| \text{ for } K_2. \end{aligned}$$

Then, for  $K_1$  and  $K_2$  we have the following estimates

$$\begin{aligned} K_1 &\leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)} (T-t)^{1-\delta} \varphi(x,t)} \right) dx dt \\ &\quad + \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \frac{\Gamma(\eta + 1)}{\Gamma(\eta - \delta + 2)} (T - t)^{1-\delta} \varphi(x, t) dx dt, \end{aligned}$$



and

$$K_2 \leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e^{\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} (T-t)^{1-\gamma}} \varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} (T-t)^{1-\gamma} \varphi(x,t) dxdt.$$

Whereupon, (2.2) allows us to write

$$M_1 \leq l\mathcal{C}_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1}$$

$$\times \ln \left( \frac{4l\mathcal{C}_1 |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1}}{e^{\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)} (T-t)^{1-\delta}} \varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)} (T-t)^{1-\delta} \varphi(x,t) dxdt,$$

and

$$K_1 \leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e^{\frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)} (T-t)^{1-\delta}} \varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)} (T-t)^{1-\delta} \varphi(x,t) dxdt,$$

where  $\mathcal{C}_1 := \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha)}$ .

On the other hand, we have

$$M_2 \leq l\mathcal{C}_2 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1}$$

$$\times \ln \left( \frac{4l\mathcal{C}_2 |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1}}{e^{\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} (T-t)^{1-\gamma}} \varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} (T-t)^{1-\gamma} \varphi(x,t) dxdt,$$

and

$$K_2 \leq \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e^{\frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} (T-t)^{1-\gamma}} \varphi(x,t)} \right) dxdt$$

$$+ \frac{1}{4} \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)} (T-t)^{1-\gamma} \varphi(x,t) dxdt,$$

where  $\mathcal{C}_2 := \frac{\Gamma(\eta+1)}{\Gamma(\eta+\beta)}$ .

Collecting estimates and taking into account (2.1), we finally get

$$\begin{aligned}
 & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \mathcal{C}_3 \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} \varphi(x,t) (T-t)^{1-\gamma} dx dt \\
 & \leq \mathcal{L}_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1} \\
 & \times \ln\left(\frac{4\mathcal{L}_1 |(-\Delta)\varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1}}{e\mathcal{C}_4 (T-t)^{1-\delta} \varphi(x,t)}\right) dx dt \\
 & + \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln\left(\frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e\mathcal{C}_4 (T-t)^{1-\delta} \varphi(x,t)}\right) dx dt \\
 & + \frac{1}{2} \mathcal{C}_4 \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} (T-t)^{1-\delta} \varphi(x,t) dx dt,
 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
 & \int_{\mathbb{R}^N} v_0 \varphi_1^l(x) dx + \mathcal{C}_4 \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} (T-t)^{1-\delta} \varphi(x,t) dx dt \\
 & \leq \mathcal{L}_2 \int_0^T \int_{\mathbb{R}^N} |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1} \\
 & \times \ln\left(\frac{4\mathcal{L}_1 |(-\Delta)\varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1}}{e\mathcal{C}_3 (T-t)^{1-\gamma} \varphi(x,t)}\right) dx dt \\
 & + \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln\left(\frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e\mathcal{C}_3 (T-t)^{1-\gamma} \varphi(x,t)}\right) dx dt \\
 & + \frac{1}{2} \mathcal{C}_3 \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} (T-t)^{1-\gamma} \varphi(x,t) dx dt,
 \end{aligned} \tag{4.12}$$

where

$$\mathcal{C}_3 := \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+2)}, \quad \mathcal{C}_4 := \frac{\Gamma(\eta+1)}{\Gamma(\eta-\delta+2)}.$$

Now, combining (4.11) and (4.12), also reminding that  $u_0, v_0 \geq 0$ , these estimates lead to

$$\begin{aligned}
 & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \frac{3}{4} \mathcal{C}_3 \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} (T-t)^{1-\gamma} \varphi(x,t) dx dt \\
 & \leq l \mathcal{C}_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta) \varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta) \varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1}}{e \mathcal{C}_4 (T-t)^{1-\delta} \varphi(x,t)} \right) dx dt \\
 & \quad + \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \mathcal{C}_4 (T-t)^{1-\delta} \varphi(x,t)} \right) dx dt \\
 & \quad + \frac{l}{2} \mathcal{C}_2 \int_0^T \int_{\mathbb{R}^N} |(-\Delta) \varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta) \varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1}}{e \mathcal{C}_3 (T-t)^{1-\gamma} \varphi(x,t)} \right) dx dt \\
 & \quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \mathcal{C}_3 (T-t)^{1-\gamma} \varphi(x,t)} \right) dx dt,
 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
 & \int_{\mathbb{R}^N} v_0 \varphi_1^l(x) dx + \frac{3}{4} \mathcal{C}_4 \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} (T-t)^{1-\delta} \varphi(x,t) dx dt \\
 & \leq l \mathcal{C}_2 \int_0^T \int_{\mathbb{R}^N} |(-\Delta) \varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta) \varphi_1(x)| T^{\beta-1} \left(1 - \frac{t}{T}\right)^{\eta+\beta-1}}{e \mathcal{C}_3 (T-t)^{1-\gamma} \varphi(x,t)} \right) dx dt \\
 & \quad + \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \mathcal{C}_3 (T-t)^{1-\gamma} \varphi(x,t)} \right) dx dt \\
 & \quad + \frac{l}{2} \mathcal{C}_1 \int_0^T \int_{\mathbb{R}^N} |(-\Delta) \varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta) \varphi_1(x)| T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\eta+\alpha-1}}{e \mathcal{C}_4 (T-t)^{1-\delta} \varphi(x,t)} \right) dx dt \\
 & \quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} \left| \frac{d}{dt} \varphi_2(t) \right| \ln \left( \frac{4 \left| \frac{d}{dt} \varphi_2(t) \right|}{e \mathcal{C}_4 (T-t)^{1-\delta} \varphi(x,t)} \right) dx dt.
 \end{aligned} \tag{4.14}$$

Let us take the scaled variables  $\tau = \frac{t}{T}$  and  $y = \frac{x}{T^{\theta/2}}$ ,  $\theta := \{\alpha; \beta\}$ . Then, inequalities (4.13) and (4.14) become

$$\begin{aligned}
 & \int_{\mathbb{R}^N} u_0 \varphi_1^l(x) dx + \frac{3}{4} \mathcal{C}_3 \int_0^T \int_{\mathbb{R}^N} e^{v(x,t)} (T-t)^{1-\gamma} \varphi(x,t) dx dt \\
 & \leq l \mathcal{C}_1 T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} |(-\Delta_y) \varphi_1(y)| (1-\tau)^{\eta+\alpha-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta_y) \varphi_1(y)| T^{\alpha-1} (1-\tau)^{\eta+\alpha-1}}{e \mathcal{C}_4 T^{1-\delta} (1-\tau)^{1-\delta} \varphi(y,\tau)} \right) dy d\tau \\
 & \quad + \eta T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} (1-\tau)^{\eta-1} \ln \left( \frac{4\eta T^{-1} (1-\tau)^{\eta-1}}{e \mathcal{C}_4 T^{1-\delta} (1-\tau)^{1-\delta} \varphi(y,\tau)} \right) dy d\tau \\
 & \quad + \frac{l}{2} \mathcal{C}_2 T^{\frac{\beta N}{2}} \int_0^1 \int_{\mathbb{R}^N} |(-\Delta_y) \varphi_1(y)| (1-\tau)^{\eta+\beta-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta_y) \varphi_1(y)| T^{\beta-1} (1-\tau)^{\eta+\beta-1}}{e \mathcal{C}_3 T^{1-\gamma} (1-\tau)^{1-\gamma} \varphi(y,\tau)} \right) dy d\tau \\
 & \quad + \frac{\eta}{2} T^{\frac{\beta N}{2}} \int_0^1 \int_{\mathbb{R}^N} (1-\tau)^{\eta-1} \ln \left( \frac{4\eta T^{-1} (1-\tau)^{\eta-1}}{e \mathcal{C}_3 T^{1-\gamma} (1-\tau)^{1-\gamma} \varphi(y,\tau)} \right) dy d\tau,
 \end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
 & \int_{\mathbb{R}^N} v_0 \varphi_1^l(x) dx + \frac{3}{4} \mathcal{C}_4 \int_0^T \int_{\mathbb{R}^N} e^{u(x,t)} (T-t)^{1-\delta} \varphi(x,t) dx dt \\
 & \leq l \mathcal{C}_2 T^{\frac{\beta N}{2}} \int_0^1 \int_{\mathbb{R}^N} |(-\Delta_y) \varphi_1(y)| (1-\tau)^{\eta+\beta-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta_y) \varphi_1(y)| T^{\beta-1} (1-\tau)^{\eta+\beta-1}}{e \mathcal{C}_3 T^{1-\gamma} (1-\tau)^{1-\gamma} \varphi(y,\tau)} \right) dy d\tau \\
 & \quad + \eta T^{\frac{\beta N}{2}} \int_0^1 \int_{\mathbb{R}^N} (1-\tau)^{\eta-1} \ln \left( \frac{4\eta T^{-1} (1-\tau)^{\eta-1}}{e \mathcal{C}_3 T^{1-\gamma} (1-\tau)^{1-\gamma} \varphi(y,\tau)} \right) dy d\tau \\
 & \quad + \frac{l}{2} \mathcal{C}_1 T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} |(-\Delta_y) \varphi_1(y)| (1-\tau)^{\eta+\alpha-1} \\
 & \quad \times \ln \left( \frac{4l \mathcal{C}_1 |(-\Delta_y) \varphi_1(y)| T^{\alpha-1} (1-\tau)^{\eta+\alpha-1}}{e \mathcal{C}_4 T^{1-\delta} (1-\tau)^{1-\delta} \varphi(y,\tau)} \right) dy d\tau \\
 & \quad + \frac{\eta}{2} T^{\frac{\alpha N}{2}} \int_0^1 \int_{\mathbb{R}^N} (1-\tau)^{\eta-1} \ln \left( \frac{4\eta T^{-1} (1-\tau)^{\eta-1}}{e \mathcal{C}_4 T^{1-\delta} (1-\tau)^{1-\delta} \varphi(y,\tau)} \right) dy d\tau.
 \end{aligned} \tag{4.16}$$

Similarly, using Lebesgue’s dominated convergence theorem, we deduce that the right-hand sides of (4.15)-(4.16) diverge to  $-\infty$  when  $T \rightarrow +\infty$ , while the left-hand sides are positive.

This leads to a contradiction. □

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