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Curvature identities for Einstein manifolds of dimensions 5 and 6

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Abstract: Patterson discussed the curvature identities on Riemannian manifolds based on the skew-symmetric properties of the generalized Kronecker delta, and a curvature identity for any 6-dimensional Riemannian manifold was independently derived from the Chern-Gauss-Bonnet Theorem. In this paper, we provide the explicit formulae of Patterson's curvature identity that holds on 5-dimensional and 6-dimensional Einstein manifolds. We confirm that the curvature identities on the Einstein manifold derived from the Chern-Gauss-Bonnet Theorem are the same as the curvature identities deduced from Patterson's result. We also provide examples that support the theorems.

Key words: Einstein, curvature identity, curvature tensor

1. Introduction

Let $M = (M, g)$ be an m -dimensional Riemannian manifold and ∇ be the Levi-Civita connection of g . The curvature tensor R on M is defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M . The Ricci tensor of M is defined by $\rho(X, Y) = \text{Tr}(Z \rightarrow R(Z, X)Y)$ and the scalar curvature of M is obtained by $\tau = \text{Tr} \rho$. A symmetric (0,2)-tensor field \bar{R} on M is defined by $\bar{R}(X, Y) = \sum_{i,j,k=1}^m R(X, e_i, e_j, e_k)R(Y, e_i, e_j, e_k)$ for a local orthonormal frame $\{e_i\}$. An m -dimensional Riemannian manifold (M, g) is said to be *Einstein* if $\rho = \frac{\tau}{m}g$. The m -dimensional Einstein manifold (M, g) is said to be *super-Einstein* if the following condition is satisfied on M

$$\bar{R}(X, Y) = \frac{\|R\|^2}{m}g(X, Y) \quad (1.1)$$

with constant $\|R\|^2$ (see [3, 6, 11]). The condition (1.1) has some geometric meanings: For a compact manifold, an Einstein metric is critical for the functional $\int_M \|R\|^2 dv_M$ restricted to $\text{vol}(M) = 1$ if and only if (1.1) satisfies. (see [2, Corollary 4.72]). Boeckx and Vanhecke [3] showed that an Einstein manifold M is super-Einstein if and only if the unit tangent sphere bundle T_1M equipped with the standard contact metric structure has constant scalar curvature ([3, Proposition 3.6]).

Let $M_q(r, f)$ denote the mean-value of a real-valued function f over a geodesic sphere $S(q; r)$ with center

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q and radius r in an m -dimensional Riemannian manifold M :

$$M_q(r, f) := \frac{1}{\int_{S(q;r)} dv_{S(q;r)}} \int_{p \in S(q;r)} f(p) dv_{S(q;r)}.$$

Gray and Willmore [11] showed the mean-value properties for an Einstein manifold and a super-Einstein manifold: They proved, from the expansions of $M_q(r, f)$, that the harmonic function f near q has the mean-value properties

$$M_q(r, f) = f(q) + O(r^6), \quad \text{as } r \rightarrow 0$$

for an Einstein manifold ([11, Theorem 1.1]) and

$$M_q(r, f) = f(q) + O(r^8), \quad \text{as } r \rightarrow 0$$

for a super-Einstein manifold with $\dim M \geq 4$ ([11, Theorem 6.1]), respectively.

A Riemannian manifold M is said to be 2-stein if there exist two functions μ_1, μ_2 on M such that $\text{Tr } R_X = \mu_1(p) \|X\|^2$ and $\text{Tr}(R_X^2) = \mu_2(p) \|X\|^4$, for all $p \in M$ and all $X \in T_p M$. Here, the Jacobi operator R_X is defined by $R_X Y = R(Y, X)X$ for tangent vectors X, Y at a point $p \in M$. A unit vector field V on M is said to be a *harmonic vector field* if it is a critical point for the energy functional in the set of all unit vector fields of M [16]. A contact metric manifold whose characteristic vector field ξ is a harmonic vector field is called an *H-contact manifold*. Nikolayevsky and Park [13] showed that for a Riemannian manifold M , $T_1 M$ equipped with the standard contact metric structure is *H-contact* if and only if M is 2-stein. Gilkey, Swann, and Vanhecke [10] showed that a 4-dimensional manifold M is 2-stein if and only if locally there is a choice of orientation of M for which the metric is self-dual and Einstein ([10, Theorem 2.6]). From the definition of a 2-stein manifold, we can derive the super-Einstein conditions (see [1, Chapter 6, §E]). Thus, a 2-stein manifold is necessarily super-Einstein.

Euh, Park, and Sekigawa [6] derived a curvature identity on any 4-dimensional manifold from the Chern-Gauss-Bonnet theorem. There are many applications of this identity (see [5, 7, 9]). On the other hand, Deszcz, Hotloś, and Sentürk [4] gave some curvature properties of 4-dimensional semi-Riemannian manifolds as an application of Patterson’s curvature identity.

In this paper, we introduce the curvature identities on some 5- and 6-dimensional Riemannian manifolds such as Einstein, super-Einstein manifolds. Throughout the paper, we assume that the components of the tensor fields are with respect to a local orthonormal frame $\{e_i\}$ and we also adopt the Einstein convention on sum over repeated indices unless otherwise specified. Our main results are the following.

Theorem A *Let $M = (M, g)$ be a 5-dimensional Riemannian manifold.*

(a) *If M is Einstein, then the following curvature identity holds on M :*

$$2\tau \bar{R}_{ij} + 4\check{R}_{ij} + 4\hat{R}_{ij} - 8\mathring{R}_{ij} = \left(\frac{\tau}{5} \|R\|^2 + \frac{\tau^3}{25}\right) g_{ij}.$$

(b) *If M is super-Einstein, then the following curvature identity holds on M :*

$$4\mathring{R}_{ij} - 2\hat{R}_{ij} = \left(\frac{9}{50} \tau \|R\|^2 - \frac{\tau^3}{50}\right) g_{ij}.$$

Theorem B *Let $M = (M, g)$ be a 6-dimensional Riemannian manifold.*

(a) *If M is Einstein, then the following curvature identity holds on M :*

$$4\tau\bar{R}_{ij} + 12\check{R}_{ij} + 12\hat{R}_{ij} - 24\overset{\circ}{R}_{ij} = (\tau\|R\|^2 - 4\overset{\circ}{R} + 2\hat{R})g_{ij}.$$

(b) *If M is super-Einstein, then the following curvature identity holds on M :*

$$2\overset{\circ}{R}_{ij} - \hat{R}_{ij} = \frac{1}{6}(2\overset{\circ}{R} - \hat{R})g_{ij}.$$

Here, we set

$$\begin{aligned} \check{R}_{ij} &= R_{iuvj}R_{abcu}R_{abcv}, & \hat{R}_{ij} &= R_{ibcd}R_{jbuv}R_{cduv}, & \overset{\circ}{R}_{ij} &= R_{ibcd}R_{jucv}R_{budv}, \\ \hat{R} &= R_{abcd}R_{abuv}R_{cduv}, & \overset{\circ}{R} &= R_{abcd}R_{aucv}R_{budv}. \end{aligned}$$

We derive the curvature identities from Patterson’s curvature identity. In Section 2, we recall the previous results for a 6-dimensional Riemannian manifold [8] and introduce Patterson’s curvature identity. In Sections 3 and 4, we prove Theorem A and Theorem B, respectively. We also give some examples supporting the theorems. In Appendix, we attach the detailed computation for the proof of Lemma 4.1.

2. Curvature identities

On a 6-dimensional compact oriented Riemannian manifold, Chern-Gauss-Bonnet theorem states that Euler characteristic $\chi(M)$ of M is given by the following integral formula.

Proposition 2.1 ([15]) *Let $M = (M, g)$ be a 6-dimensional compact oriented Riemannian manifold. Then the Euler characteristic $\chi(M)$ of M is given by*

$$\begin{aligned} \chi(M) &= \frac{1}{384\pi^3} \int_M \{ \tau^3 - 12\tau\|\rho\|^2 + 3\tau\|R\|^2 + 16\rho_{ab}\rho_{ac}\rho_{bc} \\ &\quad - 24\rho_{ab}\rho_{cd}R_{abcd} - 24\rho_{uv}R_{abcu}R_{abcv} \\ &\quad + 8R_{abcd}R_{aucv}R_{bvdu} - 2R_{abcd}R_{abuv}R_{cduv} \} dv_M. \end{aligned}$$

In [8], the authors consider the one-parameter deformation $g(t)$ of g , using the fact that Euler characteristic is a topological invariant for the deformation, they got the universal curvature identity which holds on any 6-dimensional Riemannian manifold. In particular, they obtained the following.

Theorem 2.2 ([8]) *Let $M = (M, g)$ be a 6-dimensional Einstein manifold. Then the following identity holds on M :*

$$(-\tau\|R\|^2 + 4\overset{\circ}{R} - 2\hat{R})g_{ij} + 12\check{R}_{ij} + 12\hat{R}_{ij} - 24\overset{\circ}{R}_{ij} + 4\tau\bar{R}_{ij} = 0.$$

For an integer $N \geq 1$, the *generalized Kronecker delta* is given by

$$\delta_{i_1 i_2 \dots i_N}^{j_1 j_2 \dots j_N} = \begin{vmatrix} \delta_{i_1}^{j_1} & \delta_{i_1}^{j_2} & \dots & \delta_{i_1}^{j_N} \\ \delta_{i_2}^{j_1} & \delta_{i_2}^{j_2} & \dots & \delta_{i_2}^{j_N} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{i_N}^{j_1} & \delta_{i_N}^{j_2} & \dots & \delta_{i_N}^{j_N} \end{vmatrix}.$$

By using the skew-symmetric properties of the generalized Kronecker delta, Patterson [14] obtained the curvature identity for an m -dimensional Riemannian manifold as follows:

$$\delta_{i_1 i_2 \dots i_m}^{j_1 j_2 \dots j_m} R^{i_1 i_2}_{j_1 j_2} \dots R^{i_{2r-1} i_{2r}}_{j_{2r-1} j_{2r}} = 0, \tag{2.1}$$

where r is any integer such that $1 \leq r \leq \frac{m}{2}$ (m even) or $1 \leq r \leq \frac{m-1}{2}$ (m odd).

We note that the identity (2.1) holds with respect to the Weyl curvature tensor W even if we replace R by W in (2.1) ([14, Section 8]).

3. Proof of Theorem A

Let (M, g) be a 5-dimensional Riemannian manifold. Making use of (2.1), we have the curvature identity with respect to the Weyl curvature tensor W for the case when $r = 2$ as follows:

$$\begin{aligned} & \|W\|^2(g_{ik}g_{jl} - g_{il}g_{jk}) - 4\{W_{abcj}W_{abcl}g_{ik} - W_{abcj}W_{abck}g_{il} + W_{abci}W_{abck}g_{jl} \\ & - W_{abci}W_{abcl}g_{jk} - 2W_{iabl}W_{kabj} + 2W_{iabk}W_{labj} - W_{abij}W_{abkl}\} = 0. \end{aligned} \tag{3.1}$$

Since the Weyl tensor W on a 5-dimensional Riemannian manifold is given by

$$W_{pqrs} = R_{pqrs} - \frac{1}{3}(\rho_{ps}g_{qr} + \rho_{qr}g_{ps} - \rho_{pr}g_{qs} - \rho_{qs}g_{pr}) + \frac{\tau}{12}(g_{ps}g_{qr} - g_{pr}g_{qs}),$$

we obtain the explicit formula of the curvature identity on 5-dimensional Einstein manifolds.

Lemma 3.1 *Let $M = (M, g)$ be a 5-dimensional Einstein manifold. The following curvature identity holds on M :*

$$\begin{aligned} & \left(\|R\|^2 + \frac{\tau^2}{5}\right)(g_{ik}g_{jl} - g_{il}g_{jk}) - 4(\bar{R}_{ik}g_{jl} + \bar{R}_{jl}g_{ik} - \bar{R}_{il}g_{jk} - \bar{R}_{jk}g_{il}) \\ & + 8(R_{iabl}R_{kabj} - R_{iabk}R_{labj}) + 4R_{abij}R_{abkl} + \frac{12}{5}\tau R_{ijkl} = 0. \end{aligned} \tag{3.2}$$

Proof Since $\rho_{ps} = \frac{\tau}{5}g_{ps}$, we obtain $W_{pqrs} = R_{pqrs} - \frac{\tau}{20}(g_{ps}g_{qr} - g_{pr}g_{qs})$. Now we apply the Weyl tensor to the identity (3.1). Then we have

$$\begin{aligned} \|W\|^2 &= \|R\|^2 - \frac{\tau^2}{10}, \quad W_{abcj}W_{abcl} = R_{abcj}R_{abcl} - \frac{\tau^2}{50}g_{jl}, \\ W_{iabl}W_{kabj} &= R_{iabl}R_{kabj} + \frac{\tau}{10}R_{ijkl} - \frac{\tau^2}{80}g_{il}g_{jk} + \frac{\tau^2}{400}g_{ik}g_{jl}, \\ W_{abij}W_{abkl} &= R_{abij}R_{abkl} + \frac{\tau}{5}R_{ijkl} + \frac{\tau^2}{200}(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned}$$

Similarly, we can obtain remaining terms. Then, by rearranging all terms, we complete the proof. □

We note that there are useful formulae as follows:

$$R_{ibcd}R_{jbuv}R_{cudv} = \frac{1}{2}\hat{R}_{ij}, \quad R_{ibcd}R_{jubv}R_{cudv} = \frac{1}{4}\hat{R}_{ij}, \quad R_{ibcd}R_{juv}R_{bvdu} = \hat{R}_{ij} - \frac{1}{4}\hat{R}_{ij}. \tag{3.3}$$

Now, we transvect each term of (3.2) with R_{pjkl} and use $\rho_{ip} = \frac{\tau}{5}g_{ip}$. Then, we obtain

$$\begin{aligned} & \left(\|R\|^2 + \frac{\tau^2}{5} \right) (g_{ik}g_{jl} - g_{il}g_{jk})R_{pjkl} = -\frac{2}{5}\tau \left(\|R\|^2 + \frac{\tau^2}{5} \right) g_{ip}, \\ & -4(\bar{R}_{ik}g_{jl} + \bar{R}_{jl}g_{ik} - \bar{R}_{il}g_{jk} - \bar{R}_{jk}g_{il})R_{pjkl} = -4\left(-\frac{2}{5}\tau\bar{R}_{ip} - 2\hat{R}_{ip} \right), \\ & 8(R_{iabl}R_{kabj} - R_{iabk}R_{labj})R_{pjkl} = -16\hat{R}_{ip} + 4\hat{R}_{ip}, \\ & 4R_{abij}R_{abkl}R_{pjkl} = 4\hat{R}_{ip}, \\ & \frac{12}{5}\tau R_{ijkl}R_{pjkl} = \frac{12}{5}\tau\bar{R}_{ip}. \end{aligned}$$

For the third equation above, we use (3.3). Hence, we complete the proof of Theorem A-(a).

Now we prove Theorem A-(b). Applying the condition (1.1) to (3.2), we obtain

$$R_{ijab}R_{abkl} + 2R_{iabl}R_{kabj} - 2R_{iabk}R_{labj} + \frac{3}{5}\tau R_{ijkl} = \left(\frac{3}{20}\|R\|^2 - \frac{1}{20}\tau^2 \right) (g_{ik}g_{jl} - g_{il}g_{jk}). \tag{3.4}$$

Transvecting each term of (3.4) with R_{pjkl} and using (1.1), we obtain

$$\begin{aligned} R_{ijab}R_{abkl}R_{pjkl} &= \hat{R}_{ip}, \\ 2R_{iabl}R_{kabj}R_{pjkl} &= -2\left(\hat{R}_{ip} - \frac{1}{4}\hat{R}_{ip} \right), \\ -2R_{iabk}R_{labj}R_{pjkl} &= -2\left(\hat{R}_{ip} - \frac{1}{4}\hat{R}_{ip} \right), \\ \frac{3}{5}\tau R_{ijkl}R_{pjkl} &= \frac{3}{25}\tau\|R\|^2 g_{ip}, \\ (g_{ik}g_{jl} - g_{il}g_{jk})R_{pjkl} &= -\frac{2}{5}\tau g_{ip}. \end{aligned}$$

For the second and third equations, we use (3.3). Therefore, we complete the proof of Theorem A-(b).

For a 5-dimensional 2-stein manifold, there is the following orthonormal basis introduced by Nikolayevsky [12].

Proposition 3.2 ([12, Proposition 4]) *Let $M = (M, g)$ be a 2-stein manifold of dimension 5. Then there exists an orthonormal basis $\{e_i\}$ at each point $p \in M$ such that*

$$\begin{aligned} R_{1212} &= R_{1313} = R_{2323} = R_{2424} = R_{3434} = \alpha - \beta, \quad R_{1414} = \alpha - 4\beta, \\ R_{1515} &= R_{4545} = \alpha, \quad R_{2525} = R_{3535} = \alpha - 3\beta, \\ R_{1234} &= \beta, \quad R_{1235} = \sqrt{3}\beta, \quad R_{1324} = -\beta, \quad R_{1325} = \sqrt{3}\beta, \\ R_{1423} &= -2\beta, \quad R_{2425} = \sqrt{3}\beta, \quad R_{3435} = -\sqrt{3}\beta, \end{aligned}$$

and all the other components of R vanish.

By using Nikolayevsky’s basis, we show the following theorem.

Theorem 3.3 *Let $M = (M, g)$ be a 5-dimensional 2-stein manifold. The identity (3.4) holds on M .*

Proof Each term of the left hand side of (3.4) in the case of $i = 1, j = 2, k = 3, l = 4$ is as follows:

$$\begin{aligned} R_{12ab}R_{ab34} &= 2(2\alpha - 5\beta)\beta, \\ 2R_{1ab4}R_{3ab2} &= 2(2\alpha - 5\beta)\beta, \\ -2R_{1ab3}R_{4ab2} &= 2(2\alpha + \beta)\beta, \\ \frac{3}{5}\tau R_{1234} &= -6(2\alpha - 3\beta)\beta. \end{aligned}$$

Since $R_{12ab}R_{ab34} + 2R_{1ab4}R_{3ab2} - 2R_{1ab3}R_{4ab2} + \frac{3}{5}\tau R_{1234} = 0$ and $(g_{13}g_{24} - g_{14}g_{23}) = 0$ in (3.4), Equation (3.4) holds for $i = 1, j = 2, k = 3, l = 4$. For other choice of i, j, k, l , similar processes can be done for the proof of Theorem 3.3. □

Now we give the examples for Theorem A.

Example 3.4 *Let M be the Riemannian product manifold of a 3-dimensional Riemannian manifold M_1 of constant sectional curvature k and a surface M_2 of constant sectional curvature $2k$ ($k \neq 0$). Then the manifold M is Einstein, but not super-Einstein. Let $\{e_i\}, i = 1, \dots, 5$ be an orthonormal basis of T_pM at any point $p = (p_1, p_2) \in M$, where $\{e_1, e_2, e_3\}$ and $\{e_4, e_5\}$ are bases for $T_{p_1}M_1$ and $T_{p_2}M_2$, respectively. Then we have*

$$R_{1221} = R_{1331} = R_{2332} = k, \quad R_{4554} = 2k, \tag{3.5}$$

and all the other components of R vanish. From (3.5) we have $\tau = 10k, \|R\|^2 = 28k^2$, and

$$\begin{aligned} \bar{R}_{ij} &= \begin{cases} 4k^2 & \text{if } i = j = 1, 2, 3 \\ 8k^2 & \text{if } i = j = 4, 5 \\ 0 & \text{otherwise} \end{cases}, & \check{R}_{ij} &= \begin{cases} 8k^3 & \text{if } i = j = 1, 2, 3 \\ 16k^3 & \text{if } i = j = 4, 5 \\ 0 & \text{otherwise} \end{cases}, \\ \hat{R}_{ij} &= \begin{cases} -8k^3 & \text{if } i = j = 1, 2, 3 \\ -32k^3 & \text{if } i = j = 4, 5 \\ 0 & \text{otherwise} \end{cases}, & \mathring{R}_{ij} &= \begin{cases} -2k^3 & \text{if } i = j = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Therefore, we find that the curvature identity of Theorem A-(b) does not hold, but that of Theorem A-(a) holds on M .

Example 3.5 *Let $M = SL(3)/SO(3)$. Then M is a 5-dimensional 2-stein manifold. The inner product and the curvature tensor are given by*

$$\langle X, Y \rangle = \text{Tr } XY, \quad R(X, Y)Z = -[[X, Y], Z]$$

for $X, Y, Z \in \mathfrak{X}(M)$. Then, with orthonormal basis of $SL(3)/SO(3)$

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$X_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_5 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

we have the components of the curvature tensor:

$$R_{1221} = R_{1331} = R_{2332} = R_{2442} = R_{3443} = -\frac{1}{2}, \quad R_{1441} = -2,$$

$$R_{1551} = R_{4554} = 0, \quad R_{2552} = R_{3553} = -\frac{3}{2},$$

$$R_{1234} = -\frac{1}{2}, \quad R_{1235} = -\frac{\sqrt{3}}{2}, \quad R_{1324} = \frac{1}{2}, \quad R_{1325} = -\frac{\sqrt{3}}{2},$$

$$R_{1423} = 1, \quad R_{2425} = -\frac{\sqrt{3}}{2}, \quad R_{3435} = \frac{\sqrt{3}}{2},$$
(3.6)

and all the other components vanish. From (3.6), we compute the Ricci tensor ρ_{ij} . For $i \neq j$, $\rho_{ij} = 0$ and $\rho_{11} = \sum_{a=1}^5 R_{1aa1} = -3$. Similarly, $\rho_{22} = \rho_{33} = \rho_{44} = \rho_{55} = -3$ and so we get the scalar curvature τ is -15 . Making use of (3.6), $\|R\|^2 = 75$, $\hat{R}_{ij} = 75\delta_{ij}$, and $\hat{R}_{ij} = \frac{15}{4}\delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Therefore, we find that the curvature identity of Theorem A-(b) holds on M .

4. Proof of Theorem B

We obtain the explicit formula of the curvature identity on 6-dimensional Einstein manifolds for the case when $m = 6$ and $r = 2$ of the identity (2.1). To prove Theorem B, we need Lemma 4.1.

Lemma 4.1 *Let $M = (M, g)$ be a 6-dimensional Einstein manifold. The following curvature identity holds on M :*

$$\left\{ \frac{1}{8} \left(\|R\|^2 + \frac{\tau^2}{3} \right) \left(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj} \right) \right\}$$

$$+ \left\{ -\frac{1}{2} \left(\bar{R}_{ij}(g_{hk}g_{lm} - g_{hm}g_{lk}) + \bar{R}_{ik}(g_{hm}g_{lj} - g_{hj}g_{lm}) + \bar{R}_{im}(g_{hj}g_{lk} - g_{hk}g_{lj}) \right) \right.$$

$$+ \bar{R}_{hj}(g_{im}g_{lk} - g_{ik}g_{lm}) + \bar{R}_{hk}(g_{ij}g_{lm} - g_{im}g_{lj}) + \bar{R}_{hm}(g_{ik}g_{lj} - g_{ij}g_{lk})$$

$$\left. + \bar{R}_{lj}(g_{ik}g_{hm} - g_{im}g_{hk}) + \bar{R}_{lk}(g_{im}g_{hj} - g_{ij}g_{hm}) + \bar{R}_{lm}(g_{ij}g_{hk} - g_{ik}g_{hj}) \right\}$$
(4.1)

$$\begin{aligned}
 & + \left\{ \left(-T_{ijkh} + T_{ikjh} + \frac{1}{2}S_{ihjk} + \frac{\tau}{3}R_{ihjk} \right) g_{lm} - \left(-T_{ijkl} + T_{ikjl} + \frac{1}{2}S_{iljk} + \frac{\tau}{3}R_{iljk} \right) g_{hm} \right. \\
 & \quad - \left(-T_{ijmh} + T_{imjh} + \frac{1}{2}S_{ihjm} + \frac{\tau}{3}R_{ihjm} \right) g_{lk} + \left(-T_{ijml} + T_{imjl} + \frac{1}{2}S_{iljm} + \frac{\tau}{3}R_{iljm} \right) g_{hk} \\
 & \quad + \left(-T_{ikmh} + T_{imkh} + \frac{1}{2}S_{ihkm} + \frac{\tau}{3}R_{ihkm} \right) g_{lj} - \left(-T_{ikml} + T_{imkl} + \frac{1}{2}S_{ilkm} + \frac{\tau}{3}R_{ilkm} \right) g_{hj} \\
 & \quad - \left(-T_{hjml} + T_{hmjl} + \frac{1}{2}S_{hljm} + \frac{\tau}{3}R_{hljm} \right) g_{ik} + \left(-T_{hjkl} + T_{hkjl} + \frac{1}{2}S_{hljk} + \frac{\tau}{3}R_{hljk} \right) g_{im} \\
 & \quad \left. + \left(-T_{hkml} + T_{hmkl} + \frac{1}{2}S_{hlkm} + \frac{\tau}{3}R_{hlkm} \right) g_{ij} \right\} \\
 & + A_{hjkml} - A_{ljkmi}h - A_{ijkmhl} + A_{ijmkhl} - A_{hjmki}l + A_{ljmkih} - A_{ikmjhl} + A_{hkmjil} - A_{lkmjih} \\
 & = 0,
 \end{aligned}$$

where $T_{pqrs} = R_{pabq}R_{rabs}$, $S_{pqrs} = R_{abpq}R_{abrs}$, and $A_{pqrstu} = R_{apqr}R_{astu}$.

Proof The proof is similar to that of Lemma 3.1 for the case when $m = 6$ and $r = 2$. Here, the Weyl tensor W is given by

$$W_{pqrs} = R_{pqrs} - \frac{1}{4}(\rho_{ps}g_{qr} + \rho_{qr}g_{ps} - \rho_{pr}g_{qs} - \rho_{qs}g_{pr}) + \frac{\tau}{20}(g_{ps}g_{qr} - g_{pr}g_{qs}).$$

Next we expand (2.1) for W and use the Einstein condition $\rho_{ij} = \frac{\tau}{6}g_{ij}$. Then we can obtain (4.1). For more details, we refer to Appendix. \square

Now, we transvect each term of (4.1) with R_{ihjk} . For the first term of (4.1), we have

$$\begin{aligned}
 & \frac{1}{8} \left(\|R\|^2 + \frac{\tau^2}{3} \right) \left(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj} \right) R_{ihjk} \\
 & = -\frac{1}{6} \left(\tau \|R\|^2 + \frac{\tau^3}{3} \right) g_{lm}.
 \end{aligned}$$

For the second term, we have

$$\begin{aligned}
 & -\frac{1}{2} \bar{R}_{ij} (g_{hk}g_{lm} - g_{hm}g_{lk}) R_{ihjk} = \frac{\tau}{12} \|R\|^2 g_{lm} - \frac{1}{2} \check{R}_{lm}, \\
 & -\frac{1}{2} \bar{R}_{ik} (g_{hm}g_{lj} - g_{hj}g_{lm}) R_{ihjk} = \frac{\tau}{12} \|R\|^2 g_{lm} - \frac{1}{2} \check{R}_{lm}, \quad -\frac{1}{2} \bar{R}_{im} (g_{hj}g_{lk} - g_{hk}g_{lj}) R_{ihjk} = -\frac{\tau}{6} \bar{R}_{lm}, \\
 & -\frac{1}{2} \bar{R}_{hj} (g_{im}g_{lk} - g_{ik}g_{lm}) R_{ihjk} = \frac{\tau}{12} \|R\|^2 g_{lm} - \frac{1}{2} \check{R}_{lm}, \\
 & -\frac{1}{2} \bar{R}_{hk} (g_{ij}g_{lm} - g_{im}g_{lj}) R_{ihjk} = \frac{\tau}{12} \|R\|^2 g_{lm} - \frac{1}{2} \check{R}_{lm}, \\
 & -\frac{1}{2} \bar{R}_{hm} (g_{ik}g_{lj} - g_{ij}g_{lk}) R_{ihjk} = -\frac{\tau}{6} \bar{R}_{lm}, \quad -\frac{1}{2} \bar{R}_{lj} (g_{ik}g_{hm} - g_{im}g_{hk}) R_{ihjk} = -\frac{\tau}{6} \bar{R}_{lm}, \\
 & -\frac{1}{2} \bar{R}_{lk} (g_{im}g_{hj} - g_{ij}g_{hm}) R_{ihjk} = -\frac{\tau}{6} \bar{R}_{lm}, \quad -\frac{1}{2} \bar{R}_{lm} (g_{ij}g_{hk} - g_{ik}g_{hj}) R_{ihjk} = \tau \bar{R}_{lm}.
 \end{aligned}$$

Thus, the second term of (4.1) transvecting with R_{ihjk} becomes $\frac{\tau}{3}||R||^2 g_{lm} - 2\mathring{R}_{lm} + \frac{\tau}{3}\bar{R}_{lm}$.

For the third term, we have

$$\begin{aligned} & \left(-T_{ijkh} + T_{ikjh} + \frac{1}{2}S_{ihjk} + \frac{\tau}{3}R_{ihjk}\right)g_{lm}R_{ihjk} \\ &= \left(-R_{iabj}R_{kabh}R_{ihjk} + R_{iabk}R_{jabh}R_{ihjk} + \frac{1}{2}R_{abih}R_{abjk}R_{ihjk} + \frac{1}{3}\tau R_{ihjk}R_{ihjk}\right)g_{lm} \\ &= \left(-2\mathring{R} + \hat{R} + \frac{1}{3}\tau||R||^2\right)g_{lm}, \\ & -\left(-T_{ijkl} + T_{ikjl} + \frac{1}{2}S_{iljk} + \frac{\tau}{3}R_{iljk}\right)g_{hm}R_{ihjk} \\ &= \left(R_{iabj}R_{kabl}R_{imjk} - R_{iabk}R_{jabl}R_{imjk} - \frac{1}{2}R_{abil}R_{abjk}R_{imjk} - \frac{\tau}{3}R_{iljk}R_{imjk}\right) \\ &= 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\bar{R}_{lm}, \\ & -\left(-T_{ijmh} + T_{imjh} + \frac{1}{2}S_{ihjm} + \frac{\tau}{3}R_{ihjm}\right)g_{lk}R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\bar{R}_{lm}, \\ & \left(-T_{ijml} + T_{imjl} + \frac{1}{2}S_{iljm} + \frac{\tau}{3}R_{iljm}\right)g_{hk}R_{ihjk} \\ &= \left(-R_{iabj}R_{mabl} + R_{iabm}R_{jabl} + \frac{1}{2}R_{abil}R_{abjm} + \frac{\tau}{3}R_{iljm}\right)(-\rho_{ij}) \\ &= -\frac{\tau}{6}\left(-\frac{\tau^2}{36}g_{lm} + \bar{R}_{lm} + \frac{1}{2}\bar{R}_{lm} - \frac{\tau^2}{18}g_{lm}\right) \\ &= \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\bar{R}_{lm}, \\ & \left(-T_{ikmh} + T_{imkh} + \frac{1}{2}S_{ihkm} + \frac{\tau}{3}R_{ihkm}\right)g_{lj}R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\bar{R}_{lm}, \\ & -\left(-T_{ikml} + T_{imkl} + \frac{1}{2}S_{ilkm} + \frac{\tau}{3}R_{ilkm}\right)g_{hj}R_{ihjk} = \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\bar{R}_{lm}, \\ & -\left(-T_{hjml} + T_{hmjl} + \frac{1}{2}S_{hljm} + \frac{\tau}{3}R_{hljm}\right)g_{ik}R_{ihjk} = \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\bar{R}_{lm}, \\ & \left(-T_{hjkl} + T_{hkjl} + \frac{1}{2}S_{hljk} + \frac{\tau}{3}R_{hljk}\right)g_{im}R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{\tau}{3}\bar{R}_{lm}, \\ & \left(-T_{hkml} + T_{hmkl} + \frac{1}{2}S_{hlkm} + \frac{\tau}{3}R_{hlkm}\right)g_{ij}R_{ihjk} = \frac{\tau^3}{72}g_{lm} - \frac{\tau}{4}\bar{R}_{lm}. \end{aligned}$$

Thus, the third term of (4.1) transvecting with R_{ihjk} becomes

$$\left(-2\mathring{R} + \hat{R} + \frac{\tau}{3}||R||^2 + \frac{\tau^3}{18}\right)g_{lm} + 8\mathring{R}_{lm} - 4\hat{R}_{lm} - \frac{7}{3}\tau\bar{R}_{lm}.$$

Making use of (3.3), the remaining terms in (4.1) transvecting with R_{ihjk} , we have

$$\begin{aligned} & (A_{hjkmil} - A_{ljkmi h} - A_{ijk mhl} + A_{ijm khl} - A_{hjm kil} + A_{l j m kih} - A_{ikm jhl} + A_{hkm jil} - A_{lkm jih})R_{ihjk} \\ &= R_{ahjk}R_{amil}R_{ihjk} - R_{aljk}R_{amih}R_{ihjk} - R_{aijk}R_{amhl}R_{ihjk} + R_{aijm}R_{akhl}R_{ihjk} - R_{ahjm}R_{akil}R_{ihjk} \\ & \quad + R_{aljm}R_{akih}R_{ihjk} - R_{aikm}R_{ajhl}R_{ihjk} + R_{ahkm}R_{ajil}R_{ihjk} - R_{alkm}R_{ajih}R_{ihjk} \\ &= -4\check{R}_{lm} - 2\hat{R}_{lm} + 4\mathring{R}_{lm}. \end{aligned}$$

Summing up all terms, then we complete the proof of Theorem B-(a).

Now, we prove Theorem B-(b). Applying (1.1) to (4.1), we have

$$\begin{aligned} & -\frac{1}{8}\left(\|R\|^2 - \frac{\tau^2}{3}\right)\left(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}\right) \\ & + \left(-T_{ijkh} + T_{ikjh} + \frac{1}{2}S_{ihjk} + \frac{\tau}{3}R_{ihjk}\right)g_{lm} - \left(-T_{ijkl} + T_{ikjl} + \frac{1}{2}S_{iljk} + \frac{\tau}{3}R_{iljk}\right)g_{hm} \\ & - \left(-T_{ijmh} + T_{imjh} + \frac{1}{2}S_{ihjm} + \frac{\tau}{3}R_{ihjm}\right)g_{lk} + \left(-T_{ijml} + T_{imjl} + \frac{1}{2}S_{iljm} + \frac{\tau}{3}R_{iljm}\right)g_{hk} \\ & + \left(-T_{ikmh} + T_{imkh} + \frac{1}{2}S_{ihkm} + \frac{\tau}{3}R_{ihkm}\right)g_{lj} - \left(-T_{ikml} + T_{imkl} + \frac{1}{2}S_{ilkm} + \frac{\tau}{3}R_{ilkm}\right)g_{hj} \\ & - \left(-T_{hjml} + T_{hmjl} + \frac{1}{2}S_{hljm} + \frac{\tau}{3}R_{hljm}\right)g_{ik} + \left(-T_{hjkl} + T_{hkjl} + \frac{1}{2}S_{hljk} + \frac{\tau}{3}R_{hljk}\right)g_{im} \\ & + \left(-T_{hkml} + T_{hmkl} + \frac{1}{2}S_{hlkm} + \frac{\tau}{3}R_{hlkm}\right)g_{ij} + A_{hjkmil} - A_{ljkmi h} - A_{ijk mhl} + A_{ijm khl} \\ & - A_{hjm kil} + A_{l j m kih} - A_{ikm jhl} + A_{hkm jil} - A_{lkm jih} = 0. \end{aligned} \tag{4.2}$$

We transvect each term of (4.2) with R_{ihjk} . Here, we give a few representative terms as follows:

$$\begin{aligned} & -\frac{1}{8}\left(\|R\|^2 - \frac{\tau^2}{3}\right)\left(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}\right)R_{ihjk} \\ &= \frac{1}{6}\tau\left(\|R\|^2 - \frac{\tau^2}{3}\right)g_{lm}, \\ & -\left(-T_{ijkl} + T_{ikjl} + \frac{1}{2}S_{iljk} + \frac{\tau}{3}R_{iljk}\right)g_{hm}R_{ihjk} = 2\mathring{R}_{lm} - \hat{R}_{lm} - \frac{1}{18}\tau\|R\|^2g_{lm}, \\ & \left(-T_{ijml} + T_{imjl} + \frac{1}{2}S_{iljm} + \frac{\tau}{3}R_{iljm}\right)g_{hk}R_{ihjk} = \left(\frac{\tau^3}{72} - \frac{\tau}{24}\|R\|^2\right)g_{lm}, \\ & (A_{hjkmil} - A_{ljkmi h} - A_{ijk mhl} + A_{ijm khl} - A_{hjm kil} + A_{l j m kih} - A_{ikm jhl} + A_{hkm jil} - A_{lkm jih})R_{ihjk} \\ &= -4\check{R}_{lm} - 2\hat{R}_{lm} + 4\mathring{R}_{lm} \\ &= -\frac{\tau}{9}\|R\|^2g_{lm} - 2\hat{R}_{lm} + 4\mathring{R}_{lm}. \end{aligned}$$

Similarly, we can obtain remaining terms by transvecting with R_{ihjk} . Then, by rearranging all terms, we have Theorem B-(b).

Now we give an example of Theorem B.

Example 4.2 Let M be the Riemannian product manifold of 3-dimensional Riemannian manifolds $M_1(k)$ and $M_2(k)$ of constant sectional curvature k ($k \neq 0$). Then we can easily check that M is an Einstein manifold. But M can never be 2-stein: Let $X = (X_1, X_2)$ is a tangent vector with X_1, X_2 its components tangent to $M_1(k)$ and $M_2(k)$, respectively. Then, $\text{Tr}(R_X^2) = 2k^2(\|X_1\|^4 + \|X_2\|^4)$ which cannot be equal to $\mu_2(\|X_1\|^2 + \|X_2\|^2)^2$. Let $\{e_i\}$, $i = 1, 2, \dots, 6$ be an orthonormal basis of T_pM at any point $p = (p_1, p_2) \in M$, where $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$ are bases for $T_{p_1}M_1(k)$ and $T_{p_2}M_2(k)$, respectively. Then, we have

$$R_{1221} = R_{1331} = R_{2332} = R_{4554} = R_{4664} = R_{5665} = k, \tag{4.3}$$

and all the other components of R vanish. From (4.3) we have

$$\begin{aligned} \tau &= 12k, \quad \|R\|^2 = 24k^2, \quad \mathring{R} = -12k^3, \quad \hat{R} = -48k^3, \\ \check{R}_{ij} &= 8k^3\delta_{ij}, \quad \hat{R}_{ij} = -8k^3\delta_{ij}, \quad \mathring{R}_{ij} = -2k^3\delta_{ij}, \quad \bar{R}_{ij} = 4k^2\delta_{ij}. \end{aligned}$$

Therefore, we find that the curvature identities of Theorem B hold on M . Here we note that M is a super-Einstein manifold.

5. Appendix

In this appendix, we give the proof of Lemma 4.1.

By replacing the curvature tensor R by the Weyl curvature tensor W in (2.1), we can also obtain the curvature identity of W . In the case of $m = 6$ and $r = 2$, making use of the fact that the Weyl curvature tensor is traceless, we have the following curvature identity.

Proposition 5.1 The Weyl curvature tensor W of any 6-dimensional Riemannian manifold satisfies the following identity:

$$\begin{aligned} 0 &= \|W\|^2(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}) \\ &- 4\{W_{iabc}W_{jabc}(g_{hk}g_{lm} - g_{hm}g_{lk}) + W_{iabc}W_{kabc}(g_{hm}g_{lj} - g_{hj}g_{lm}) + W_{iabc}W_{mabc}(g_{hj}g_{lk} - g_{hk}g_{lj}) \\ &\quad + W_{habc}W_{jabc}(g_{im}g_{lk} - g_{ik}g_{lm}) + W_{habc}W_{kabc}(g_{ij}g_{lm} - g_{im}g_{lj}) + W_{habc}W_{mabc}(g_{ik}g_{lj} - g_{ij}g_{lk}) \\ &\quad + W_{iabc}W_{jabc}(g_{ik}g_{hm} - g_{im}g_{hk}) + W_{iabc}W_{kabc}(g_{im}g_{hj} - g_{ij}g_{hm}) + W_{iabc}W_{mabc}(g_{ij}g_{hk} - g_{ik}g_{hj})\} \\ &- 8\{(W_{iabj}W_{kabh} - W_{iabk}W_{jabh})g_{lm} - (W_{iabj}W_{kabl} - W_{iabk}W_{jabl})g_{hm} \\ &\quad - (W_{iabj}W_{mabh} - W_{iabm}W_{jabh})g_{lk} + (W_{iabj}W_{mabl} - W_{iabm}W_{jabl})g_{hk} \\ &\quad + (W_{iabk}W_{mabh} - W_{iabm}W_{kabh})g_{lj} - (W_{iabk}W_{mabl} - W_{iabm}W_{kabl})g_{hj} \\ &\quad + (W_{habj}W_{kabl} - W_{habk}W_{jabl})g_{im} - (W_{habj}W_{mabl} - W_{habm}W_{jabl})g_{ik} \\ &\quad + (W_{habk}W_{mabl} - W_{habm}W_{kabl})g_{ij}\} \\ &+ 4\{W_{abih}W_{abjk}g_{lm} - W_{abih}W_{abjm}g_{lk} + W_{abih}W_{abkm}g_{lj} - W_{abil}W_{abjk}g_{hm} \\ &\quad + W_{abil}W_{abjm}g_{hk} - W_{abil}W_{abkm}g_{hj} + W_{abhl}W_{abjk}g_{im} - W_{abhl}W_{abjm}g_{ik} + W_{abhl}W_{abkm}g_{ij}\} \\ &+ 8\{W_{ahjk}W_{amil} - W_{aljk}W_{amih} - W_{ajhl}W_{aikm} - W_{aijk}W_{amhl} + W_{ajih}W_{atmk} \\ &\quad + W_{ajil}W_{ahkm} - W_{ahjm}W_{akil} + W_{aljm}W_{akih} + W_{aijm}W_{akhl}\}. \end{aligned} \tag{5.1}$$

Proof of Lemma 4.1. The Weyl tensor W on a 6-dimensional Riemannian manifold is given by

$$W_{pqrs} = R_{pqrs} - \frac{1}{4}(\rho_{ps}g_{qr} + \rho_{qr}g_{ps} - \rho_{pr}g_{qs} - \rho_{qs}g_{pr}) + \frac{\tau}{20}(g_{ps}g_{qr} - g_{pr}g_{qs}).$$

Since M is Einstein, we have $W_{pqrs} = R_{pqrs} - \frac{\tau}{30}(g_{ps}g_{qr} - g_{pr}g_{qs})$. We substitute the Weyl tensor into (5.1). Then, we have

$$\begin{aligned} \|W\|^2 &= \|R\|^2 - \frac{\tau^2}{15}, \\ W_{iabc}W_{jabc} &= \bar{R}_{ij} - \frac{\tau^2}{90}g_{ij}, \\ W_{iabj}W_{kabh} - W_{iabk}W_{jabh} &= R_{iabj}R_{kabh} - R_{iabk}R_{jabh} - \frac{2}{15}\tau R_{ihjk} - \frac{7}{900}\tau^2(g_{ij}g_{hk} - g_{ik}g_{hj}), \\ W_{abih}W_{abjk} &= R_{abih}R_{abjk} + \frac{2}{15}\tau R_{ihjk} + \frac{\tau^2}{450}(g_{ij}g_{hk} - g_{ik}g_{hj}), \\ W_{ahjk}W_{amil} &= R_{ahjk}R_{amil} - \frac{\tau}{30}(R_{kmil}g_{hj} - R_{jmil}g_{hk} + R_{lhjk}g_{im} - R_{ihjk}g_{lm}) \\ &\quad + \frac{\tau^2}{900}(g_{im}g_{hj}g_{lk} - g_{ik}g_{hj}g_{lm} - g_{im}g_{hk}g_{lj} + g_{ij}g_{hk}g_{lm}). \end{aligned} \tag{5.2}$$

Making use of (5.2), the right-hand side of (5.1) is the following:

$$\begin{aligned} & \left(\|R\|^2 - \frac{\tau^2}{15} \right) (g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}) \\ & - 4 \left\{ \bar{R}_{ij}(g_{hk}g_{lm} - g_{hm}g_{lk}) + \bar{R}_{ik}(g_{hm}g_{lj} - g_{hj}g_{lm}) + \bar{R}_{im}(g_{hj}g_{lk} - g_{hk}g_{lj}) \right. \\ & \quad + \bar{R}_{hj}(g_{im}g_{lk} - g_{ik}g_{lm}) + \bar{R}_{hk}(g_{ij}g_{lm} - g_{im}g_{lj}) + \bar{R}_{hm}(g_{ik}g_{lj} - g_{ij}g_{lk}) \\ & \quad + \bar{R}_{lj}(g_{ik}g_{hm} - g_{im}g_{hk}) + \bar{R}_{lk}(g_{im}g_{hj} - g_{ij}g_{hm}) + \bar{R}_{lm}(g_{ij}g_{hk} - g_{ik}g_{hj}) \\ & \quad \left. - \frac{\tau^2}{30}(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}) \right\} \\ & - 8 \left\{ (R_{iabj}R_{kabh} - R_{iabk}R_{jabh})g_{lm} - (R_{iabj}R_{kabl} - R_{iabk}R_{jabl})g_{hm} \right. \\ & \quad - (R_{iabj}R_{mabh} - R_{iabm}R_{jabh})g_{lk} + (R_{iabj}R_{mabl} - R_{iabm}R_{jabl})g_{hk} \\ & \quad + (R_{iabk}R_{mabh} - R_{iabm}R_{kabh})g_{lj} - (R_{iabk}R_{mabl} - R_{iabm}R_{kabl})g_{hj} \\ & \quad + (R_{habj}R_{kabl} - R_{habk}R_{jabl})g_{im} - (R_{habj}R_{mabl} - R_{habm}R_{jabl})g_{ik} \\ & \quad + (R_{habk}R_{mabl} - R_{habm}R_{kabl})g_{ij} \\ & \quad - \frac{2}{15}\tau(R_{ihjk}g_{lm} - R_{iljk}g_{hm} - R_{ihjm}g_{lk} + R_{iljm}g_{hk} + R_{ihkm}g_{lj} \\ & \quad \quad - R_{ilkm}g_{hj} + R_{hljk}g_{im} - R_{hljm}g_{ik} + R_{hlkm}g_{ij}) \\ & \quad \left. - \frac{7}{300}\tau^2(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj}) \right\} \end{aligned}$$

$$\begin{aligned}
 &+4\left\{R_{abih}R_{abjk}g_{lm} - R_{abil}R_{abjk}g_{hm} - R_{abih}R_{abjm}g_{lk} + R_{abil}R_{abjm}g_{hk} + R_{abih}R_{abkm}g_{lj} \right. \\
 &\quad - R_{abil}R_{abkm}g_{hj} + R_{abhl}R_{abjk}g_{im} - R_{abhl}R_{abjm}g_{ik} + R_{abhl}R_{abkm}g_{ij} \\
 &\quad + \frac{2}{15}\tau(R_{ihjk}g_{lm} - R_{iljk}g_{hm} - R_{ihjm}g_{lk} + R_{iljm}g_{hk} + R_{ihkm}g_{lj} \\
 &\quad \quad - R_{ilkm}g_{hj} + R_{hljk}g_{im} - R_{hljm}g_{ik} + R_{hlkm}g_{ij}) \\
 &\quad \left. + \frac{\tau^2}{150}(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj})\right\} \\
 &+8\left\{R_{ahjk}R_{amil} - R_{aljk}R_{amih} - R_{aijk}R_{amhl} + R_{aijm}R_{akhl} - R_{ahjm}R_{akil} \right. \\
 &\quad + R_{aljm}R_{akih} - R_{aikm}R_{ajhl} + R_{ahkm}R_{ajil} - R_{alkm}R_{ajih} \\
 &\quad - \frac{2}{15}\tau(R_{ilkm}g_{hj} - R_{iljm}g_{hk} - R_{hljk}g_{im} - R_{ihjk}g_{lm} - R_{ihkm}g_{lj} \\
 &\quad \quad + R_{ihjm}g_{lk} + R_{iljk}g_{hm} - R_{hlkm}g_{ij} + R_{hljm}g_{ik}) \\
 &\quad \left. + \frac{\tau^2}{150}(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj})\right\}.
 \end{aligned}$$

Rearranging these terms, we have

$$\begin{aligned}
 0 = &\left\{(\|R\|^2 + \frac{\tau^2}{3})(g_{ij}g_{hk}g_{lm} - g_{ij}g_{hm}g_{lk} - g_{ik}g_{hj}g_{lm} + g_{ik}g_{hm}g_{lj} + g_{im}g_{hj}g_{lk} - g_{im}g_{hk}g_{lj})\right\} \\
 &-4\left\{\bar{R}_{ij}(g_{hk}g_{lm} - g_{hm}g_{lk}) + \bar{R}_{ik}(g_{hm}g_{lj} - g_{hj}g_{lm}) + \bar{R}_{im}(g_{hj}g_{lk} - g_{hk}g_{lj}) \right. \\
 &\quad + \bar{R}_{hj}(g_{im}g_{lk} - g_{ik}g_{lm}) + \bar{R}_{hk}(g_{ij}g_{lm} - g_{im}g_{lj}) + \bar{R}_{hm}(g_{ik}g_{lj} - g_{ij}g_{lk}) \\
 &\quad \left. + \bar{R}_{lj}(g_{ik}g_{hm} - g_{im}g_{hk}) + \bar{R}_{lk}(g_{im}g_{hj} - g_{ij}g_{hm}) + \bar{R}_{lm}(g_{ij}g_{hk} - g_{ik}g_{hj})\right\} \\
 &+ \left(-8R_{iabj}R_{kabh} + 8R_{iabk}R_{jabh} + 4R_{abih}R_{abjk} + \frac{8}{3}\tau R_{ihjk}\right)g_{lm} \\
 &- \left(-8R_{iabj}R_{kabl} + 8R_{iabk}R_{jabl} + 4R_{abil}R_{abjk} + \frac{8}{3}\tau R_{iljk}\right)g_{hm} \\
 &- \left(-8R_{iabj}R_{mabh} + 8R_{iabm}R_{jabh} + 4R_{abih}R_{abjm} + \frac{8}{3}\tau R_{ihjm}\right)g_{lk} \\
 &+ \left(-8R_{iabj}R_{mabl} + 8R_{iabm}R_{jabl} + 4R_{abil}R_{abjm} + \frac{8}{3}\tau R_{iljm}\right)g_{hk} \\
 &+ \left(-8R_{iabk}R_{mabh} + 8R_{iabm}R_{kabh} + 4R_{abih}R_{abkm} + \frac{8}{3}\tau R_{ihkm}\right)g_{lj} \\
 &- \left(-8R_{iabk}R_{mabl} + 8R_{iabm}R_{kabl} + 4R_{abil}R_{abkm} + \frac{8}{3}\tau R_{ilkm}\right)g_{hj} \\
 &+ \left(-8R_{habj}R_{kabl} + 8R_{habk}R_{jabl} + 4R_{abhl}R_{abjk} + \frac{8}{3}\tau R_{hljk}\right)g_{im} \\
 &- \left(-8R_{habj}R_{mabl} + 8R_{habm}R_{jabl} + 4R_{abhl}R_{abjm} + \frac{8}{3}\tau R_{hljm}\right)g_{ik}
 \end{aligned}$$

$$\begin{aligned}
 & + \left(-8R_{habk}R_{mabl} + 8R_{habm}R_{kabl} + 4R_{abhl}R_{abkm} + \frac{8}{3}\tau R_{hlkm} \right) g_{ij} \\
 & + 8(R_{ahjk}R_{amil} - R_{aljk}R_{amih} - R_{aijk}R_{amhl} + R_{aijm}R_{akhl} - R_{ahjm}R_{akil} + R_{aljm}R_{akih} \\
 & \quad - R_{aikm}R_{ajhl} + R_{ahkm}R_{ajil} - R_{alkm}R_{ajih}).
 \end{aligned}$$

This completes the proof of Lemma 4.1. □

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