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Bernstein–Walsh-type inequalities for derivatives of algebraic polynomials on the regions of complex plane

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Abstract: In this paper, we study Bernstein–Walsh-type estimates for the derivatives of an arbitrary algebraic polynomial on some general regions of the complex plane.

Key words: Algebraic polynomial, quasiconformal mapping, smooth curve, quasismooth curve, asymptotic conformal curve

1. Introduction

Let \mathbb{C} be a complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ such that $0 \in G$; $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = \text{ext}L$; $\Delta := \Delta(0,1) := \{w : |w| > 1\}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ such that $\Phi(\infty) = \infty$ and $\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} > 0$; $\Psi := \Phi^{-1}$. For $R > 1$, we take $L_R := \{z : |\Phi(z)| = R\}$, $G_R := \text{int}L_R$ and $\Omega_R := \text{ext}L_R$.

Let \wp_n denote the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $\{z_j\}_{j=1}^l$ be the fixed system of distinct points on curve L . For some fixed R_0 , $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0}$, consider generalized Jacobi weight function $h(z)$, which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^l |z - z_j|^{\gamma_j}, \quad (1.1)$$

where $\gamma_j > -2$, for all $j = 1, 2, \dots, l$, and h_0 is uniformly separated from zero in G_{R_0} , i.e. there exists a constant $c_1(G) > 0$ such that for all $z \in G_{R_0}$, $h_0(z) \geq c_1(G) > 0$.

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Let $0 < p \leq \infty$ and σ be the two-dimensional Lebesgue measure. For the Jordan region G , we introduce:

$$\begin{aligned} \|P_n\|_p & : = \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) |P_n(z)|^p d\sigma_z \right)^{1/p}, \quad 0 < p < \infty, \\ \|P_n\|_\infty & : = \|P_n\|_{A_\infty(1,G)} := \max_{z \in \bar{G}} |P_n(z)|, \quad p = \infty, \\ A_p(1, G) & = : A_p(G), \end{aligned} \tag{1.2}$$

and when L is rectifiable:

$$\begin{aligned} \|P_n\|_{\mathcal{L}_p(h,L)} & : = \left(\int_L h(z) |P_n(z)|^p |dz| \right)^{1/p} < \infty, \quad 0 < p < \infty, \\ \|P_n\|_{\mathcal{L}_\infty(1,L)} & : = \max_{z \in L} |P_n(z)|, \quad p = \infty, \\ \mathcal{L}_p(1, L) & = : \mathcal{L}_p(L). \end{aligned} \tag{1.3}$$

Well known Bernstein–Walsh inequality [49] shows that for any $P_n(z) \in \wp_n$

$$\|P_n\|_{C(\bar{G}_R)} \leq |\Phi(z)|^n \|P_n\|_{C(\bar{G})} \tag{1.4}$$

holds. Thus, for the points $z \in \bar{G}_R$, $R = 1 + \frac{1}{n}$, the $\|P_n\|_C$ -norm in \bar{G}_R and \bar{G} have the same order of growth. Also, in [49] some similar estimates for various norms were given for the right hand side of (1.3).

For the weight function $h(z) \equiv 1$ and arbitrary polynomial $P_n(z) \in \wp_n$ in [33] it was shown:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq |\Phi(z)|^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \quad p > 0. \tag{1.5}$$

The estimate (1.5) has been generalized in [12, Lemma 2.4] for the weight function $h(z) \neq 1$, defined as in (1.1) for the $\gamma_j > -1$, $j = 1, 2, \dots, l$, and it was obtained as follows:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \quad \gamma^* = \max \{0; \gamma_j : 1 \leq j \leq l\}. \tag{1.6}$$

To give a similar estimation to (1.6) for the $A_p(h, G)$ -norm, first of all, we will give the following definition.

Let the function φ maps G conformally and univalently onto $B := B(0, 1) := \{w : |w| < 1\}$ which is normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$ and let $\psi := \varphi^{-1}$.

Following [41, p. 286], a bounded Jordan region G is called a κ -quasidisk, $0 \leq \kappa < 1$ if any conformal mapping ψ can be extended to a K -quasiconformal homeomorphism of the plane $\bar{\mathbb{C}}$ on the $\bar{\mathbb{C}}$ with $K = \frac{1+\kappa}{1-\kappa}$. In that case the curve $L := \partial G$ is called a κ -quasircle. The region G (curve L) is called a quasidisk (quasircle), if it is κ -quasidisk (κ -quasircle) with some $0 \leq \kappa < 1$.

A Jordan curve L is called a quasircle or quasiconformal curve if it is the image of the unit circle under a quasiconformal mapping of $\bar{\mathbb{C}}$ to $\bar{\mathbb{C}}$ ([34]). On the other hand, some geometric criteria of quasiconformality of

the curves was given in [15, p. 81] (also, see [42, p. 107]). It is well known that quasicircles can be nonrectifiable (see, for example, [24], [34, p. 104]).

In [8] (also, see [6]) it was given an analog of the estimates (1.4) and (1.6) for the quasidisks in the norm (1.2), with the weight function $h(z)$ which is defined by (1.1), as follows:

$$\|P_n\|_{A_p(h, G_R)} \leq c_1 R^{*n+\frac{1}{p}} \|P_n\|_{A_p(h, G)}, \quad p > 0,$$

where $R^* := 1 + c_2(R - 1)$ and $c_2 > 0$, $c_1 := c_1(G, p, c_2) > 0$ are constants independent of n and R .

Further, for an arbitrary Jordan region G , any $P_n \in \wp_n$, $R_1 = 1 + \frac{1}{n}$, in [9, Theorem1.1] it was obtained that

$$\|P_n\|_{A_p(G_R)} \leq cR^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, \quad p > 0,$$

is true for arbitrary $R > R_1 = 1 + \frac{1}{n}$, where $c = \left(\frac{2}{e^p-1}\right)^{\frac{1}{p}} [1 + O(\frac{1}{n})]$, $n \rightarrow \infty$.

N. Stylianopoulos in [45] replaced the norm $\|P_n\|_{C(\bar{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (1.4) and found a new version of the Bernstein–Walsh Lemma: Assume that L is quasiconformal and rectifiable. Then there exists a constant $c = c(L) > 0$ depending only on L such that

$$|P_n(z)| \leq c \frac{\sqrt{n}}{d(z, L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega,$$

where $d(z, L) := \inf \{|\zeta - z| : \zeta \in L\}$, holds for every $P_n \in \wp_n$.

In this paper, we continue the study of the problem on pointwise estimates of the derivatives $|P_n^{(m)}(z)|$ for $m \geq 1$, in unbounded regions of the complex plane and we obtained the estimates as the following type:

$$|P_n^{(m)}(z)| \leq \eta_n(G, h, p, m, z) \|P_n\|_p, \quad z \in \Omega, \tag{1.7}$$

where $\eta_n(\cdot) \rightarrow \infty$ as $n \rightarrow \infty$, depending on the properties of the G, h .

Analogous results of (1.7)-type for $m = 0$, a different weight function h , an unbounded region and some norms were obtained in [28, p. 418–428], [36]–[40] and the others.

To get an estimate for the $|P_n^{(m)}(z)|$ on the whole complex plane, we will need an estimate for the $|P_n^{(m)}(z)|$ in the bounded region G . To do this, we will use Bernstein–Markov–Nikolsky type estimates for $|P_n^{(m)}(z)|$, $z \in \bar{G}$, as the following type:

$$\|P_n^{(m)}\|_\infty \leq \lambda_n(G, h, p) \|P_n\|_p, \quad m = 0, 1, 2, \dots, \tag{1.8}$$

where $\lambda_n := \lambda_n(G, h, p, m) > 0$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, is a constant depending on the properties of the region G and the weight function h .

Estimates of (1.8)- type were studied since the beginning of the 20th century([22], [23], [46]). In recent years, (1.8)- type inequalities for various spaces have been studied in [5], [6], [7], [14] (see also the references

cited therein), [18], [26], [27], [28, pp. 418–428], [36], [37, subsection 5.3], [38], [39, pp. 122–133], [44] and the others. For $m = 0$ the similar estimates were continued to be studied in [3]–[8], [10], [13], [20], [21] and the others.

Therefore, combining the estimates (1.7) and (1.8), we will obtain the estimate of $|P_n^{(m)}(z)|$ for any $m = 1, 2, \dots$, in whole complex plane $\mathbb{C} = \overline{G}_R \cup \Omega_R$, $R > 1$:

$$|P_n^{(m)}(z)| \leq c_4 \|P_n\|_p \begin{cases} \lambda_n(G, h, p), & z \in \overline{G}_R \\ \eta_n(G, h, p, d(z, L)) |\Phi(z)|^{n+1}, & z \in \Omega_R, \end{cases} \tag{1.9}$$

where $c_4 = c_4(G, p) > 0$ is a constant independent of n, h, P_n , and $\lambda_n(G, h, p) \rightarrow \infty$, $\eta_n(\cdot) \rightarrow \infty$ as $n \rightarrow \infty$, depending on the properties of the G, h .

2. Definitions and main results

Throughout this paper, c, c_0, c_1, c_2, \dots are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ are sufficiently small positive constants (generally, different in different relations), which in general, depends on G and parameters that inessential for the argument. Otherwise, the dependence will be explicitly stated. For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k + 1, \dots, m$.

Let z_1, z_2 be arbitrary points on L and $L(z_1, z_2)$ denotes the subarc of L of shorter diameter with endpoints z_1 and z_2 . The curve L is a quasicircle if and only if the quantity

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|}$$

is bounded for all $z_1, z_2 \in L$ and $z \in L(z_1, z_2)$ (three point property). A curve L is called "c–quasiconformal" by Lesley (see [35, p. 341]), if there exists the positive constant c , independent of points z_1, z_2 and z such that

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \leq c.$$

The Jordan curve L is called asymptotically conformal [25], [42], if

$$\max_{z \in L(z_1, z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \rightarrow 1, \quad |z_1 - z_2| \rightarrow 0.$$

Some various properties of the asymptotically conformal curves have been studied in [16], [29], [30], [31], [32], [43], and references provided therein. According to the geometric criteria of quasiconformality of the curves ([15, p. 81], [42, p. 107]) every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. The asymptotically conformal curves can be nonrectifiable.

As stated in [41, p. 163], we say that a bounded Jordan curve L is λ –quasismooth or Lavrentiev curve if for every pair $z_1, z_2 \in L$, where $l(z_1, z_2)$ denote the shorter subarc of L , joining $z_1 \in L$ and $z_2 \in L$ and $|l(z_1, z_2)|$ is the linear measure (length) of $l(z_1, z_2)$, there exists a constant $\lambda := \lambda(L) \geq 1$, such that

$$|l(z_1, z_2)| \leq \lambda |z_1 - z_2|, \quad z_1, z_2 \in L.$$

In this case, the inner region $intL$ of a Lavrentiev curve L is called a Lavrentiev region.

Let S be a rectifiable Jordan curve or an arc and let $z = z(s)$, $s \in [0, |S|]$, $|S| := \text{mes } S$, be the natural parametrization of S .

A Jordan curve or an arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. We will write $G \in C_\theta$, if $\partial G \in C_\theta$.

Following [41, p. 48](see also [19, p. 32]), we say that a Jordan curve S called Dini-smooth, if it has a parametrization $z = z(s)$, $0 \leq s \leq |S| := \text{mes } S$ such that $z'(s) \neq 0$, $0 \leq s \leq |S|$ and $|z'(s_2) - z'(s_1)| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

A Jordan region G has a piecewise Dini-smooth boundary, if $L := \partial G$ consists of the union of finite Dini-smooth arcs L_j , $j = \overline{1, m}$, such that they have exterior (with respect to \overline{G}) angles $\lambda_j \pi$, $0 < \lambda_j < 2$, at the corner points $\{z_j\}$, $j = \overline{1, m}$, where two arcs meet.

According to the "three-point" criterion [34, p. 100], every piecewise C_θ -curve and Dini-smooth curve (without cusps) is quasiconformal.

Now, we give the definitions of quasidisks with some general functional conditions.

Definition 2.1 We say that $G \in Q_\alpha$ if G is a quasicircle and $\Phi \in H^\alpha(\overline{\Omega})$ for some $0 < \alpha \leq 1$.

We note that the class Q_α is sufficiently large. A detailed account on it and the related topics are contained in [35], [42], [47] (see also the references cited therein). We consider only some cases:

- a) If L is a Dini-smooth curve [42], then $G \in Q_1$.
- b) If L is a piecewise Dini-smooth curve and largest exterior angle on L has opening $\alpha\pi$, $0 < \alpha < 1$, [42, p.52], then $G \in Q_\alpha$.
- c) If $L =: \partial G$ is a smooth curve having continuous tangent line, then $G \in Q_\alpha$ for all $0 < \alpha < 1$.
- d) If G is "L-shaped" region, then $\Phi \in Lip \frac{2}{3}$.
- e) If L is quasismooth (in the sense of Lavrentiev), then $\Phi \in Lip \alpha$ for $\alpha = \frac{1}{2}(1 - \frac{1}{\pi} \arcsin \frac{1}{c})^{-1}$ [47], [48].
- f) If L is "c-quasiconformal" (see, for example, [35]), then $\Phi \in Lip \alpha$ for $\alpha = \frac{\pi}{2(\pi - \arcsin \frac{1}{c})}$.
- g) If L is an asymptotic conformal curve, then $\Phi \in Lip \alpha$ for all $0 < \alpha < 1$ [35].

Now, we can state the corresponding results for the class of regions $G \in Q_\alpha$. First, we give one more theorem that we will use in this case, and after that we give estimate for $|P_n^{(m)}(z)|$, $z \in \overline{G}$, for each $m \geq 0$.

Theorem A [14, Th. 1] Let $0 < p \leq \infty$; $L \in Q_\alpha$ for some $0 < \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 0, 1, 2, \dots$

$$\|P_n^{(m)}\|_\infty \leq c_4 \|P_n\|_p \begin{cases} n^{\frac{1}{\alpha}(\frac{\gamma^*+2}{p}+m)}, & \frac{1}{2} \leq \alpha \leq 1, \\ n^{\delta(\frac{\gamma^*+2}{p}+m)}, & 0 < \alpha < \frac{1}{2}, \end{cases} \tag{2.1}$$

where here and throughout this paper

$$\gamma^* := \max \{0; \gamma_j, j = \overline{1, l}\}, \tag{2.2}$$

and $\delta = \delta(G)$, $1 \leq \delta \leq 2$, is a certain number depending on the specified region.

Recall that for any $m \geq 1$ and for $m = 0$, the sharpness of estimate (2.1) was given in [13, Th. 2.1].

For $0 < \delta_j < \delta_0 := \frac{1}{4} \min \{|z_i - z_j| : i, j = 1, 2, \dots, l, i \neq j\}$, let $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min_{1 \leq j \leq l} \delta_j$; for $L = \partial G$ we set: $U_\infty(L, \delta) := \bigcup_{\zeta \in L} U(\zeta, \delta)$ —infinite open cover of the curve L ; $U_N(L, \delta) := \bigcup_{j=1}^N U_j(L, \delta) \subset U_\infty(L, \delta)$ —finite open cover of the curve L ; $\Omega(\delta) := \Omega(L, \delta) := \Omega \cap U_N(L, \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$; $\Omega_R(\delta) := \Omega(L_R, \delta) := \Omega_R \cap U_N(L_R, \delta)$, $\widehat{\Omega}_R := \Omega_R \setminus \Omega_R(\delta)$.

Now, we start to formulate the new results.

2.1. The general estimate (recurrence formula)

First, we present a general estimate of $|P_n^{(m)}(z)|$ for which it will be possible to obtain estimates for the derivative for each order $m = 1, 2, \dots$

Theorem 2.2 *Let $p \geq 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, \dots$, we have:*

$$|P_n^{(m)}(z)| \leq c_1 |\Phi^{n+1}(z)| \left\{ \frac{\|P_n\|_p}{d(z, L)} A_{n,p}^1(z, m) + \sum_{j=1}^m C_m^j B_{n,j}^1(z) |P_n^{(m-j)}(z)| \right\}, \tag{2.3}$$

where $c_1 = c_1(G, \gamma, m, p) > 0$ is a constant independent of n and z ;

$$A_{n,p}^1(z, m) : = \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+m-1\right)\frac{1}{\alpha}}, & p \geq 2, & m \geq 2, & \gamma > -2, \\ n^{\frac{\gamma^*+2}{p\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha, \end{cases}$$

$$B_{n,j}^1(z) : = n^{\frac{j}{\alpha}}, j = 1, 2, \dots, m,$$

if $z \in \Omega(\delta)$, and

$$A_{n,p}^1(z, m) : = \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & 2 \leq p < \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma \leq \alpha, \end{cases}$$

$$B_{n,j}^1(z) : = n, j = \overline{1, m},$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 2.3 Let $1 < p < 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, \dots$, we have:

$$\left| P_n^{(m)}(z) \right| \leq c_2 \left| \Phi^{n+1}(z) \right| \left\{ \frac{\|P_n\|_p}{d(z, L)} A_{n,p}^2(z, m) + \sum_{j=1}^m C_m^j B_{n,j}^1(z) \left| P_n^{(m-j)}(z) \right| \right\}, \tag{2.4}$$

where $c_2 = c_2(G, \gamma, m, p) > 0$ is a constant independent of n and z ;

$$A_{n,p}^2(z, m) := n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}},$$

if $z \in \Omega(\delta)$,

$$A_{n,p}^2(z, m) := \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, & p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}} (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & p > 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma \leq 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$ and $B_{n,j}^1(z)$ is defined as in Theorem 2.2.

In order to get the final estimates for $\left| P_n^{(m)}(z) \right|$, $m \geq 2$, we need an estimate for the $\left| P_n^{(m-j)}(z) \right|$ for each $j = \overline{1, m}$. Consider the case $m = 2$ and $j = 1, 2$. Cases of $m \geq 3$ are carried out sequentially by applying the estimates obtained from (2.3) and (2.4).

2.2. Estimate for $|P_n(z)|$

Theorem 2.4 Let $p > 1$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and $z \in \Omega_R$, we have:

$$\left| P_n(z) \right| \leq c_3 \frac{\left| \Phi^{n+1}(z) \right|}{d(z, L)} A_{n,p}^3 \|P_n\|_p, \tag{2.5}$$

where $c_3 = c_3(G, \gamma, p) > 0$ is a constant independent of n and z ;

$$A_{n,p}^3 := \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha, \\ n^{(1-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}-1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

for any $p \geq 2$ and

$$A_{n,p}^3 := \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}} (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

for $1 < p < 2$.

The following remark shows the sharpness of Theorem 2.4.

Remark 2.5 For any $n \in \mathbb{N}$ there exists $Q_n \in \wp_n$ and $G^* \subset \mathbb{C}$ such that

$$|Q_n(z)| \geq c \frac{\sqrt{n}}{d(z, L)} \|Q_n\|_{A_2(G^*)} |\Phi(z)|^{n+1}, \quad z \in F \Subset \Omega^* := CG^*,$$

where $c = c(G^*) > 0$.

We note that the estimate for $|P_n(z)|$ was previously obtained in our paper [13, Th. 3] for $p > 0$. But this result is better for $p \geq 2$ and coincides with that of [13, Th. 3] for $1 < p < 2$.

2.3. Estimate for $|P'_n(z)|$

Now, using Theorems 2.2, 2.3 and 2.4, we can give an estimate for $|P'_n(z)|$ for $z \in \Omega_R$.

Theorem 2.6 Let $p \geq 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P'_n(z)| \leq c_4 \frac{|\Phi^{2(n+1)}(z)|}{d(z, L)} \|P_n\|_p A_{n,p}^4(z), \tag{2.6}$$

where $c_4 = c_4(G, \gamma, p) > 0$ is a constant independent of n and z ;

$$A_{n,p}^4(z) := \begin{cases} n^{\frac{\gamma+2}{p\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{\frac{1}{\alpha}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{\frac{1}{\alpha}+1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{\frac{1}{\alpha}+1-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha, \\ n^{\frac{3}{p}+(2-\frac{2}{p})\frac{1}{\alpha}-1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, and

$$A_{n,p}^4(z) := \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}+1}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{2-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{2-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{2-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha, \\ n^{(1-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 2.7 Let $1 < p < 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P'_n(z)| \leq c_5 \frac{|\Phi^{2(n+1)}(z)|}{d(z, L)} \|P_n\|_p A_{n,p}^5(z), \tag{2.7}$$

where $c_5 = c_5(G, \gamma, p) > 0$ is a constant independent of n and z ;

$$A_{n,p}^5(z) := \begin{cases} n^{\frac{\gamma+2}{p\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{\frac{\gamma+2}{p\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\frac{2}{p\alpha}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\frac{2}{p\alpha}+1-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{\frac{2}{p\alpha}+1-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, and

$$A_{n,p}^5(z) = \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}+1}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}+1}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+1} (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+2-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+2-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

2.4. Estimate for $|P_n''(z)|$

Considering the estimates obtained in Theorems 2.6, 2.7 for $|P_n'(z)|$ and Theorem 2.4 for $|P_n(z)|$ in the Theorems 2.2, 2.3 respectively, we can give the following:

Theorem 2.8 Let $p \geq 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P_n''(z)| \leq c_6 \frac{|\Phi^{3(n+1)}(z)|}{d(z, L)} \|P_n\|_p A_{n,p}^6(z), \tag{2.8}$$

where $c_6 = c_6(G, \gamma, p) > 0$ is a constant independent of n and z ;

$$A_{n,p}^6(z) := \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & 0 \leq \gamma < \alpha, \\ n^{(3-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}-1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, and

$$A_{n,p}^6(z) := \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha \\ n^{(1-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}+1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 2.9 Let $1 < p < 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P_n''(z)| \leq c_7 \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} \|P_n\|_p A_{n,p}^7(z), \tag{2.9}$$

where $c_7 = c_7(G, \gamma, p) > 0$ is a constant independent of n and z ;

$$A_{n,p}^7(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}+1\right)\frac{1}{\alpha}} (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}+1\right)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{\left(\frac{2}{p}+1\right)\frac{1}{\alpha}+1-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, and

$$A_{n,p}^7(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{\left(\frac{\gamma+2}{p}+1\right)\frac{1}{\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+3-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

2.5. Estimates for $|P_n'(z)|$ and $|P_n''(z)|$ in whole plane

According to (1.4) (applied to the polynomial $Q_{n-1}(z) := P_n'(z)$), the estimation (2.1) is also true for the points $z \in \overline{G}_R$, $R = 1 + \varepsilon_0 n^{-1}$, with a different constant. Therefore, combining the estimates (2.1) (for the $z \in \overline{G}_R$) with (2.6), (2.7), (2.8), (2.9), we will obtain the estimate on the growth of $|P_n'(z)|$ and $|P_n''(z)|$ respectively, in the whole complex plane:

Theorem 2.10 Let $p \geq 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P_n'(z)| \leq c_8 \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+1\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{|\Phi^{2(n+1)}(z)|}{d(z,L)} A_{n,p}^4(z), & z \in \Omega_R, \end{cases}$$

where $c_8 = c_8(G, \gamma, p) > 0$ is a constant independent of n and z ; $A_{n,p}^4(z)$ is defined as in Theorem 2.6 for all $z \in \Omega_R$.

Theorem 2.11 Let $1 < p < 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P_n'(z)| \leq c_9 \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+1\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{|\Phi^{2(n+1)}(z)|}{d(z,L)} A_{n,p}^5(z), & z \in \Omega_R, \end{cases}$$

where $c_9 = c_9(G, \gamma, p) > 0$ is a constant independent of n and z ; $A_{n,p}^5(z)$ is defined as in Theorem 2.7 for all $z \in \Omega_R$.

Theorem 2.12 Let $p \geq 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P_n''(z)| \leq c_{10} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+2\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A_{n,p}^6(z), & z \in \Omega_R, \end{cases}$$

where $c_{10} = c_{10}(G, \gamma, p) > 0$ is a constant independent of n and z ; $A_{n,p}^6(z)$ is defined as in Theorem 2.8 for all $z \in \Omega_R$.

Theorem 2.13 Let $1 < p < 2$; $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, and $h(z)$ be defined by (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$|P_n''(z)| \leq c_{11} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+2\right)\frac{1}{\alpha}}, & z \in \overline{G}_R, \\ \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A_{n,p}^7(z), & z \in \Omega_R, \end{cases}$$

where $c_{11} = c_{11}(G, \gamma, p) > 0$ is a constant independent of n and z ; $A_{n,p}^7(z)$ is defined as in Theorem 2.9 for all $z \in \Omega_R$.

Thus, using Theorems 2.2, 2.3 and estimating the $|P_n^{(m)}(z)|$ sequentially for each $m \geq 3$, and combining the obtained estimates with Theorem A, we can obtain estimates for the $|P_n^{(m)}(z)|$ on the whole complex plane.

3. Some auxiliary results

Throughout this paper we denote “ $a \preceq b$ ” and “ $a \asymp b$ ” are equivalent to $a \leq cb$ and $c_1a \leq b \leq c_2a$ for some constants c, c_1, c_2 respectively.

Lemma 3.1 [1] Let G be a quasidisk, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then

- a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.
- b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{c_1} \asymp \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \asymp \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{c_2},$$

where $0 < r_0 < 1$ is a constant depending on G .

Corollary 3.2 Under the conditions of Lemma 3.1, we have:

$$|w_1 - w_2|^{c_1} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^\varepsilon,$$

where $\varepsilon = \varepsilon(G) < 1$.

Lemma 3.3 *Let $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$. Then, for all $w_1, w_2 \in \overline{\Omega}'$, we have:*

$$|\Psi(w_1) - \Psi(w_2)| \geq |w_1 - w_2|^{\frac{1}{\alpha}}.$$

This fact follows from of an appropriate result for the mapping $f \in \Sigma(\kappa)$ [41, p. 287] and estimation for the Ψ' [17, Th. 2.8]:

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{3.1}$$

Lemma 3.4 [11] *Let L be a K -quasiconformal curve; $R = 1 + \frac{c}{n}$. Then, for any fixed $\varepsilon \in (0, 1)$ there exists a level curve $L_{1+\varepsilon(R-1)}$ such that the following holds for any polynomial $P_n(z) \in \wp_n$,*

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{|\Phi'|}, L_{1+\varepsilon(R-1)}\right)} \preceq n^{\frac{1}{p}} \|P_n\|_p, \quad p > 0.$$

Let $\{z_j\}_{j=1}^l$ be a fixed system of the points on L and the weight function $h(z)$ be defined as in (1.1). The following result is the integral analog of the familiar lemma of Bernstein–Walsh [49, p. 101] for the $A_p(h, G)$ -norm.

Lemma 3.5 [5] *Let G be a quasidisk and $P_n(z)$, $\deg P_n \leq n, n = 1, 2, \dots$, is arbitrary polynomial and weight function $h(z)$ satisfied the condition(1.1). Then for any $R > 1, p > 0$ and $n = 1, 2, \dots$*

$$\|P_n\|_{A_p(h, G_R)} \leq c_3 (1 + c(R - 1))^{n + \frac{1}{p}} \|P_n\|_{A_p(h, G)}$$

where c, c_3 are independent of n and G .

4. Proof of Theorems

Proof [of Theorems 2.2 and 2.3]. The proofs of Theorem 2.2 and 2.3 will be simultaneously. Let $G \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$ and let $R = 1 + \frac{1}{n}, R_1 := 1 + \frac{R-1}{2}$. For $z \in \Omega$ assume that:

$$H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}.$$

We represent the m -th derivative of $H_{n,p}(z)$ as follows:

$$\begin{aligned} H_n^{(m)}(z) &= \left(\frac{P_n(z)}{\Phi^{n+1}(z)} \right)^{(m)} \\ &= \sum_{j=0}^m C_m^j \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} P_n^{(m-j)}(z) \\ &= \frac{P_n^{(m)}(z)}{\Phi^{n+1}(z)} + \sum_{j=1}^m C_m^j \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} P_n^{(m-j)}(z), \end{aligned}$$

where $C_m^j := \frac{m(m-1)\dots(m-j+1)}{j!}$. Therefore,

$$P_n^{(m)}(z) = \Phi^{n+1}(z) \left\{ \left(\frac{P_n(z)}{\Phi^{n+1}(z)} \right)^{(m)} - \sum_{j=1}^m C_m^j \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} P_n^{(m-j)}(z) \right\}, z \in \Omega.$$

Then,

$$\left| P_n^{(m)}(z) \right| \leq |\Phi^{n+1}(z)| \left\{ \left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)} \right)^{(m)} \right| + \sum_{j=1}^m C_m^j \left| \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} \right| \left| P_n^{(m-j)}(z) \right| \right\}. \tag{4.1}$$

As can be seen from (4.1), the following two statements on the right side need to be evaluated for $z \in \Omega$ in order to obtain the evaluation for $\left| P_n^{(m)}(z) \right|$:

$$\text{A) } \left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)} \right)^{(m)} \right|; \text{ B) } \left| \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} \right|.$$

Now, we start to evaluate them.

A) Since the function $H_n(z)$, $H_n(\infty) = 0$, is analytic in Ω , continuous on $\bar{\Omega}$, then Cauchy integral representation for the derivatives gives:

$$H_n^{(m)}(z) = -\frac{1}{2\pi i} \int_{L_{R_1}} H_n(\zeta) \frac{d\zeta}{(\zeta - z)^{m+1}}, z \in \Omega_R, m \geq 1.$$

Then,

$$\left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)} \right)^{(m)} \right| \leq \frac{1}{2\pi} \int_{L_{R_1}} \left| \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|^{m+1}} \leq \frac{1}{2\pi d(z, L_{R_1})} \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m}. \tag{4.2}$$

Denote by

$$A_n(z) := \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m}, \tag{4.3}$$

and estimate these integrals separately.

For this we give some notations.

Let $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. For $\eta := \min \{ \eta_j, j = \overline{1, l} \}$, where $\eta_j = \min_{t \in \partial \Phi(\Omega(z_j, \delta_j))} |t - w_j| > 0$, we set that:

$$\Delta_j(\eta_j) : = \{ t : |t - w_j| \leq \eta_j \} \subset \Phi(\Omega(z_j, \delta_j)),$$

$$\Delta(\eta) : = \bigcup_{j=1}^l \Delta_j(\eta), \widehat{\Delta}_j = \Delta \setminus \Delta(\eta_j); \widehat{\Delta}(\eta) := \Delta \setminus \Delta(\eta); \Delta'_1 := \Delta'_1(1),$$

$$\Delta'_1(\rho) : = \left\{ t = Re^{i\theta} : R \geq \rho > 1, \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},$$

$$\Delta'_j : = \Delta'_j(1), \Delta'_j(\rho) := \left\{ t = Re^{i\theta} : R \geq \rho > 1, \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_0}{2} \right\}, j = 2, 3, \dots, l,$$

where $\varphi_0 = 2\pi - \varphi_l$; $\Omega_j := \Psi(\Delta'_j)$, $L^j_{R_1} := L_{R_1} \cap \Omega_j$; $\Omega = \bigcup_{j=1}^l \Omega_j$.

For the simplicity of the calculations, we can limit ourselves to only one point on the boundary where the weight function has singularities, i.e. let $h(z)$ be defined as in (1.1) for $l = 1$ and we put: $\gamma_1 =: \gamma$. To estimate $A_n(z)$, first of all replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrand by $|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}$ $|\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$\begin{aligned} A_n(z) &= \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m} = \sum_{i=1}^2 \int_{F^i_{R_1}} \frac{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma_i}{p}} |P_n(\Psi(\tau)) (\Psi'(\tau))^{\frac{2}{p}}| |\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}} |\Psi(\tau) - \Psi(w)|^m} |d\tau| \\ &\leq \sum_{i=1}^2 \left(\int_{F^i_{R_1}} |\Psi(\tau) - \Psi(w_1)|^\gamma |P_n(\Psi(\tau))|^p |\Psi'(\tau)|^2 |d\tau| \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{F^i_{R_1}} \left(\frac{|\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}} |\Psi(\tau) - \Psi(w)|^m} \right)^q |d\tau| \right)^{\frac{1}{q}} =: \sum_{i=1}^2 A_n^i(z), \end{aligned}$$

where $F^1_{R_1} := \Phi(L^1_{R_1}) = \Delta'_1 \cap \{\tau : |\tau| = R_1\}$, $F^2_{R_1} := \Phi(L_{R_1}) \setminus F^1_{R_1}$ and

$$\begin{aligned} A_n^i(z) &: = \left(\int_{F^i_{R_1}} |f_{n,p}(\tau)|^p |d\tau| \right)^{\frac{1}{p}} \left(\int_{F^i_{R_1}} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \right)^{\frac{1}{q}} \\ &= : J^i_{n,1} \cdot J^i_{n,2}(z), \\ f_{n,p}(\tau) &: = (\Psi(\tau) - \Psi(w_1))^{\frac{\gamma}{p}} P_n(\Psi(\tau)) (\Psi'(\tau))^{\frac{2}{p}}, \quad |\tau| = R_1. \end{aligned}$$

Applying to Lemma 3.4, we get:

$$J^i_{n,1} \preceq n^{\frac{1}{p}} \|P_n\|_p, \quad i = 1, 2.$$

For the estimation of the integral

$$(J^i_{n,2}(z))^q = \int_{F^i_{R_1}} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \tag{4.4}$$

for $i = 1, 2$, we set:

$$\begin{aligned} E^{11}_{R_1} &: = \{\tau : \tau \in F^1_{R_1}, |\tau - w_1| < c_1(R_1 - 1)\}, \\ E^{12}_{R_1} &: = \{\tau : \tau \in F^1_{R_1}, c_1(R_1 - 1) \leq |\tau - w_1| < \eta\}, \\ E^{13}_{R_1} &: = \{\tau : \tau \in \Phi(L_{R_1}), \eta \leq |\tau - w_1| < \eta^*\}, \eta^* = \eta^*(G), \end{aligned} \tag{4.5}$$

where $0 < c_1 < \eta$ is chosen so that $\{\tau : |\tau - w_1| < c_1(R_1 - 1)\} \cap \Delta \neq \emptyset$ and $\Phi(L_{R_1}) = \bigcup_{k=1}^3 E_{R_1}^{1k}$. Taking into consideration these notations, from (4.4) we have:

$$J_{n,2}^1(z) + J_{n,2}^2(z) =: J_2(z) = J_2(E_{R_1}^{11}) + J_2(E_{R_1}^{12}) + J_2(E_{R_1}^{13}) =: J_2^1(z) + J_2^2(z) + J_2^3(z)$$

and, consequently,

$$\begin{aligned} A_n(z) &= A_n^1(z) + A_n^2(z) \leq n^{\frac{1}{p}} \|P_n\|_p \cdot [J_2^1(z) + J_2^2(z) + J_2^3(z) + J_2^4(z)] \\ &=: A_{n,1}(z) + A_{n,2}(z) + A_{n,3}(z), \end{aligned} \tag{4.6}$$

where

$$A_{n,k}(z) := n^{\frac{1}{p}} \|P_n\|_p \cdot J_2^k(z), \quad k = 1, 2, 3,$$

$$(J_2^k(z))^q := \int_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}}, \quad k = 1, 2, 3.$$

For any $k = 1, 2$, denote by

$$E_{R_1,1}^{1k} := \{\tau \in E_{R_1}^{1k} : |\Psi(\tau) - \Psi(w_1)| \geq |\Psi(\tau) - \Psi(w)|\}, \quad E_{R_1,2}^{1k} := E_{R_1}^{1k} \setminus E_{R_1,1}^{1k},$$

$$\begin{aligned} (I(E_{R_1,1}^{1k}))^q &: = \begin{cases} \int_{E_{R_1,1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}}, & \text{if } \gamma \geq 0, \\ \int_{E_{R_1,1}^{1k}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}}, & \text{if } \gamma < 0, \end{cases} \\ (I(E_{R_1,2}^{1k}))^q &: = \int_{E_{R_1,2}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+qm}}, \quad k = 1, 2, \end{aligned} \tag{4.7}$$

and estimate the last integrals.

Given the possible values of q ($q > 2$ and $q \leq 2$) and γ ($-2 < \gamma < 0$ and $\gamma \geq 0$), we will consider the cases separately.

Case 1. Let $1 < q \leq 2$ ($p \geq 2$). Then,

$$(I(E_{R_1,1}^{1k}))^q = \int_{E_{R_1,1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}}.$$

1.1. Let $\gamma \geq 0$. If $z \in \Omega(\delta)$, applying Lemma 3.3 to (3.1), we get:

$$(I(E_{R_1,1}^{11}))^q \leq \int_{E_{R_1,1}^{11}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}} \tag{4.8}$$

$$\begin{aligned} & \asymp \int_{E_{R_1,1}^{11}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}} \preceq n^{2-q} \int_{E_{R_1,1}^{11}} \frac{|d\tau|}{|\tau - w|^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}} \\ & \asymp n^{2-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}} mes E_{R_1,1}^{11} \preceq n^{1-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}; \end{aligned}$$

$$I(E_{R_1,1}^{11}) \preceq n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}.$$

$$\left(I(E_{R_1,2}^{11})\right)^q \preceq \int_{E_{R_1,2}^{11}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+qm}}$$

$$\preceq n^{2-q} \int_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}} \preceq n^{2-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}} mes E_{R_1,2}^{11}$$

$$\preceq n^{1-q+\frac{\gamma(q-1)+qm-(2-q)}{\alpha}};$$

$$I(E_{R_1,2}^{11}) \preceq n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}.$$

$$\left(I(E_{R_1,1}^{12})\right)^q \preceq \int_{E_{R_1,1}^{12}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}} \tag{4.9}$$

$$\preceq \int_{E_{R_1,1}^{12}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}} \preceq n^{2-q} \int_{E_{R_1,1}^{11}} \frac{|d\tau|}{|\tau - w|^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha}}}$$

$$\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha} - 1}, & \gamma(q-1) + qm - (2-q) > \alpha, \\ \ln n, & \gamma(q-1) + qm - (2-q) = \alpha, \\ 1, & \gamma(q-1) + qm - (2-q) < \alpha; \end{cases}$$

$$I(E_{R_1,1}^{12}) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}, & \gamma > -2, \quad m \geq 2, \\ n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, \quad m = 1, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, \quad m = 1, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, \quad m = 1. \end{cases}$$

$$\left(I(E_{R_1,2}^{12})\right)^q \preceq \int_{E_{R_1,2}^{12}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+qm}} \preceq n^{2-q} \int_{E_{R_1,2}^{12}} \frac{|d\tau|}{|\tau - w_1|^{\gamma(q-1)+qm-(2-q)}} \tag{4.10}$$

$$\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)+qm-(2-q)}{\alpha} - 1}, & \gamma(q-1) + qm - (2-q) > \alpha, \\ \ln n, & \gamma(q-1) + qm - (2-q) = \alpha, \\ 1, & \gamma(q-1) + qm - (2-q) < \alpha; \end{cases}$$

$$I(E_{R_1,2}^{12}) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}, & \gamma > -2, \quad m \geq 2, \\ n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, \quad m = 1, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, \quad m = 1, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, \quad m = 1, \end{cases}$$

$$I(E_{R_1,1}^{12}) + I(E_{R_1,2}^{12}) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > -2, & m \geq 2, \\ n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1. \end{cases}$$

For $\tau \in E_{R_1}^{13}$ we see that $\eta < |\tau - w_1| < 2\pi\hat{R}_1$, $|\tau - w| \geq \eta - c_1$. Therefore, $|\Psi(\tau) - \Psi(w_1)| \geq 1$, from Lemma 3.1 and for $|\tau - w_1| \geq \eta$, $|\Psi(\tau) - \Psi(w)| \geq |\tau - w|^{\frac{1}{\alpha}}$, from Lemma 3.3. Then, for $w \in \Delta(w_1, \eta)$, applying (3.1), we get:

$$\begin{aligned} (J_2^3(z))^q &\preceq \int_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \preceq \int_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \\ &\preceq n^{2-q} \int_{E_{R_1,1}^1} \frac{|d\tau|}{|\tau - w|^{\frac{qm-(2-q)}{\alpha}}} \\ &\preceq n^{2-q} \begin{cases} n^{\frac{qm-(2-q)}{\alpha}-1}, & qm - (2-q) > \alpha, & m \geq 2, \\ n^{\frac{qm-(2-q)}{\alpha}-1}, & qm - (2-q) > \alpha, & m = 1, \\ \ln n, & qm - (2-q) = \alpha, & m = 1, \\ 1, & qm - (2-q) < \alpha, & m = 1; \end{cases} \\ J_2^3(z) &\preceq \begin{cases} n^{\left(\frac{2}{p}+m-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & p \geq 2, & m \geq 2, \\ n^{\frac{2}{p\alpha}-\frac{1}{p}}, & p < 1 + \frac{2}{\alpha}, & m = 1, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{2}{\alpha}, & m = 1, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{2}{\alpha}, & m = 1, \end{cases} \quad z \in \Omega(\delta), \tag{4.11} \\ J_2^3(z) &\preceq n^{\left(\frac{1}{\alpha}-1\right)\left(1-\frac{2}{p}\right)}, \quad z \in \widehat{\Omega}(\delta). \end{aligned}$$

Combining (4.8–4.11), for $p \geq 2, \gamma \geq 0$ and $z \in \Omega(\delta)$, we get:

$$\sum_{k=1}^3 J_2^k(z) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > -2, & m \geq 2, \\ n^{\left(\frac{\gamma}{p}+1\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1. \end{cases} \tag{4.12}$$

If $z \in \widehat{\Omega}(\delta)$, then

$$\begin{aligned} (J_2^1(z))^q &\preceq \int_{E_{R_1}^{11}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq \int_{E_{R_1}^{11}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)}{\alpha}}} \\ &\preceq n^{2-q} \int_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}} \preceq n^{2-q+\frac{\gamma(q-1)-(2-q)}{\alpha}} \text{mes} E_{R_1}^{11} \\ &\preceq n^{1-q+\frac{\gamma(q-1)-(2-q)}{\alpha}}; \\ J_2^1(z) &\preceq n^{\frac{\gamma}{p\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}. \end{aligned} \tag{4.13}$$

$$\begin{aligned}
 (J_2^2(z))^q &\preceq \int_{E_{R_1}^{12}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq \int_{E_{R_1}^{12}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)}{\alpha}}} \\
 &\preceq n^{2-q} \int_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}} \\
 &\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)-(2-q)}{\alpha} - 1}, & \gamma(q-1) - (2-q) > \alpha, \\ \ln n, & \gamma(q-1) - (2-q) = \alpha, \\ 1, & \gamma(q-1) - (2-q) < \alpha; \end{cases} \\
 J_2^2(z) &\preceq \begin{cases} n^{\frac{\gamma}{p\alpha} - (1-\frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}, & 2 \leq p < \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{(1-\frac{2}{p})(\ln n)^{1-\frac{1}{p}}}, & p = \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p > \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & -2 < \gamma \leq \alpha. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (J_2^3(z))^q &\preceq \int_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq \int_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(2-q)}; \\
 J_2^3(z) &\preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})},
 \end{aligned}$$

and, from (4.6)–(4.13), we obtain:

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, & \gamma > -2, p \geq 2 & m \geq 2, \\ n^{\frac{\gamma+2}{p\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha} & m = 1, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha} & m = 1, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, \end{cases} \tag{4.14}$$

if $z \in \Omega(\delta)$ and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma \leq \alpha, \end{cases} \tag{4.15}$$

if $z \in \widehat{\Omega}(\delta)$.

1.2. If $\gamma < 0$, for $w \in \Delta(w_1, \eta) \cap \Omega_R(\delta)$, such that $|\Psi(\tau) - \Psi(w_1)| \preceq |\Psi(\tau) - \Psi(w)|$, according to Lemma 3.1,

analogously we have:

$$\begin{aligned}
 (I(E_{R_1,1}^{11}))^q &\preceq \int_{E_{R_1,1}^{11}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \tag{4.16} \\
 &\preceq n^{2-q} \int_{E_{R_1,1}^{11}} \frac{|d\tau|}{|\tau - w|^{\frac{qm-(2-q)}{\alpha}}} \preceq n^{2-q + \frac{qm-(2-q)}{\alpha}} \text{mes} E_{R_1}^{11} \\
 &\preceq n^{2-q + \frac{qm-(2-q)}{\alpha} - 1}, \\
 I(E_{R_1,1}^{11}) &\preceq n^{\frac{m}{\alpha} - (1 - \frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,2}^{11}))^q &\asymp \int_{E_{R_1,2}^{11}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1) + qm}} \tag{4.17} \\
 &\preceq n^{2-q} \int_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1) + qm - (2-q)}{\alpha}}} \\
 &\preceq n^{2-q + \frac{\gamma(q-1) + qm - (2-q)}{\alpha}} \text{mes} E_{R_1,2}^{11} \preceq n^{2-q + \frac{\gamma(q-1) + qm - (2-q)}{\alpha} - 1}, \\
 I(E_{R_1,2}^{11}) &\preceq n^{(\frac{\gamma}{p} + m)\frac{1}{\alpha} - (1 - \frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

For $\tau \in E_{R_1}^{12}$ we see that $|\tau - w_1| < \eta$ and from Lemma 3.1, $|\Psi(\tau) - \Psi(w_1)| \preceq 1$.

Then, for $w \in \Delta(w_1, \eta) \cap \Omega_R(\delta)$, such that $|\Psi(\tau) - \Psi(w_1)| \preceq |\Psi(\tau) - \Psi(w)|$, applying Lemma 3.3, we get:

$$\begin{aligned}
 (I(E_{R_1,1}^{12}))^q &= \int_{E_{R_1,1}^{12}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \tag{4.18} \\
 &\preceq n^{2-q} \int_{E_{R_1,1}^{12}} \frac{|d\tau|}{|\tau - w|^{\frac{qm-(2-q)}{\alpha}}} \preceq n^{2-q} \int_{E_{R_1,1}^{12}} \frac{1}{|\tau - w|^{\frac{qm-(2-q)}{\alpha}}} |d\tau| \preceq n^{2-q + \frac{qm-(2-q)}{\alpha} - 1}, \\
 I(E_{R_1,1}^{12}) &\preceq n^{\frac{m}{\alpha} - (1 - \frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,2}^{12}))^q &\asymp \int_{E_{R_1,2}^{12}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm + \gamma(q-1)}} \tag{4.19} \\
 &\preceq n^{2-q} \int_{E_{R_1,2}^{12}} \frac{|d\tau|}{|\tau - w|^{\frac{qm + \gamma(q-1) - (2-q)}{\alpha}}} \\
 &\preceq n^{2-q} \begin{cases} n^{\frac{qm + \gamma(q-1) - (2-q)}{\alpha} - 1}, & qm + \gamma(q-1) - (2-q) > \alpha, \\ \ln n, & qm + \gamma(q-1) - (2-q) = \alpha, \\ 1, & qm + \gamma(q-1) - (2-q) < \alpha; \end{cases}
 \end{aligned}$$

$$I(E_{R_1,2}^{12}) \preceq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p \geq 2, & m \geq 2, & \gamma > -2, \\ n^{\left(\frac{\gamma}{p}+1\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha. \end{cases}$$

For $\tau \in E_{R_1}^{13}$ and each $w \in \Delta(w_1, \eta) \cap \Omega_R(\delta)$ we see that $\eta < |\tau - w_1| < 2\pi\dot{R}_1$. Therefore, from Lemma 3.1 and applying (3.1), we get:

$$\begin{aligned} (I(E_{R_1}^{13}))^q & : = \int_{E_{R_1,1}^{13}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \tag{4.20} \\ & \asymp \int_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \\ & \preceq n^{2-q} \int_{E_{R_1}^{13}} \frac{|d\tau|}{|\tau - w|^{\frac{qm-(2-q)}{\alpha}}} \preceq n^{1-q + \frac{qm-(2-q)}{\alpha}}; \\ I(E_{R_1}^{13}) & \preceq n^{\frac{m}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}. \end{aligned}$$

Therefore, combining (4.16)–(4.20) in case of $\gamma < 0$ for $z \in \Omega(\delta)$, we have:

$$\sum_{k=1}^3 J_2^k(z) \preceq n^{\frac{m}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}} \tag{4.21}$$

$$\begin{aligned} & + \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p \geq 2, & m \geq 2, & \gamma > -2, \\ n^{\left(\frac{\gamma}{p}+1\right)\frac{1}{\alpha}-\left(1-\frac{2}{p}\right)\frac{1}{\alpha}-\frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha, \end{cases} \\ & \preceq \begin{cases} n^{\frac{m}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}, & p \geq 2, & m \geq 2, & \gamma > -2, \\ n^{\frac{1}{\alpha} - \left(1-\frac{2}{p}\right)\frac{1}{\alpha} - \frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha. \end{cases} \end{aligned}$$

If $z \in \widehat{\Omega}(\delta)$, then $|w - w_1| \geq \eta$, from Lemma 3.3 and from (3.1), we get:

$$\begin{aligned}
 (J_2^1(z))^q &= \int_{E_{R_1}^{11}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} \\
 &\preceq \int_{E_{R_1}^{11}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau| \\
 &\preceq n^{2-q} \int_{E_{R_1}^{11}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)+2-q} |d\tau| \preceq n^{2-q} \text{mes} E_{R_1}^{11} \preceq n^{1-q} \preceq 1; \\
 J_2^1(z) &\preceq 1.
 \end{aligned}
 \tag{4.22}$$

$$\begin{aligned}
 (J_2^2(z))^q &\preceq \int_{E_{R_1}^{12}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau| \\
 &\preceq \int_{E_{R_1}^{12}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |d\tau| \preceq n^{2-q}; \\
 J_2^2(z) &\preceq n^{1-\frac{2}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (J_2^3(z))^q &\preceq \int_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq \int_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |d\tau| \preceq n^{2-q}; \\
 J_2^3(z) &\preceq n^{1-\frac{2}{p}}.
 \end{aligned}$$

Combining the last three estimates, in case of $\gamma < 0$ for $z \in \widehat{\Omega}(\delta)$, we have:

$$\sum_{k=1}^3 J_2^k(z) \preceq n^{1-\frac{2}{p}}.
 \tag{4.23}$$

Then, for the $\gamma < 0$, from (4.21-4.23), we obtain:

$$\sum_{k=1}^3 J_2^k(z) \preceq \begin{cases} n^{\frac{m}{\alpha} - (1-\frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}, & p \geq 2, & m \geq 2, & \gamma > -2, & z \in \Omega(\delta), \\ n^{\frac{1}{\alpha} - (1-\frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, & z \in \Omega(\delta), \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, & z \in \Omega(\delta), \\ n^{1-\frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, & z \in \Omega(\delta), \\ n^{1-\frac{2}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha, & z \in \Omega(\delta), \\ n^{1-\frac{2}{p}}, & p \geq 2, & & & z \in \widehat{\Omega}(\delta), \end{cases}$$

and, consequently, in this case from (4.6), we have:

$$A_n(z) \preceq n^{\frac{1}{p}} \|P_n\|_p \cdot \begin{cases} n^{\frac{m}{\alpha} - (1 - \frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}, & p \geq 2, & m \geq 2, & \gamma > -2, & z \in \Omega(\delta), \\ n^{\frac{1}{\alpha} - (1 - \frac{2}{p})\frac{1}{\alpha} - \frac{1}{p}}, & p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, & z \in \Omega(\delta), \\ n^{1 - \frac{2}{p}} (\ln n)^{1 - \frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, & z \in \Omega(\delta), \\ n^{1 - \frac{2}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & 0 > \gamma > -2 + \alpha, & z \in \Omega(\delta), \\ n^{1 - \frac{2}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha, & z \in \Omega(\delta), \\ n^{1 - \frac{2}{p}}, & p \geq 2, & & & z \in \widehat{\Omega}(\delta). \end{cases} \quad (4.24)$$

Therefore, combining (4.12) and (4.24), for any $\gamma > -2$, $p \geq 2$, we obtain:

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma^*+2}{p} + m - 1)\frac{1}{\alpha}}, & p \geq 2, & m \geq 2, & \gamma > -2, \\ n^{(\frac{\gamma^*+2}{p})\frac{1}{\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ (n \ln n)^{1 - \frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1 - \frac{1}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1 - \frac{1}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha, \end{cases} \quad (4.25)$$

if $z \in \Omega(\delta)$, and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p} + m - 1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ (n \ln n)^{1 - \frac{1}{p}}, & p = \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1 - \frac{1}{p}}, & p > \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1 - \frac{1}{p}}, & p \geq 2, & 0 \leq \gamma \leq \alpha, \\ n^{1 - \frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases} \quad (4.26)$$

if $z \in \widehat{\Omega}(\delta)$.

Case 2. Let $q > 2$ ($p < 2$). Then, $2 - q < 0$, and so

$$\begin{aligned} (I(E_{R_1,1}^{1k}))^q & : = \begin{cases} \int_{E_{R_1,1}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}}, & \text{if } \gamma \geq 0, \\ \int_{E_{R_1,1}^{1k}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w)|^{qm}}, & \text{if } \gamma < 0, \end{cases} \quad (4.27) \\ (I(E_{R_1,2}^{1k}))^q & : = \int_{E_{R_1,2}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+q}}, \quad k = 1, 2, \\ (J_2^3(z))^q & : = \int_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}}. \end{aligned}$$

2.1. If $\gamma \geq 0$ and $z \in \Omega(\delta)$, applying Lemmas 3.1 and 3.3 to (4.27), we obtain:

$$\begin{aligned}
 (I(E_{R_1,1}^{11}))^q &\preceq \int_{E_{R_1,1}^{11}} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}} & (4.28) \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,1}^{11}} \frac{|d\tau|}{|\tau - w|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha} + (\frac{1}{\alpha}-1)(q-2)} mes E_{R_1,1}^{11} \\
 &\preceq n^{\frac{\gamma(q-1)+qm}{\alpha} + (\frac{1}{\alpha}-1)(q-2)-1}; \\
 I(E_{R_1,1}^{11}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,2}^{11}))^q &\preceq \int_{E_{R_1,2}^{11}} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+qm}} & (4.29) \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha} + (\frac{1}{\alpha}-1)(q-2)} mes E_{R_1,2}^{11} \\
 &\preceq n^{\frac{\gamma(q-1)+qm}{\alpha} + (\frac{1}{\alpha}-1)(q-2)-1}; \\
 I(E_{R_1,2}^{11}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,1}^{12}))^q &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,1}^{12}} \frac{|d\tau|}{|\tau - w|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha} + (\frac{1}{\alpha}-1)(q-2)-1}; \\
 I(E_{R_1,1}^{12}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,2}^{12}))^q &\preceq \int_{E_{R_1,2}^{12}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+qm}} & (4.30) \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,2}^{12}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)+qm}{\alpha}}} \preceq n^{\frac{\gamma(q-1)+qm}{\alpha} + (\frac{1}{\alpha}-1)(q-2)-1}; \\
 I(E_{R_1,2}^{12}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

For $\tau \in E_{R_1}^{13}$ we see that $\eta < |\tau - w_1| < 2\pi\dot{R}_1$. Therefore, from Lemma 3.1, we have $|\Psi(\tau) - \Psi(w_1)| \succeq 1$. For $w \in \Delta(w_1, \eta)$ $|\Psi(\tau) - \Psi(w)| \succeq |\tau - w|^{\frac{1}{\alpha}}$. Then, applying Lemma 3.3, we get:

$$\begin{aligned}
 (J_2^3(z))^q &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{13}} \frac{|d\tau|}{|\tau - w|^{\frac{qm}{\alpha}}} \preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{qm}{\alpha} - 1}; & (4.31) \\
 J_2^3(z) &\preceq n^{\frac{m}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

If $z \in \widehat{\Omega}(\delta)$, then $|w - w_1| \geq \eta$, from (3.1) we have:

$$\begin{aligned} (J_2^1(z))^q &\preceq \int_{E_{R_1}^{11}} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)}{\alpha}}} \\ &\preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{\gamma(q-1)}{\alpha}} \text{mes} E_{R_1}^{11} \preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{\gamma(q-1)}{\alpha} - 1}; \\ J_2^1(z) &\preceq n^{\frac{\gamma}{p\alpha} + (\frac{2}{p}-1)(\frac{1}{\alpha}-1) - \frac{1}{p}}. \end{aligned}$$

$$\begin{aligned} (J_2^2(z))^q &\preceq \int_{E_{R_1}^{12}} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)}{\alpha}}} \\ &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \begin{cases} n^{\frac{\gamma(q-1)}{\alpha} - 1}, & \gamma(q-1) > \alpha, \\ \ln n, & \gamma(q-1) = \alpha, \\ 1, & \gamma(q-1) < \alpha; \end{cases} \\ J_2^2(z) &\preceq \begin{cases} n^{\frac{\gamma}{p\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}, & \gamma > \alpha(p-1), \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)} (\ln n)^{1-\frac{1}{p}}, & \gamma = \alpha(p-1), \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}, & \gamma < \alpha(p-1). \end{cases} \end{aligned}$$

$$\begin{aligned} (J_2^3(z))^q &= \int_{E_{R_1}^{13}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} \\ &\preceq \int_{E_{R_1}^{13}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}} \preceq \int_{E_{R_1}^{13}} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(q-2)}; \\ J_2^3(z) &\preceq n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}. \end{aligned}$$

From (4.28–4.31) and (4.6), for $\gamma \geq 0, 1 < p < 2, m \geq 1$, we have:

$$A_n(z) \preceq \|P_n\|_p n^{(\frac{\gamma+2}{p} + m - 1)\frac{1}{\alpha}}, \tag{4.32}$$

if $z \in \Omega(\delta)$ and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p} + m - 1)\frac{1}{\alpha}}, & \gamma > (p-1)\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha} + 1 - \frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & \gamma = (p-1)\alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha} + 1 - \frac{1}{p}}, & \gamma < (p-1)\alpha, \end{cases} \tag{4.33}$$

if $z \in \widehat{\Omega}(\delta)$.

2.2. Let $\gamma < 0$. For $z \in \Omega(\delta)$, according to Lemma 3.1, we have:

$$\begin{aligned}
 (I(E_{R_1,1}^{11}))^q &= \int_{E_{R_1,1}^{11}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w)|^{qm+\gamma(q-1)}} \tag{4.34} \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,1}^{11}} \left(\frac{|\tau - w_1|}{|\tau - w|}\right)^{-\gamma(q-1)\frac{1}{\alpha_1}} \frac{|d\tau|}{|\tau - w|^{\frac{qm+\gamma(q-1)}{\alpha}}} \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{\gamma(q-1)+qm}{\alpha}} \text{mes} E_{R_1,1}^{11} \preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{\gamma(q-1)+qm}{\alpha} - 1}; \\
 I(E_{R_1,1}^{11}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,2}^{11}))^q &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{qm+\gamma(q-1)}{\alpha}}} \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{qm+\gamma(q-1)}{\alpha}} \text{mes} E_{R_1,2}^{11} \preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{qm+\gamma(q-1)}{\alpha} - 1}; \\
 I(E_{R_1,2}^{11}) &\preceq n^{(\frac{\gamma}{p}+m)\frac{1}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,1}^{12}))^q &\preceq \int_{E_{R_1,1}^{12}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w)|^{qm}} \preceq \int_{E_{R_1,1}^{12}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w)|^{qm}} \tag{4.35} \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,1}^{12}} \frac{|d\tau|}{|\tau - w|^{\frac{qm}{\alpha}}} \preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{qm}{\alpha} - 1}; \\
 I(E_{R_1,1}^{12}) &\preceq n^{\frac{m}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (I(E_{R_1,2}^{12}))^q &\preceq \int_{E_{R_1,2}^{12}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w)|^{qm}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,2}^{12}} \frac{|d\tau|}{|\tau - w|^{\frac{qm}{\alpha}}} \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{qm}{\alpha} - 1}; \\
 I(E_{R_1,2}^{12}) &\preceq n^{\frac{m}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (J_2^3(z))^q &\preceq \int_{E_{R_1}^{13}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w)|^{qm}} \tag{4.36} \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{13}} \frac{|d\tau|}{|\tau - w|^{\frac{qm}{\alpha}}} \preceq n^{(\frac{1}{\alpha}-1)(q-2) + \frac{qm}{\alpha} - 1}; \\
 J_2^3(z) &\preceq n^{\frac{m}{\alpha} + (\frac{2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

For the $z \in \widehat{\Omega}(\delta)$, analogously we obtain:

$$J_2^i(z) \preceq n^{\left(\frac{2}{p}-1\right)\left(\frac{1}{\alpha}-1\right)}, i = 1, 2, 3. \tag{4.37}$$

So, for $\gamma < 0$, from (4.6), we have:

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{2}{p}+m\right)\frac{1}{\alpha}}, & \text{if } z \in \Omega(\delta), \\ n^{\left(\frac{2}{p}-1\right)\left(\frac{1}{\alpha}-1\right)+\frac{1}{p}}, & \text{if } z \in \widehat{\Omega}(\delta). \end{cases} \tag{4.38}$$

Therefore, for any $\gamma \geq -2, 1 < p < 2, m \geq 1$, from (4.32), (4.33) and (4.38), we get:

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & \gamma \geq 0, \\ n^{\left(\frac{2}{p}+m-1\right)\frac{1}{\alpha}}, & \gamma < 0, \end{cases} \tag{4.39}$$

if $z \in \Omega(\delta)$, and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & \gamma > (p-1)\alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & \gamma = (p-1)\alpha, \\ n^{\left(\frac{2}{p}-1\right)\frac{1}{\alpha}+1-\frac{1}{p}}, & -2 < \gamma < (p-1)\alpha, \end{cases} \tag{4.40}$$

if $z \in \widehat{\Omega}(\delta)$.

B) Now, we begin to estimate the $\left| \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} \right|$. Since $\Phi(\infty) = \infty$, then Cauchy integral representation for the region Ω_R gives:

$$\left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} = -\frac{1}{2\pi i} \int_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^{j+1}}, \quad z \in \Omega_R.$$

If we take the modulus of both sides, we get:

$$\left| \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} \right| \leq \frac{1}{2\pi} \int_{L_{R_1}} \frac{1}{|\Phi^{n+1}(\zeta)|} \frac{|d\zeta|}{|\zeta - z|^{j+1}} \leq \frac{1}{2\pi} \int_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^{j+1}}.$$

Replacing the variable $\tau = \Phi(\zeta)$ and according to (3.1), we obtain:

$$\left| \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} \right| \preceq \int_{|\tau|=R_1} \frac{d(\Psi(\tau), L)}{|\tau| - 1} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{j+1}} \tag{4.41}$$

$$\preceq n \int_{|\tau|=R_1} \frac{|d\tau|}{|\tau - w|^{\frac{j}{\alpha}}} \preceq \begin{cases} n^{\frac{j}{\alpha}}, & \text{if } z \in \Omega(\delta), \\ n, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases} \quad j = \overline{1, m}.$$

Combining estimates (4.1)–(4.6), (4.25), (4.26), (4.39), (4.40) and (4.41), we get:

$$\left| P_n^{(m)}(z) \right| \leq |\Phi^{n+1}(z)| \cdot \left[\frac{A_n(z)}{d(z, L)} + \sum_{j=1}^m C_m^j \left| P_n^{(m-j)}(z) \right| \right] \begin{cases} n^{\frac{j}{\alpha}}, & \text{if } z \in \Omega(\delta), \\ n, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases} \tag{4.42}$$

where for any $\gamma > -2, p \geq 2, m \geq 1$

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+m-1\right)\frac{1}{\alpha}}, & p \geq 2, & m \geq 2, & \gamma > -2, \\ n^{\left(\frac{\gamma^*+2}{p}\right)\frac{1}{\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & m = 1, & \gamma > -2 + \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & m = 1, & -2 < \gamma \leq -2 + \alpha, \end{cases}$$

if $z \in \Omega(\delta)$ and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)\frac{1}{\alpha}}, & 2 \leq p < \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\gamma+2}{1+\alpha} + \frac{\alpha}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma \leq \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$; for any $\gamma \geq -2, 1 < p < 2, m \geq 1$,

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*+2}{p}+m-1\right)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > -2, & \text{if } z \in \Omega(\delta), \\ n^{\left(\frac{\gamma+2}{p}-1\right)\frac{1}{\alpha}+\frac{1}{p}}, & \gamma > (p-1)\alpha, & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)}(\ln n)^{1-\frac{1}{p}}, & \gamma = (p-1)\alpha, & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\left(\frac{1}{\alpha}-1\right)\left(\frac{2}{p}-1\right)}, & -2 < \gamma < (p-1)\alpha, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases}$$

and $\gamma^* := \max\{0; \gamma\}$. Therefore, the proofs of Theorems 2.2 and 2.3 are completed.

Now, we start to evaluate $|P_n(z)|$. For this, we will make the necessary evaluations by writing $m = 0$ in the above proof. Since the function $H_n(z) = \frac{P_n(z)}{\Phi^{n+1}(z)}$, $H_n(\infty) = 0$, is analytic in Ω , continuous on $\overline{\Omega}$, then Cauchy integral representation for the region Ω_{R_1} gives

$$H_n(z) = -\frac{1}{2\pi i} \int_{L_{R_1}} H_n(\zeta) \frac{d\zeta}{\zeta - z}, z \in \Omega_{R_1}.$$

Then,

$$\left| \frac{P_n(z)}{\Phi^{n+1}(z)} \right| \leq \frac{1}{2\pi} \int_{L_{R_1}} \left| \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|} \leq \frac{1}{2\pi d(z, L_{R_1})} \int_{L_{R_1}} |P_n(\zeta)| |d\zeta| \tag{4.43}$$

and so,

$$|P_n(z)| \preceq \frac{|\Phi^{n+1}(z)|}{d(z, L_{R_1})} \int_{L_{R_1}} |P_n(\zeta)| |d\zeta|.$$

Denote by

$$A_n := \int_{L_{R_1}} |P_n(\zeta)| |d\zeta|, \tag{4.44}$$

and estimate this integral. To estimate A_n , first of all replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrand by $|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}} |\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$\begin{aligned} A_n &= \int_{L_{R_1}} |P_n(\zeta)| |d\zeta| = \sum_{i=1}^2 \int_{F_{R_1}^i} \frac{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma_i}{p}} |P_n(\Psi(\tau)) (\Psi'(\tau))^{\frac{2}{p}}| |\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}} |d\tau| \\ &\leq \sum_{i=1}^2 \left(\int_{F_{R_1}^i} |\Psi(\tau) - \Psi(w_1)|^\gamma |P_n(\Psi(\tau))|^p |\Psi'(\tau)|^2 |d\tau| \right)^{\frac{1}{p}} \times \left(\int_{F_{R_1}^i} \left(\frac{|\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}} \right)^q |d\tau| \right)^{\frac{1}{q}} \\ &= : \sum_{i=1}^2 A_n^i, \end{aligned}$$

where $F_{R_1}^1 := \Phi(L_{R_1}^1) = \Delta_1' \cap \{\tau : |\tau| = R_1\}$, $F_{R_1}^2 := \Phi(L_{R_1}) \setminus F_{R_1}^1$ and

$$\begin{aligned} A_n^i &: = \left(\int_{F_{R_1}^i} |f_{n,p}(\tau)|^p |d\tau| \right)^{\frac{1}{p}} \left(\int_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} |d\tau| \right)^{\frac{1}{q}} \\ &= : J_{n,1}^i \cdot J_{n,2}^i, \\ f_{n,p}(\tau) &:= (\Psi(\tau) - \Psi(w_1))^{\frac{\gamma}{p}} P_n(\Psi(\tau)) (\Psi'(\tau))^{\frac{2}{p}}, \quad |\tau| = R_1. \end{aligned}$$

Applying Lemma 3.4, we get:

$$J_{n,1}^i \leq n^{\frac{1}{p}} \|P_n\|_p, \quad i = 1, 2.$$

Therefore, we need to evaluate the following integrals:

$$(J_{n,2}^i)^q = \int_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} |d\tau|, \quad i = 1, 2.$$

For the estimation of the integral $J_{n,2}^i$, for $i = 1, 2$, we use notations (4.5) and (4.6) and, consequently, we need to evaluate the following statement:

$$A_n = n^{\frac{1}{p}} \|P_n\|_p \cdot (J_2^1 + J_2^2 + J_2^3), \tag{4.45}$$

where

$$(J_2^k)^q := \int_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}}, \quad k = 1, 2, 3.$$

So, for any $k = 1, 2, 3$ we will estimate the integrals J_2^k . Given possible values of q ($q > 2$ and $q \leq 2$) and γ ($-2 < \gamma < 0$ and $\gamma \geq 0$), we will consider the cases separately.

Case 1. Let $1 < q \leq 2$ ($p \geq 2$).

1.1. Let $\gamma \geq 0$. Applying Lemma 3.3, we get:

$$\begin{aligned}
 (J_2^1)^q &= \int_{E_{R_1}^{11}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq \int_{E_{R_1}^{11}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \tag{4.46} \\
 &\preceq n^{2-q} \int_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}} \preceq n^{2-q + \frac{\gamma(q-1)-(2-q)}{\alpha}} \text{mes} E_{R_1}^{11} \\
 &\preceq n^{2-q + \frac{\gamma(q-1)-(2-q)}{\alpha} - 1}; \\
 J_2^1 &\preceq n^{(\frac{\gamma+2}{p} - 1)\frac{1}{\alpha} - \frac{1}{p}}.
 \end{aligned}$$

$$\begin{aligned}
 (J_2^2)^q &\preceq \int_{E_{R_1}^{12}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(2-q)} \int_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)-(2-q)}{\alpha}}} \tag{4.47} \\
 &\preceq n^{2-q} \begin{cases} n^{\frac{\gamma(q-1)-(2-q)}{\alpha} - 1}, & \gamma(q-1) - (2-q) > \alpha, \\ \ln n, & \gamma(q-1) - (2-q) = \alpha, \\ 1, & \gamma(q-1) - (2-q) < \alpha; \end{cases} \\
 J_2^2 &\preceq \begin{cases} n^{(\frac{\gamma+2}{p} - 1)\frac{1}{\alpha} - \frac{1}{p}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & -2 < \gamma \leq \alpha. \end{cases}
 \end{aligned}$$

For $\tau \in E_{R_1}^{13}$ we see that $\eta < |\tau - w_1| < 2\pi R_1$. Therefore, $|\Psi(\tau) - \Psi(w_1)| \succeq 1$, from Lemma 3.1 and applying (3.1), we get:

$$\begin{aligned}
 (J_2^3)^q &\preceq \int_{E_{R_1}^{13}} |\Psi'(\tau)|^{2-q} |d\tau| \preceq \int_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |d\tau| \preceq n^{(\frac{1}{\alpha} - 1)(2-q)}; \tag{4.48} \\
 J_2^3 &\preceq n^{(\frac{1}{\alpha} - 1)(1 - \frac{2}{p})}.
 \end{aligned}$$

Combining (4.46–4.48), for $p \geq 2, \gamma \geq 0$ and $z \in \Omega_R$, we get:

$$\sum_{k=1}^3 J_2^k \preceq \begin{cases} n^{(\frac{\gamma+2}{p} - 1)\frac{1}{\alpha} - \frac{1}{p}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & 0 < \gamma \leq \alpha. \end{cases} \tag{4.49}$$

From (4.45)–(4.49), we obtain:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p} - 1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2, 0 < \gamma \leq \alpha. \end{cases} \tag{4.50}$$

1.2. If $\gamma < 0$, analogously we have:

$$(J_2^1)^q \preceq \int_{E_{R_1}^{11}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |d\tau| \tag{4.51}$$

$$\preceq n^{2-q} \int_{E_{R_1}^{11}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)+(2-q)} |d\tau| \preceq n^{2-q} \int_{E_{R_1}^{11}} |d\tau|$$

$$\preceq n^{2-q} \text{mes} E_{R_1}^{11} \preceq n^{1-q};$$

$$J_2^1 \preceq 1.$$

For $\tau \in E_{R_1}^{12}$, we get:

$$(J_2^2)^q \preceq \int_{E_{R_1}^{12}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(2-q)}; \tag{4.52}$$

$$J_2^2 \preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})}.$$

$$(J_2^3)^q \preceq \int_{E_{R_1}^{12}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1} \right)^{2-q} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(2-q)}; \tag{4.53}$$

$$J_2^3 \preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})}.$$

Therefore, combining (4.51)–(4.53) in case of $\gamma < 0$ for $z \in \Omega_R$, we have:

$$\sum_{k=1}^3 J_2^k \preceq n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})},$$

and, consequently, in this case from (4.45), we have:

$$A_n \preceq \|P_n\|_p \cdot n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})+\frac{1}{p}}, \quad z \in \Omega_R. \tag{4.54}$$

Therefore, combining (4.49) and (4.54), for any $\gamma > -2, p \geq 2, z \in \Omega_R$, we obtain:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 < \gamma \leq \alpha, \\ n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})+\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0. \end{cases} \tag{4.55}$$

Case 2. Let $q > 2$ ($p < 2$). Then, $2 - q < 0$, and so

$$(J_2^k(z))^q := \int_{E_{R_1}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}}, k = 1, 2, 3. \tag{4.56}$$

2.1. If $\gamma \geq 0$, applying Lemmas 3.1, 3.3 and (3.1), we obtain:

$$\begin{aligned} (J_2^1)^q &\preceq \int_{E_{R_1,2}^{11}} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1,2}^{11}} \frac{|d\tau|}{|\tau - w_1|^{\gamma(q-1)}} \\ &\preceq n^{\gamma(q-1)\frac{1}{\alpha} + (\frac{1}{\alpha}-1)(q-2)} \text{mes} E_{R_1,2}^{11} \preceq n^{\gamma(q-1)\frac{1}{\alpha} + (\frac{1}{\alpha}-1)(q-2)-1}, \\ J_2^1 &\preceq n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}. \end{aligned} \tag{4.57}$$

$$\begin{aligned} (J_2^2)^q &\preceq \int_{E_{R_1}^{12}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma(q-1)}{\alpha}}} \\ &\preceq \begin{cases} n^{\frac{\gamma(q-1)}{\alpha} + (\frac{1}{\alpha}-1)(q-2)-1}, & \gamma(q-1) > \alpha, \\ n^{(\frac{1}{\alpha}-1)(q-2)} \ln n, & \gamma(q-1) = \alpha, \\ n^{(\frac{1}{\alpha}-1)(q-2)}, & \gamma(q-1) < \alpha; \end{cases} \\ J_2^2 &\preceq \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}, & 1 + \frac{\gamma}{\alpha} < p < 2, \quad 0 \leq \gamma < \alpha, \\ n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha} - \frac{1}{p}}, & 1 < p < 2, \quad \gamma \geq \alpha. \end{cases} \end{aligned} \tag{4.58}$$

For $\tau \in E_{R_1}^{13}$ $\eta < |\tau - w_1| < 2\pi\dot{R}_1$ and from Lemma 3.1 $|\Psi(\tau) - \Psi(w_1)| \asymp 1$. Then, we get:

$$\begin{aligned} (J_2^3)^q &= \int_{E_{R_1}^{13}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{13}} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(q-2)}; \\ J_2^3 &\preceq n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}. \end{aligned} \tag{4.59}$$

From (4.56–4.59) and (4.45), for $\gamma \geq 0, 1 < p < 2, z \in \Omega_R$, we have:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1) + \frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, \quad 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1) + \frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, \quad 0 \leq \gamma < \alpha, \\ n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 1 < p < 2, \quad \gamma \geq \alpha. \end{cases} \tag{4.60}$$

2.2. Let $\gamma < 0$. For $z \in \Omega_R$, according to Lemma 3.1, we have:

$$\begin{aligned}
 (J_2^1)^q &\preceq \int_{E_{R_1}^{11}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} |d\tau| \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{11}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{11}} |d\tau| \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \text{mes} E_{R_1}^{11} \preceq n^{(\frac{1}{\alpha}-1)(q-2)-1}; \\
 J_2^1 &\preceq n^{(\frac{2}{p}-1)(\frac{1}{\alpha}-1)-1+\frac{1}{p}}.
 \end{aligned}
 \tag{4.61}$$

$$\begin{aligned}
 (J_2^2)^q &\preceq \int_{E_{R_1}^{12}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} \left(\frac{|\tau| - 1}{d(\Psi(\tau), L)} \right)^{q-2} |d\tau| \\
 &\preceq n^{(\frac{1}{\alpha}-1)(q-2)} \int_{E_{R_1}^{12}} |\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |d\tau| \preceq n^{(\frac{1}{\alpha}-1)(q-2)}; \\
 J_2^2 &\preceq n^{(\frac{2}{p}-1)(\frac{1}{\alpha}-1)}.
 \end{aligned}
 \tag{4.62}$$

$$\begin{aligned}
 (J_2^3)^q &\preceq \int_{E_{R_1}^{13}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)} |d\tau|}{|\Psi'(\tau)|^{q-2}} \preceq n^{(\frac{1}{\alpha}-1)(q-2)}; \\
 J_2^3 &\preceq n^{(\frac{2}{p}-1)(\frac{1}{\alpha}-1)}.
 \end{aligned}
 \tag{4.63}$$

So, for $\gamma < 0$, from (4.45), we have:

$$A_n \preceq n^{(\frac{2}{p}-1)(\frac{1}{\alpha}-1)+\frac{1}{p}} \|P_n\|_p, \quad z \in \Omega_R.
 \tag{4.64}$$

Therefore, for any $\gamma \geq -2, 1 < p < 2$, from (4.60) and (4.64), we get:

$$A_n \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0. \end{cases}
 \tag{4.65}$$

Combining estimates (4.1)–(4.45), (4.55), (4.55), (4.65), we get:

$$|P_n(z)| \preceq \frac{|\Phi^{n+1}(z)|}{d(z, L_{R_1})} A_n,$$

where for $p \geq 2$

$$A_n \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}}, & p > \max\left\{2; \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}\right\}, & \gamma \geq 0, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

and for $1 < p < 2$

$$A_n \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}}, & 1 < p < \min\left\{2; 1 + \frac{\gamma}{\alpha}\right\}, & \gamma > 0, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)+\frac{1}{p}}, & \max\left\{1; 1 + \frac{\gamma}{\alpha}\right\} < p < 2, & -2 < \gamma < \alpha, \end{cases}$$

and therefore, the proof of Theorem 2.4 is completed. □

Proof [of Theorems 2.6 and 2.7.] From (4.25), (4.26) and (4.42), we get:

$$|P'_n(z)| \preceq |\Phi^{n+1}(z)| \cdot \left[\frac{A_n(z)}{d(z, L)} + |P_n(z)| \begin{cases} n^{\frac{1}{\alpha}} & \text{if } z \in \Omega(\delta), \\ n, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases} \right] \tag{4.66}$$

where for any $\gamma > -2, p \geq 2, m = 1$

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{\gamma^*+2}{p\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & \gamma > -2 + \alpha, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma+2}{\alpha}, & \gamma > -2 + \alpha, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{\gamma+2}{\alpha}, & \gamma > -2 + \alpha, \\ n^{1-\frac{1}{p}}, & p \geq 2, & -2 < \gamma \leq -2 + \alpha, \end{cases}$$

if $z \in \Omega(\delta)$, and

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}-\frac{1}{p}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{(1-\frac{2}{p})(\frac{1}{\alpha}-\frac{1}{p})} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma > \alpha, \\ n^{1-\frac{2}{p}}, & p \geq 2, & 0 \leq \gamma \leq \alpha, \\ n^{(\frac{1}{\alpha}-1)(1-\frac{2}{p})}, & p \geq 2, & \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$; for any $\gamma \geq -2, 1 < p < 2, m = 1$,

$$A_n(z) \preceq \|P_n\|_p \begin{cases} n^{\frac{\gamma^*+2}{p}\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > -2, & \text{if } z \in \Omega(\delta), \\ n^{(\frac{\gamma+2}{p}-1)\frac{1}{\alpha}-\frac{1}{p}}, & \gamma > \alpha(p-1), & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)} (\ln n)^{1-\frac{1}{p}}, & \gamma = \alpha(p-1), & \text{if } z \in \widehat{\Omega}(\delta), \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}-1)}, & -2 < \gamma < \alpha(p-1), & \text{if } z \in \widehat{\Omega}(\delta), \end{cases}$$

and $\gamma^* := \max\{0; \gamma\}$. Taking into account the estimates for $|P_n(z)|$ from (4.39), and combining with (4.40) give a proof of the needed estimates. □

Proof [of Theorems 2.8 and 2.9.] According to the Theorems 2.2, 2.4, 2.6 and estimates (4.2), (4.3), (4.25), (4.26), (4.42), for $m = 2$ and $p \geq 2$ from (4.1), we have:

$$\begin{aligned} |P_n''(z)| &\leq \left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)} \right)'' \right| + \sum_{j=1}^2 C_2^j \left| \left(\frac{1}{\Phi^{n+1}(z)} \right)^{(j)} \right| |P_n^{(2-j)}(z)| \\ &\leq |\Phi^{n+1}(z)| \left[\left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)} \right)'' \right| + C_2^1 B_{n,1}^1 |P_n'(z)| + C_2^2 B_{n,2}^1 |P_n(z)| \right] \\ &\preceq |\Phi^{n+1}(z)| \left[\frac{\|P_n\|_p}{d(z, L)} A_n^1(z, 2) + C_2^1 B_{n,1}^1 |P_n'(z)| + C_2^2 B_{n,2}^1 |P_n(z)| \right]. \end{aligned}$$

Substituting estimates for the $B_{n,j}^1$, $j = 1, 2$, $|P_n(z)|$ and $|P_n'(z)|$ from Theorems 2.2, 2.4 and 2.6 correspondingly, we get:

$$|P_n''(z)| \preceq \frac{|\Phi^{3(n+1)}(z)|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{1-\frac{1}{p}+\frac{2}{\alpha}}, & 2 \leq p < 1 + \frac{\gamma+2}{\alpha}, & 0 \leq \gamma < \alpha, \\ n^{(3-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}-1} & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

for $z \in \Omega(\delta)$, and

$$|P_n''(z)| \preceq \frac{|\Phi^{3(n+1)}(z)|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 2 \leq p < \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}}, & p > \frac{\alpha}{1+\alpha} + \frac{\gamma+2}{1+\alpha}, & \gamma \geq \alpha, \\ n^{3-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \alpha, \\ n^{(1-\frac{2}{p})\frac{1}{\alpha}+\frac{3}{p}+1}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

for $z \in \widehat{\Omega}(\delta)$.

Analogously, from Theorems 2.3, 2.4 and 2.7, for $1 < p < 2$, we have:

$$|P_n''(z)| \preceq \frac{|\Phi^{3(n+1)}(z)|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}+1)+\frac{1}{p}+2} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}+1)+\frac{1}{p}+2}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{(\frac{1}{\alpha}-1)(\frac{2}{p}+1)+\frac{1}{p}+2}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

for $z \in \Omega(\delta)$, and

$$|P_n''(z)| \preceq \frac{|\Phi^{3(n+1)}(z)|}{d(z, L_{R_1})} \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 1 < p < 2, & \gamma \geq \alpha, \\ n^{(\frac{\gamma+2}{p}+1)\frac{1}{\alpha}}, & 1 < p < 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+3-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{\gamma}{\alpha}, & 0 < \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1 + \frac{\gamma}{\alpha} < p < 2, & 0 \leq \gamma < \alpha, \\ n^{(\frac{2}{p}-1)\frac{1}{\alpha}+3-\frac{1}{p}}, & 1 < p < 2, & -2 < \gamma < 0, \end{cases}$$

for $z \in \widehat{\Omega}(\delta)$. Therefore, the proofs of Theorems 2.8 and 2.9 are completed.

In conclusion, note that, there is a quantity $d(z, L_{R_1})$ everywhere in proofs. We show that $d(z, L_{R_1}) \succeq d(z, L)$ holds for all $z \in \Omega_R$. For the points $z \notin \Omega(L_{R_1}, d(L_{R_1}, L_R))$, we have: $d(z, L_{R_1}) \succeq \delta \succeq d(z, L)$. Now, let $z \in \Omega(L_{R_1}, d(L_{R_1}, L_R))$. Denote by $\xi_1 \in L_{R_1}$ the point for which $d(z, L_{R_1}) = |z - \xi_1|$, and point $\xi_2 \in L$, such that $d(z, L) = |z - \xi_2|$, and for $w = \Phi(z)$, $t_1 = \Phi(\xi_1)$, $t_2 = \Phi(\xi_2)$, we have: $|w - w_1| \geq ||w - w_2| - |w_2 - w_1|| \geq ||w - w_2| - \frac{1}{2}|w - w_2|| \geq \frac{1}{2}|w - w_2|$. Then, according to Lemma 3.1, we obtain: $d(z, L_{R_1}) \succeq d(z, L)$. \square

4.0.1. Proof of Remark 2.5

Proof Let $G \in Q_\alpha$, $\frac{1}{2} \leq \alpha \leq 1$, $h(z) \equiv 1$ and $\{Q_n(z)\}$, $\deg Q_n = n$, $n = 0, 1, 2, \dots$, be a system of Bergman polynomials for region G , i.e. system of polynomials, $Q_n(z) := a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_n > 0$ satisfying the conditions

$$\iint_G Q_n(z) \overline{Q_m(z)} d\sigma_z = \begin{cases} 1, & n = m, \\ 0 & n \neq m. \end{cases}$$

According to [2], for arbitrary quasidisks, we have

$$Q_n(z) = a_n \rho^{n+1} \Phi^n(z) \Phi'(z) A_n(z), \quad z \in F \Subset \Omega,$$

where $\rho = \rho(G) > 1$ and

$$\sqrt{\frac{n+1}{\pi}} \leq a_n \rho^{n+1} \leq c_1 \sqrt{\frac{n+1}{\pi}},$$

for some $c_1 = c_1(G) > 1$ and

$$c_2 \leq |A_n(z)| \leq 1 + \frac{c_3}{\sqrt{|\Phi(z)| - 1}},$$

for some $c_i = c_i(G) > 0$, $i = 2, 3$. Therefore, since $\|Q_n\|_{A_2(G)} = 1$, then:

$$\begin{aligned} |Q_n(z)| &\geq c_2 \sqrt{\frac{n+1}{\pi}} |\Phi(z)|^n \frac{|\Phi(z)| - 1}{d(z, L)} \\ &\geq c_3 \frac{\sqrt{n}}{d(z, L)} |\Phi(z)|^{n+1} \left(1 - \frac{1}{|\Phi(z)|}\right) \\ &\geq c_4 \frac{\sqrt{n}}{d(z, L)} |\Phi(z)|^{n+1} \|Q_n\|_{A_2(G)}, \end{aligned}$$

and the proof is completed. \square

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References

- [1] Abdullayev FG, Andrievskii VV. On the orthogonal polynomials in the domains with K -quasiconformal boundary. *Izvestiya Akademii Nauk Azerbaidzhanskoi SSR Seriya Fiziko-Tekhnicheskikh I Matematicheskikh Nauk* 1983; 4 (1): 7-11 (in Russian).
- [2] Abdullayev FG. On the orthogonal polynomials in domains with quasiconformal boundary. PhD, Institute of Applied Mathematics and Mechanics National Academy of Science of Ukraine, Donetsk, Ukraine, 1986 (in Russian).
- [3] Abdullayev FG. On some properties of the orthogonal polynomials over the region of the complex plane (Part I). *Ukrainian Mathematical Journal* 2000; 52 (12): 1807-1821.
- [4] Abdullayev FG. On some properties of the orthogonal polynomials over the region of the complex plane (Part II). *Ukrainian Mathematical Journal* 2001; 53 (1): 1-14.
- [5] Abdullayev FG. On some properties of the orthogonal polynomials over the region of the complex plane (Part III). *Ukrainian Mathematical Journal* 2001; 53 (12): 1934-1948.
- [6] Abdullayev FG. On the interference of the weight and boundary contour for orthogonal polynomials over the region. *Journal of Computational Analysis and Applications* 2004; 6 (1): 31-42.
- [7] Abdullayev FG. The properties of the orthogonal polynomials with weight having singularity on the boundary contour. *Journal of Computational Analysis and Applications* 2004; 6 (1): 43-59.
- [8] Abdullayev FG, Aral D. On the Bernstein-Walsh type lemmas in regions of the complex plane. *Ukrainian Mathematical Journal* 2011; 63 (3): 337-350.
- [9] Abdullayev FG, Özkartepe P. An analogue of the Bernstein-Walsh lemma in Jordan regions of the complex plane. *Journal of Inequalities and Applications* 2013; 2013 (570): 1-7. doi: 10.1186/1029-242X-2013-570
- [10] Abdullayev FG, Gün CD. On the behavior of the algebraic polynomials in regions with piecewise smooth boundary without cusps. *Annales Polonici Mathematici* 2014; 111: 39-58.
- [11] Abdullayev FG, Özkartepe P. On the behavior of the algebraic polynomial in unbounded regions with piecewise Dini-smooth boundary. *Ukrainian Mathematical Journal* 2014; 66 (5): 645-665.
- [12] Abdullayev FG, Özkartepe P. On the growth of algebraic polynomials in the whole complex plane. *Journal of the Korean Mathematical Society* 2015; 52 (4): 699-725.
- [13] Abdullayev FG, Tunc T, Abdullayev GA. Polynomial inequalities in quasidisks on weighted Bergman space. *Ukrainian Mathematical Journal* 2017; 69 (5): 675-695.
- [14] Abdullayev FG, Gün CD. Bernstein-Nikolskii type inequalities for algebraic polynomials in Bergman space in regions of complex plane. *Ukrainian Mathematical Journal* 2021; 73 (4): 439-454.
- [15] Ahlfors L. *Lectures on Quasiconformal Mappings*. Princeton, NJ: Van Nostrand, 1966.
- [16] Anderson JM, Gehring FW, Hinkkanen A. Polynomial approximation in quasidisks. *Differential Geometry and Complex Analysis* 1985; A Volume Dedicated to the Memory of Harry Ernest Rauch: 75-86.
- [17] Andrievskii VV, Belyi VI, Dzyadyk VK. *Conformal Invariants in Constructive Theory of Functions of Complex Plane*. Atlanta: World Federation Publishers Company, 1995.
- [18] Andrievskii VV. Weighted polynomial inequalities in the complex plane. *Journal of Approximation Theory* 2012; 164 (9): 1165-1183.
- [19] Andrievskii VV, Blatt HP. *Discrepancy of Signed Measures and Polynomial Approximation*. New York: Springer-Verlag, 2010.
- [20] Balcı S, Imashkyzy M, Abdullayev FG. Polynomial inequalities in regions with interior zero angles in the Bergman space. *Ukrainian Mathematical Journal* 2018; 70 (3): 362-384.

- [44] Pritsker I. Comparing norms of polynomials in one and several variables. *Journal of Mathematical Analysis and Applications* 1997; 216: 685-695. doi:10.1006/jmaa.1997.5699
- [45] Stylianopoulos N. Strong asymptotics for Bergman polynomials over domains with corners and applications. *Constructive Approximation* 2012; 38 (1): 59-100.
- [46] Szegő G, Zygmund A. On certain mean values of polynomials. *Journal d'Analyse Mathématique*. 1953; 3 (1): 225-244.
- [47] Warschawski SE. On differentiability at the boundary in conformal mapping. *Proceedings of the American Mathematical Society* 1961; 12: 614-620.
- [48] Warschawski SE. On Hölder continuity at the boundary in conformal maps. *Journal of Mathematics and Mechanics* 1968; 18: 423-427.
- [49] Walsh JL. *Interpolation and Approximation by Rational Functions in the Complex Domain*. AMS Colloquium Publications, 1960.