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A fixed point theorem using condensing operators and its applications to Erdélyi–Kober bivariate fractional integral equations

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Abstract: The primary aim of this article is to discuss and prove fixed point results using the operator type condensing map, and to obtain the existence of solution of Erdélyi–Kober bivariate fractional integral equation in a Banach space. An instance is given to explain the results obtained, and we construct an iterative algorithm by sinc interpolation to find an approximate solution of the problem with acceptable accuracy.

Key words: Measure of noncompactness, fractional quadratic integral equations, fixed point theorem, sinc interpolation, iterative algorithm

1. Introduction

Fractional calculus is a mathematical analysis branch that explores the several different scenarios of taking the differentiation operator D to real number powers or complex number powers. A fractional derivative in applied mathematics and mathematical analysis is a derivative of any noninteger order, real or complex. The first existence is in a letter written by G.W. Leibniz in 16^{th} century to Antoine de l'Hopital [24]. In one of N. H. Abel's early papers, [2], fractional calculus was adopted, where those elements can be considered: the definition of integration and differentiation of fractional order, the strictly inverse connection among them, the perception that differentiation and integration of fractional order can be perceived as being in the same generalized operation, and indeed the coherent form for arbitrary real order differentiation and integration. Over the 19th and early 20th centuries, the theory and applications of fractional calculus developed greatly, and countless contributors have provided interpretations for fractional derivatives and integrals. The applications of fractional calculus can be found in almost all disciplines of modern engineering and science, e.g., rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc. The major reason for the success of fractional calculus applications is that fractional-order models are often more accurate than integer-order. The researchers are able to model the nonlocal and distributed effects often encountered in natural and technical phenomena with the help of different types of fractional operators. The Erdélyi–Kober fractional integral is used in many branches of mathematics such as porous media, viscoelasticity and electrochemistry, etc. (see [14, 23]). Different types of fractional integral and

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fractional differential equations were solved by many researchers; see for instance [1, 5, 28, 32, 33] and references therein.

Schauder and Darbo's fixed point theorems play a key role in addressing functional integral equations. A significant role is played by the notion of a measure of noncompactness (MNC), in the fixed point theory. The essential paper of Kuratowski [25] pioneered this principle. By the mid-19th century, using the definition of a noncompactness measure, Darbo [16] proved a theorem that guarantees the existence of fixed points using condensing operators.

Fixed point theory and measure of noncompactness have many applications for solving different types of integral equations; see for instance [4, 8, 12, 13, 20–22, 27, 29–31, 37, 38] and references therein.

Let $(\mathscr{E}, \| \,.\, \|)$ be a real Banach space. Let $B(\theta, r)$ be a closed ball in $\mathscr{E}$ centered at θ and with radius r. If $\mathscr{O}$ is a nonempty subset of $\mathscr{E}$, then by $\bar{\mathscr{O}}$ and $\mathrm{Conv}\,\mathscr{O}$ we denote the closure and convex closure of $\mathscr{O}$. Moreover, let $\mathfrak{M}_{\mathscr{E}}$ denote the family of all nonempty and bounded subsets of $\mathscr{E}$ and $\mathfrak{N}_{\mathscr{E}}$ its subfamily consisting of all relatively compact sets. We denote by $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$.

Definition 1.1 [11] A function $\vartheta : \mathfrak{M}_{\mathscr{E}} \to \mathbb{R}_+$ is called a MNC in $\mathscr{E}$ if it satisfies the following conditions:

(i) for all $\mathfrak{Y} \in \mathfrak{M}_{\mathscr{E}}$, we have $\vartheta(\mathfrak{Y}) = 0$ implies that $\mathfrak{Y}$ is precompact.

(ii) the family $\ker \vartheta = \{\mathfrak{Y} \in \mathfrak{M}_{\mathscr{E}} : \vartheta(\mathfrak{Y}) = 0\}$ is nonempty and $\ker \vartheta \subset \mathfrak{N}_{\mathscr{E}}$.

(iii) $\mathfrak{Y} \subseteq \mathfrak{Z} \implies \vartheta(\mathfrak{Y}) \leq \vartheta(\mathfrak{Z})$.

(iv) $\vartheta\left(\bar{\mathfrak{Y}}\right) = \vartheta(\mathfrak{Y})$.

(v) $\vartheta(\mathrm{Conv}\mathfrak{Y}) = \vartheta(\mathfrak{Y})$.

(vi) $\vartheta(\lambda\mathfrak{Y} + (1-\lambda)\mathfrak{Z}) \leq \lambda\vartheta(\mathfrak{Y}) + (1-\lambda)\vartheta(\mathfrak{Z})$ for $\lambda \in [0, 1]$.

(vii) if $\mathfrak{Y}_n \in \mathfrak{M}_{\mathscr{E}}$, $\mathfrak{Y}_n = \bar{\mathfrak{Y}}_n$, $\mathfrak{Y}_{n+1} \subset \mathfrak{Y}_n$ for $n = 1, 2, 3, ...$ and $\lim\limits_{n\to\infty} \vartheta(\mathfrak{Y}_n) = 0$ then $\bigcap\limits_{n=1}^{\infty} \mathfrak{Y}_n \neq \varphi$.

The family $\ker \vartheta$ is said to be the *kernel of measure* ϑ. Observe that the intersection set $\mathfrak{Y}_\infty$ from (vii) is a member of the family $\ker \vartheta$. In fact, since $\vartheta(\mathfrak{Y}_\infty) \leq \vartheta(\mathfrak{Y}_n)$ for any n, we infer that $\vartheta(\mathfrak{Y}_\infty) = 0$. This gives $\mathfrak{Y}_\infty \in \ker \vartheta$.

Definition 1.2 [11] Let $\mathscr{O}$ be a nonempty subset of a Banach space $\mathscr{E}$ and $\mathscr{B} : \mathscr{O} \to \mathscr{E}$ be a continuous operator transforming bounded subsets of $\mathscr{O}$ to bounded ones. We say that $\mathscr{B}$ satisfies the Darbo condition with a constant k with respect to the measure ϑ provided $\vartheta(\mathfrak{T}\mathfrak{Y}) \leq k\vartheta(\mathfrak{Y})$ for each $\mathfrak{Y} \in \mathfrak{M}_{\mathscr{E}}$ such that $\mathfrak{Y} \subset \mathscr{O}$.

We recall the Schauder and Darbo fixed point theorems:

Theorem 1.3 [3, Schauder] Let $\mathfrak{D}$ be a nonempty, closed and convex subset of a Banach space $\mathscr{E}$. Then every compact, continuous map $\mathscr{B} : \mathfrak{D} \to \mathfrak{D}$ has at least one fixed point.

Theorem 1.4 [16, Darbo] Let $\mathfrak{Z}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathscr{E}$. Let $\mathfrak{S} : \mathfrak{Z} \to \mathfrak{Z}$ be a continuous mapping. Assume that there is a constant $k \in [0, 1)$ such that

$$\vartheta(\mathfrak{S}\mathfrak{U}) \leq k\vartheta(\mathfrak{U}),\ \mathfrak{U} \subseteq \mathfrak{Z}.$$

Then $\mathfrak{S}$ has a fixed point.

In order to establish our fixed point theorem, we need some of the following related concepts.

Definition 1.5 [7] Let $F([0,\infty))$ be class of all function $f:[0,\infty)\to[0,\infty)$ and let Θ be class of all operators

$$\mathfrak{A}(.;.):F([0,\infty))\to F([0,\infty)),\ f\to\mathfrak{A}(f;.)$$

satisfying the following conditions:

(1) $\mathfrak{A}(f;t)>0$ for $t>0$ and $\mathfrak{A}(f;0)=0$,

(2) $\mathfrak{A}(f;t)\le\mathfrak{A}(f;s)$ for $t\le s$,

(3) $\lim_{n\to\infty}\mathfrak{A}(f;t_n)=\mathfrak{A}(f;\lim_{n\to\infty}t_n)$,

(4) $\mathfrak{A}(f;\max\{t,s\})=\max\{\mathfrak{A}(f;t),\mathfrak{A}(f;s)\}$ for some $f\in F([0,\infty))$.

Example 1.6 If $f:[0,\infty)\to[0,\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0,\infty)$, nonnegative and such that for each $t>0$, $\int_0^t f(s)ds>0$, then the operator defined by

$$\mathfrak{A}(f;t)=\int_0^t f(s)ds$$

satisfies the conditions of Definition 1.5.

Example 1.7 If $f:[0,\infty)\to[0,\infty)$ is a nondecreasing, continuous function such that $f(0)=0$ and $f(t)>0$ for $t>0$, then the operator defined by

$$\mathfrak{A}(f;t)=\frac{f(t)}{1+f(t)}$$

satisfies the conditions of Definition 1.5.

Example 1.8 If $f:[0,\infty)\to[0,\infty)$ is any function, then the operator defined by

$$\mathfrak{A}(f;t)=t$$

satisfies the conditions of Definition 1.5.

Definition 1.9 [22] Let $\mathbf{F}$ be the class of all functions $\mathfrak{H}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ satisfying the following conditions:

(1) $\max\{\iota,\varpi\}\le\mathfrak{H}(\iota,\varpi)$ for $\iota,\varpi\ge 0$.

(2) $\mathfrak{H}$ is continuous and nondecreasing.
For example, $\mathfrak{H}(\iota,\varpi)=\iota+\varpi$.

We denote by $\mathfrak{G}$ the class of all functions $\gamma:\mathbb{R}_+\to[0,1)$ which satisfy the following condition: $t_n\to 0$ whenever $\gamma(t_n)\to 1$ (see [19]).

2. A fixed point theorem involving condensing operators

In this section, we are introducing a new fixed point theorem using the operator type condensing map.

Theorem 2.1 *Let $\mathfrak{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$. Also, let $\mathscr{B}:\mathfrak{D}\to\mathfrak{D}$ be a continuous function and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function such that*

$$\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}(\mathscr{O})\right)\right),\varphi\left(\vartheta\left(\mathscr{B}(\mathscr{O})\right)\right)\right]\leq\gamma\left(\vartheta(\mathscr{O})\right)\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{O}\right)\right),\varphi\left(\vartheta\left(\mathscr{O}\right)\right)\right] \tag{2.1}$$

for all $\mathscr{O}\subseteq\mathfrak{D}$, $\gamma\in\mathfrak{S}$, $\mathfrak{H}\in\mathbf{F}$, $\mathfrak{A}(.,.)\in\Theta$, where ϑ is an arbitrary MNC. Then $\mathscr{B}$ has at least one fixed point in $\mathfrak{D}$.

Proof Let us define a sequence $(\mathfrak{D}_l)$ such that $\mathfrak{D}_0=\mathfrak{D}$ and $\mathfrak{D}_{l+1}=\mathrm{Conv}(\mathscr{B}\mathfrak{D}_l)$ for $n\geq 0$. We observe that $\mathscr{B}\mathfrak{D}_0=\mathscr{B}\mathfrak{D}\subseteq\mathfrak{D}=\mathfrak{D}_0$, $\mathfrak{D}_1=\mathrm{Conv}(\mathscr{B}\mathfrak{D}_0)\subseteq\mathfrak{D}=\mathfrak{D}_0$. Therefore, by continuing this process, we have $\mathfrak{D}_0\supseteq\mathfrak{D}_1\supseteq\mathfrak{D}_2\supseteq\ldots\supseteq\mathfrak{D}_l\supseteq\mathfrak{D}_{l+1}\supseteq\ldots$.
If there exists a natural number m such that $\vartheta(\mathfrak{D}_m)=0$, then $\mathfrak{D}_m$ is compact. By Schauder's theorem, now we know that $\mathscr{B}$ has a fixed point.
If $\vartheta(\mathfrak{D}_l)>0$ for some $l\geq 0$, by (2.1), we have

$$\begin{aligned}
&\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_{l+1}\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_{l+1}\right)\right)\right]\\
&=\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathrm{Conv}\left(\mathscr{B}\mathfrak{D}_l\right)\right)\right),\varphi\left(\vartheta\left(\mathrm{Conv}\left(\mathscr{B}\mathfrak{D}_l\right)\right)\right)\right]\\
&=\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathscr{B}\mathfrak{D}_l\right)\right)\right]\\
&\leq\gamma(\vartheta\left(\mathfrak{D}_l\right))\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right]\\
&<\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right].
\end{aligned}$$

As the sequence $\{\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right]\}$ is decreasing and nonnegative,

$$\lim_{l\to\infty}\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right]=r.$$

If possible, assume $r>0$. As $l\to\infty$, we have

$$r\leq r\lim_{l\to\infty}\gamma(\vartheta\left(\mathfrak{D}_l\right))$$

which gives $\lim\limits_{l\to\infty}\gamma(\vartheta\left(\mathfrak{D}_l\right))\geq 1$ and it is a contradiction. Hence,

$$\lim_{l\to\infty}\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right]=0.$$

Since $\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\geq 0$,

$$0\leq\max\left\{\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right\}\leq\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right].$$

As $l\to\infty$

$$\max\left\{\lim_{l\to\infty}\mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right),\lim_{l\to\infty}\varphi\left(\vartheta\left(\mathfrak{D}_l\right)\right)\right\}=0$$

i.e. $\lim_{l\to\infty} \mathfrak{A}\left(f;\vartheta\left(\mathfrak{D}_l\right)\right)=0$ implies $\lim_{l\to\infty}\vartheta\left(\mathfrak{D}_l\right)=0$.

Since $\mathfrak{D}_l \supseteq \mathfrak{D}_{l+1}$ by the hypothesis, we have conclusion that $\mathfrak{D}_\infty = \bigcap_{l=1}^{\infty}\mathfrak{D}_l$ is a nonempty, closed and convex subset of $\mathfrak{D}$, and $\mathfrak{D}_\infty$ is invariant under $\mathscr{B}$. Thus, Schauder's theorem indicates that $\mathscr{B}$ has a fixed point in $\mathfrak{D}_\infty \subseteq \mathfrak{D}$. This completes the proof of the theorem. □

Theorem 2.2 *Let $\mathfrak{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$. Also, let $\mathscr{B}:\mathfrak{D}\to\mathfrak{D}$ be a continuous function and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function such that*

$$\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}(\mathscr{O})\right)\right)+\varphi\left(\vartheta\left(\mathscr{B}(\mathscr{O})\right)\right)\leq\gamma\left(\vartheta(\mathscr{O})\right)\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{O}\right)\right)+\varphi\left(\vartheta\left(\mathscr{O}\right)\right)\right] \tag{2.2}$$

for all $\mathscr{O}\subseteq\mathfrak{D}$, $\gamma\in\mathfrak{S}$, $\mathfrak{A}(.,.)\in\Theta$, where ϑ is an arbitrary MNC. Then $\mathscr{B}$ has at least one fixed point in $\mathfrak{D}$.

Proof The result follows by taking $\mathfrak{A}(\iota,\varpi)=\iota+\varpi$, in Theorem 2.1. □

Theorem 2.3 *Let $\mathfrak{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$. Also, let $\mathscr{B}:\mathfrak{D}\to\mathfrak{D}$ be a continuous function such that*

$$\vartheta\left(\mathscr{B}(\mathscr{O})\right)\leq\gamma\left(\vartheta(\mathscr{O})\right)\vartheta\left(\mathscr{O}\right) \tag{2.3}$$

for all $\mathscr{O}\subseteq\mathfrak{D}$, $\gamma\in\mathfrak{S}$, where ϑ is an arbitrary MNC. Then $\mathscr{B}$ has at least one fixed point in $\mathfrak{D}$.

Proof The result follows by taking $\mathfrak{A}(f;t)=t$ and $\varphi\equiv 0$, in Theorem 2.2. □

Theorem 2.4 *Let $\mathfrak{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$. Also, let $\mathscr{B}:\mathfrak{D}\to\mathfrak{D}$ be a continuous function such that*

$$\vartheta\left(\mathscr{B}(\mathscr{O})\right)\leq k\vartheta\left(\mathscr{O}\right),\ k\in[0,1) \tag{2.4}$$

for all $\mathscr{O}\subseteq\mathfrak{D}$, where ϑ is an arbitrary MNC. Then $\mathscr{B}$ has at least one fixed point in $\mathfrak{D}$.

Proof The result follows by taking $\gamma(t)=k$ for all $t\geq 0$, $k\in[0,1)$, in Theorem 2.3. □

Definition 2.5 *[15] An element $(p,q)\in\mathscr{O}\times\mathscr{O}$ is called a coupled fixed point of a mapping $\mathscr{B}:\mathscr{O}\times\mathscr{O}\to\mathscr{O}$ if $\mathscr{B}(p,q)=p$ and $\mathscr{B}(q,p)=q$.*

Theorem 2.6 *[11] Suppose $\vartheta_1,\vartheta_2,\ldots,\vartheta_n$ are MNC in $\mathscr{E}_1,\mathscr{E}_2,\ldots,\mathscr{E}_n$, respectively. Furthermore, let $\mathscr{O}:\mathbb{R}_+^n\to\mathbb{R}_+$ be convex and $\mathfrak{F}\left(p_1,p_2,\ldots,p_n\right)=0$ if and only if $p_l=0$ for $l=1,2,\ldots,n$. Then $\vartheta(\mathscr{O})=\mathfrak{F}\left(\vartheta_1(\mathscr{O}_1),\vartheta_2(\mathscr{O}_2),\ldots,\vartheta_n(\mathscr{O}_n)\right)$ define a MNC in $\mathscr{E}_1\times\mathscr{E}_2\times\ldots\times\mathscr{E}_n$, where $\mathscr{O}_l$ denotes the natural projection of $\mathscr{O}$ into $\mathscr{E}_l$ for $l=1,2,\ldots,n$.*

Example 2.7 *[11] Let ϑ be a MNC on $\mathscr{E}$. Define $\mathfrak{F}(p,q)=p+q$, $p,q\in\mathbb{R}_+$. Then $\mathfrak{F}$ has all the properties mentioned in Theorem 2.6. Hence, $\vartheta^{cf}(\mathscr{O})=\vartheta(\mathscr{O}_1)+\vartheta(\mathscr{O}_2)$ is a MNC in the space $\mathscr{E}\times\mathscr{E}$, where $\mathscr{O}_l$, $l=1,2$ denote the natural projections of $\mathscr{O}$.*

Theorem 2.8 *Let $\mathfrak{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$. Also, let $\mathscr{B}:\mathfrak{D}\times\mathfrak{D}\to\mathfrak{D}$ be a continuous function and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing continuous function satisfying $\varphi(p+q)\leq\varphi(p)+\varphi(q),\ p,q\geq 0$ such that*

$$\begin{aligned}&\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}(\mathscr{O}_1\times\mathscr{O}_2)\right)\right),\varphi\left(\vartheta\left(\mathscr{B}(\mathscr{O}_1\times\mathscr{O}_2)\right)\right)\right]\\&\leq\frac{1}{2}\gamma\left(\vartheta(\mathscr{O}_1)+\vartheta(\mathscr{O}_2)\right)\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right),\varphi\left(\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right)\right]\end{aligned}$$

for all $\mathscr{O}_1,\mathscr{O}_2\subseteq\mathfrak{D},\ \gamma\in\mathfrak{S},\ \mathfrak{H}\in\mathbf{F},\ \mathfrak{A}(.,.)\in\Theta,$ where ϑ is an arbitrary MNC. Also, $\mathfrak{A}(f;p+q)\leq\mathfrak{A}(f;p)+\mathfrak{A}(f;q),\ p,q\geq 0$. Then $\mathscr{B}$ has at least one coupled fixed point in $\mathfrak{D}$.

Proof We observe that $\vartheta^{cf}(\mathscr{O})=\vartheta(\mathscr{O}_1)+\vartheta(\mathscr{O}_2)$ is a MNC on $\mathscr{E}\times\mathscr{E}$ for any bounded subset $\mathscr{O}\subseteq\mathscr{E}\times\mathscr{E}$, where $\mathscr{O}_1,\mathscr{O}_2$ denote the natural projection of $\mathscr{O}$.
Consider a mapping $\mathscr{B}^{cf}:\mathfrak{D}\times\mathfrak{D}\to\mathfrak{D}\times\mathfrak{D}$ by $\mathscr{B}^{cf}(p,q)=\left(\mathscr{B}(p,q),\mathscr{B}(q,p)\right)$.
It is trivial that $\mathscr{B}^{cf}$ is continuous. Let $\mathscr{O}\subseteq\mathfrak{D}\times\mathfrak{D}$, then

$$\begin{aligned}&\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta^{cf}\left(\mathscr{B}^{cf}\left(\mathscr{O}\right)\right)\right),\varphi\left(\vartheta^{cf}\left(\mathscr{B}^{cf}\left(\mathscr{O}\right)\right)\right)\right]\\
&\leq\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta^{cf}\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\times\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right),\varphi\left(\vartheta^{cf}\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\times\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right)\right]\\
&=\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\right)+\vartheta\left(\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right),\varphi\left(\vartheta\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\right)+\vartheta\left(\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right)\right]\\
&\leq\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\right)\right)+\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right),\varphi\left(\vartheta\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\right)\right)+\varphi\left(\vartheta\left(\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right)\right]\\
&\leq\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\right)\right),\varphi\left(\vartheta\left(\mathscr{B}\left(\mathscr{O}_1\times\mathscr{O}_2\right)\right)\right)\right]+\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right),\varphi\left(\vartheta\left(\mathscr{B}\left(\mathscr{O}_2\times\mathscr{O}_1\right)\right)\right)\right]\\
&\leq\frac{1}{2}\gamma\left(\vartheta(\mathscr{O}_1)+\vartheta(\mathscr{O}_2)\right)\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right),\varphi\left(\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right)\right]\\
&+\frac{1}{2}\gamma\left(\vartheta(\mathscr{O}_2)+\vartheta(\mathscr{O}_1)\right)\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{O}_2\right)+\vartheta\left(\mathscr{O}_1\right)\right),\varphi\left(\vartheta\left(\mathscr{O}_2\right)+\vartheta\left(\mathscr{O}_1\right)\right)\right]\\
&=\gamma\left(\vartheta(\mathscr{O}_1)+\vartheta(\mathscr{O}_2)\right)\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right),\varphi\left(\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right)\right]\\
&=\gamma\left(\vartheta^{cf}(\mathscr{O})\right)\mathfrak{H}\left[\mathfrak{A}\left(f;\vartheta^{cf}\left(\mathscr{O}\right)\right),\varphi\left(\vartheta^{cf}\left(\mathscr{O}\right)\right)\right].\end{aligned}$$

By Theorem 2.1, we conclude that $\mathscr{B}^{cf}$ has at least one fixed point in $\mathfrak{D}\times\mathfrak{D}$, i.e. $\mathscr{B}$ has minimum of one coupled fixed point. □

Corollary 2.9 *Let $\mathfrak{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$. Also, let $\mathscr{B}:\mathfrak{D}\times\mathfrak{D}\to\mathfrak{D}$ be a continuous function and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing continuous function satisfying $\varphi(p+q)\leq\varphi(p)+\varphi(q),\ p,q\geq 0$ such that*

$$\begin{aligned}&\mathfrak{H}\left[\vartheta\left(\mathscr{B}(\mathscr{O}_1\times\mathscr{O}_2)\right),\varphi\left(\vartheta\left(\mathscr{B}(\mathscr{O}_1\times\mathscr{O}_2)\right)\right)\right]\\&\leq\frac{k}{2}\mathfrak{H}\left[\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right),\varphi\left(\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right)\right]\end{aligned}$$

for all $k\in[0,1),\ \mathscr{O}_1,\mathscr{O}_2\subseteq\mathfrak{D},\ \mathfrak{H}\in\mathbf{F},$ where ϑ is an arbitrary MNC. Then $\mathscr{B}$ has at least one coupled fixed point in $\mathfrak{D}$.

Proof The result can be obtained by taking $\mathfrak{A}(f;t)=t$ and $\gamma(t)=k,\ k\in[0,1)$, in Theorem 2.8. □

Corollary 2.10 *Let $\mathfrak{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$. Also, let $\mathscr{B}:\mathfrak{D}\times\mathfrak{D}\to\mathfrak{D}$ be a continuous function and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing continuous function satisfying $\varphi(p+q)\leq\varphi(p)+\varphi(q),\ p,q\geq 0$ such that*

$$\begin{aligned}&\vartheta\left(\mathscr{B}(\mathscr{O}_1\times\mathscr{O}_2)\right)+\varphi\left(\vartheta\left(\mathscr{B}(\mathscr{O}_1\times\mathscr{O}_2)\right)\right)\\ &\leq\frac{k}{2}\left[\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)+\varphi\left(\vartheta\left(\mathscr{O}_1\right)+\vartheta\left(\mathscr{O}_2\right)\right)\right]\end{aligned}$$

for all $k\in[0,1)$, $\mathscr{O}_1,\mathscr{O}_2\subseteq\mathfrak{D}$, where ϑ is an arbitrary MNC. Then $\mathscr{B}$ has at least one coupled fixed point in $\mathfrak{D}$.

Proof The result can be obtained by taking $\mathfrak{H}(\iota,\varpi)=\iota+\varpi$ in Corollary 2.9. □

3. Application to Erdélyi–Kober bivariate fractional integral equations

Alamo and Rodríguez [6] defined the Erdélyi–Kober fractional integral of a continuous function f as

$$I_\beta^\gamma f(t)=\frac{\beta}{\Gamma(\gamma)}\int_0^t\frac{s^{\beta-1}f(s)}{\left(t^\beta-s^\beta\right)^{1-\gamma}}ds,\ \beta>0,\ 0<\gamma<1.$$

Using the Erdélyi–Kober fractional integral, Darwish and Sadarangani [17, 18] solved functional integral equations.

Analogous to the above definition, for a continuous function g on $\mathbb{R}\times\mathbb{R}$, we define the Erdélyi–Kober type bivariate fractional integral as follows:

$$I_\beta^{\gamma_1,\gamma_2}g(x,y)=\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^x\int_0^y\frac{s^{\beta-1}t^{\beta-1}g(t,s)}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds,$$

where $\beta>0,\ 0<\gamma_1,\ \gamma_2<1,\ \Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\ z>0.$

In this article, we work in the space $\mathscr{E}=C(I\times I)$ which consists of real-valued continuous functions on $I\times I$, where $I=[0,1]$. The norm of the space $\mathscr{E}$ is given by

$$\parallel x\parallel=\sup\left\{|x(t,s)|:t,s\in I\right\},\ x\in\mathscr{E}.$$

The space $\mathscr{E}$ has the Banach algebra structure.

Let $\mathscr{O}$ be a fixed nonempty and bounded subset of the space $\mathscr{E}=C(I\times I)$ and for $x\in\mathscr{E}$ and $\epsilon>0$, denote by $\omega(x,\epsilon)$ the modulus of continuity

$$\omega(x,\epsilon)=\sup\left\{|x(t,s)-x(u,v)|:t,s,u,v\in I,|t-u|\leq\epsilon,|s-v|\leq\epsilon\right\}.$$

Further, we define

$$\omega(\mathscr{O},\epsilon)=\sup\left\{\omega(x,\epsilon):x\in\mathscr{O}\right\}.$$

$$\omega_0(\mathscr{O})=\lim_{\epsilon\to 0}\omega(\mathscr{O},\epsilon).$$

It can be shown that the function ω_0 is a measure of noncompactness in the space $\mathscr{E}$ (see [13]).

Now we discuss the existence of the solution of the following bivariate fractional integral equation

$$z(x,y)=g(x,y)+\frac{\beta^2z(x,y)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^x\int_0^y\frac{s^{\beta-1}t^{\beta-1}u(x,y,t,s,z(t,s))}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds,\tag{3.1}$$

where $\beta>0, 0<\gamma_1,\gamma_2<1,\ x,y\in I$.
We consider the following assumptions:

(1) $g\in\mathscr{E}$;

(2) $u: I\times I\times I\times I\times\mathbb{R}\to\mathbb{R}$ is a continuous function and there exists a nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such $|u(x,y,t,s,z)|\le\psi(|z|),\ x,y,t,s\in I,\ z\in\mathbb{R}$;

(3) There exists $d_0>0$ such that $\parallel g\parallel+\frac{\psi(d_0)d_0}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}<d_0$ and $\psi(d_0)<\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)$.

Let $B_{d_0}=\{z\in\mathscr{E}:\parallel z\parallel\le d_0\}$ be the closed ball with center 0 and radius d_0.

Lemma 3.1 *[17] If $\mathfrak{W}:\mathbb{R}_+\to\mathbb{R}_+$ is the function defined by $\mathfrak{W}(p)=p^\beta$, then:*

(1) If $\beta\ge 1$ and $p_1,p_2\in I$ with $p_1<p_2$, then $p_2^\beta-p_1^\beta\le\beta(p_2-p_1)$.

(2) If $0<\beta<1$ and $p_1,p_2\in I$ with $p_1<p_2$, then $p_2^\beta-p_1^\beta\le(p_2-p_1)^\beta$.

We use Lemma 3.1 to prove the following theorem.

Theorem 3.2 *Under the hypothesis (1)–(3), Equation (3.1) has at least one solution in $\mathscr{E}$.*

Proof Let the operator $\mathfrak{P}$ be defined on $\mathscr{E}$

$$(\mathfrak{P}z)(x,y)=g(x,y)+z(x,y)(\mathfrak{V}z)(x,y),$$

where

$$(\mathfrak{V}z)(x,y)=\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^x\int_0^y\frac{s^{\beta-1}t^{\beta-1}u(x,y,t,s,z(t,s))}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds \text{ for } t,s\in I.$$

To prove that $\mathfrak{P}$ maps into $\mathscr{E}$ it must be established that if $z\in\mathscr{E}$ then $(\mathfrak{V}z)(x,y)\in\mathscr{E}$.
To prove this, fixed $\epsilon>0$. Also, let $z\in\mathscr{E}$ and $x_1,x_2,y_1,y_2\in I$ with $|x_2-x_1|\le\epsilon,\ |y_2-y_1|\le\epsilon$. Without loss of generality, we can assume $x_1\le x_2,\ y_1\le y_2$. Also, let

$$\omega_u(\epsilon)=\sup\left\{\begin{array}{c}|u(x_2,y_2,t,s,l)-u(x_1,y_1,t,s,l)|: x_1,x_2,y_1,y_2,t,s\in I,\\ |x_2-x_1|\le\epsilon,|y_2-y_1|\le\epsilon,l\in[-\parallel z\parallel,\parallel z\parallel]\end{array}\right\}.$$

Then, we get

$$
\begin{aligned}
&|(\mathfrak{V}z)(x_2,y_2)-(\mathfrak{V}z)(x_1,y_1)|\\
&=\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_2,y_2,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^{x_1}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_1^\beta-t^\beta\right)^{1-\gamma_2}\left(x_1^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|\\
&\le\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_2,y_2,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|\\
&+\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^{x_1}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|\\
&+\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^{x_1}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^{x_1}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_1^\beta-t^\beta\right)^{1-\gamma_2}\left(x_1^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|\\
&=\mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{I}_1&=\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_2,y_2,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|\\
&\le\frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}\left|u(x_2,y_2,t,s,z(t,s))-u(x_1,y_1,t,s,z(t,s))\right|}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds\\
&\le\frac{\omega_u(\epsilon)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^{x_2}\frac{\beta s^{\beta-1}ds}{\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}\int_0^{y_2}\frac{\beta t^{\beta-1}dt}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_1}}\\
&=\frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}x_2^{\beta\gamma_1}y_2^{\beta\gamma_2}\\
&\le\frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)},
\end{aligned}
$$

$$\mathfrak{I}_2 = \frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^{x_2}\int_0^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^{x_1}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|$$

$$= \frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_{x_1}^{x_2}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds+\int_0^{x_2}\int_{y_1}^{y_2}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|$$

$$\leq \frac{\beta^2\psi(\| z \|)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_{x_1}^{x_2}\int_0^{y_1}s^{\beta-1}t^{\beta-1}\left(y_2^\beta-t^\beta\right)^{\gamma_2-1}\left(x_2^\beta-s^\beta\right)^{\gamma_1-1}dtds$$

$$+\frac{\beta^2\psi(\| z \|)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^{x_2}\int_{y_1}^{y_2}s^{\beta-1}t^{\beta-1}\left(y_2^\beta-t^\beta\right)^{\gamma_2-1}\left(x_2^\beta-s^\beta\right)^{\gamma_1-1}dtds$$

$$\leq \frac{\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\left[\left(x_2^\beta-x_1^\beta\right)^{\gamma_1}\left\{y_2^{\beta\gamma_2}-\left(y_2^\beta-y_1^\beta\right)^{\gamma_2}\right\}+x_2^{\beta\gamma_1}\left(y_2^\beta-y_1^\beta\right)^{\gamma_2}\right]$$

$$\leq \frac{\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\left\{\left(x_2^\beta-x_1^\beta\right)^{\gamma_1}y_2^{\beta\gamma_2}+x_2^{\beta\gamma_1}\left(y_2^\beta-y_1^\beta\right)^{\gamma_2}\right\},$$

and

$$\mathfrak{I}_3 = \frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^{x_1}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_2^\beta-t^\beta\right)^{1-\gamma_2}\left(x_2^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^{x_1}\int_0^{y_1}\frac{s^{\beta-1}t^{\beta-1}u(x_1,y_1,t,s,z(t,s))}{\left(y_1^\beta-t^\beta\right)^{1-\gamma_2}\left(x_1^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right|$$

$$\leq \frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\times$$

$$\int_0^{x_1}\int_0^{y_1}s^{\beta-1}t^{\beta-1}\left|u(x_1,y_1,t,s,z(t,s))\right|\left|\left(y_2^\beta-t^\beta\right)^{\gamma_2-1}\left(x_2^\beta-s^\beta\right)^{\gamma_1-1}-\left(y_1^\beta-t^\beta\right)^{\gamma_2-1}\left(x_1^\beta-s^\beta\right)^{\gamma_1-1}\right|dtds$$

$$\leq \frac{\beta^2\psi(\| z \|)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^{x_1}\int_0^{y_1}s^{\beta-1}t^{\beta-1}\left\{\left(y_2^\beta-t^\beta\right)^{\gamma_2-1}\left(x_1^\beta-s^\beta\right)^{\gamma_1-1}-\left(y_1^\beta-t^\beta\right)^{\gamma_2-1}\left(x_2^\beta-s^\beta\right)^{\gamma_1-1}\right\}dtds$$

$$= \frac{\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\times$$

$$\left\{x_2^{\beta\gamma_1}\left(y_2^\beta-y_1^\beta\right)^{\gamma_2}+y_2^{\beta\gamma_2}\left(x_2^\beta-x_1^\beta\right)^{\gamma_1}-\left(x_2^\beta-x_1^\beta\right)^{\gamma_1}\left(y_2^\beta-y_1^\beta\right)^{\gamma_2}+x_1^{\beta\gamma_1}y_1^{\beta\gamma_2}-x_2^{\beta\gamma_1}y_2^{\beta\gamma_2}\right\}$$

$$\leq \frac{\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\left\{x_2^{\beta\gamma_1}\left(y_2^\beta-y_1^\beta\right)^{\gamma_2}+y_2^{\beta\gamma_2}\left(x_2^\beta-x_1^\beta\right)^{\gamma_1}\right\}.$$

Therefore,

$$|(\mathfrak{V}z)(x_2,y_2)-(\mathfrak{V}z)(x_1,y_1)|$$

$$\leq \frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}+\frac{2\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\left\{x_2^{\beta\gamma_1}\left(y_2^\beta-y_1^\beta\right)^{\gamma_2}+y_2^{\beta\gamma_2}\left(x_2^\beta-x_1^\beta\right)^{\gamma_1}\right\}.$$

Case I: If $\beta \geq 1,$ then

$$|(\mathfrak{V}z)(x_2,y_2)-(\mathfrak{V}z)(x_1,y_1)|$$

$$\leq \frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}+\frac{2\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\left\{\beta^{\gamma_1}\epsilon^{\gamma_1}+\beta^{\gamma_2}\epsilon^{\gamma_2}\right\}.$$

Case II: If $0<\beta<1$, then

$$
\begin{aligned}
&|(\mathfrak{V}z)(x_2,y_2)-(\mathfrak{V}z)(x_1,y_1)| \\
&\le \frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}+\frac{2\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\left\{\epsilon^{\beta\gamma_1}+\epsilon^{\beta\gamma_2}\right\}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&|(\mathfrak{V}z)(x_2,y_2)-(\mathfrak{V}z)(x_1,y_1)| \\
&\le \frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}+\frac{2\psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\max\left\{\beta^{\gamma_1}\epsilon^{\gamma_1}+\beta^{\gamma_2}\epsilon^{\gamma_2},\epsilon^{\beta\gamma_1}+\epsilon^{\beta\gamma_2}\right\}.
\end{aligned}
$$

Since u is a continuous function on $I\times I\times I\times I\times[-\| z \|,\| z \|]$, $\omega_u(\epsilon)\to 0$ as $\epsilon\to 0$.
We have $|(\mathfrak{V}z)(x_2,y_2)-(\mathfrak{V}z)(x_1,y_1)|\to 0$. Thus, $\mathfrak{V}z\in\mathscr{E}$.
Again for $z\in\mathscr{E}$ and $x,y\in I$, we have

$$
\begin{aligned}
&|(\mathfrak{P}z)(x,y)| \\
&\le |g(x,y)|+\frac{\beta^2\,|z(x,y)|}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^x\int_0^y \frac{s^{\beta-1}t^{\beta-1}\,|u(x,y,t,s,z(t,s))|}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds \\
&\le \| g \| +\frac{\beta^2 \| z \| \psi(\| z \|)}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^x\int_0^y \frac{s^{\beta-1}t^{\beta-1}}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds \\
&= \| g \| +\frac{x^{\beta\gamma_1}y^{\beta\gamma_2} \| z \| \psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)},
\end{aligned}
$$

i.e.

$$
\| \mathfrak{P}z \| \le \| g \| +\frac{\| z \| \psi(\| z \|)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}.
$$

By assumption (3), we observe that $\mathfrak{P}$ maps B_{d_0} into itself.
Let $\epsilon>0$ and $z\in B_{d_0}$ be fixed. We consider $\bar z\in B_{d_0}$ with $\| z-\bar z \|<\epsilon$. For any $x,y\in I$,

$$
\begin{aligned}
&|(\mathfrak{V}z)(x,y)-(\mathfrak{V}\bar z)(x,y)| \\
&= \frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left|\int_0^x\int_0^y \frac{s^{\beta-1}t^{\beta-1}u(x,y,t,s,z(t,s))}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds-\int_0^x\int_0^y \frac{s^{\beta-1}t^{\beta-1}u(x,y,t,s,\bar z(t,s))}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds\right| \\
&\le \frac{\beta^2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^x\int_0^y \frac{s^{\beta-1}t^{\beta-1}\,|u(x,y,t,s,z(t,s))-u(x,y,t,s,\bar z(t,s))|}{\left(y^\beta-t^\beta\right)^{1-\gamma_2}\left(x^\beta-s^\beta\right)^{1-\gamma_1}}dtds \\
&\le \frac{\delta_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)},
\end{aligned}
$$

where

$$
\delta_u(\epsilon)=\sup\left\{\begin{array}{c}|u\left(x,y,t,s,l_1\right)-u\left(x,y,t,s,l_2\right)|:x,y,t,s\in I,\\ \| l_2-l_1 \|\le\epsilon, l_1,l_2\in[-d_0,d_0]\end{array}\right\}.
$$

Considering uniform continuity of u on $I\times I\times I\times I\times[-d_0,d_0]$, $\delta_u(\epsilon)\to 0$ as $\epsilon\to 0$. Thus, $\mathfrak{V}$ is continuous on B_{d_0}, so $\mathfrak{P}$ is continuous on B_{d_0}.

Now, let $\mathscr{O} \subseteq B_{d_0}$ be a nonempty set and $z \in \mathscr{O}$. For fixed $\epsilon > 0$ and $x_1, x_2, y_1, y_2 \in I$ such that $x_1 \leq x_2,\ y_1 \leq y_2,\ |x_1 - x_2| \leq \epsilon$ and $|y_1 - y_2| \leq \epsilon$, we have

$$
\begin{aligned}
&|(\mathfrak{P}z)(x_2,y_2) - (\mathfrak{P}z)(x_1,y_1)| \\
&\leq |g(x_2,y_2) - g(x_1,y_1)| + |(\mathfrak{V}z)(x_2,y_2) - (\mathfrak{V}z)(x_1,y_1)|\,|z(x_2,y_2)| \\
&+ |z(x_2,y_2) - z(x_1,y_1)|\,|(\mathfrak{V}z)(x_1,y_1)| \\
&\leq \omega(g,\epsilon) + d_0\left[\frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} + \frac{2\psi(d_0)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\max\left\{\beta^{\gamma_1}\epsilon^{\gamma_1} + \beta^{\gamma_2}\epsilon^{\gamma_2}, \epsilon^{\beta\gamma_1} + \epsilon^{\beta\gamma_2}\right\}\right] \\
&+ \frac{\psi(d_0)\omega(z,\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)},
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\omega(\mathfrak{P}z,\epsilon) \\
&\leq \omega(g,\epsilon) + d_0\left[\frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} + \frac{2\psi(d_0)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\max\left\{\beta^{\gamma_1}\epsilon^{\gamma_1} + \beta^{\gamma_2}\epsilon^{\gamma_2}, \epsilon^{\beta\gamma_1} + \epsilon^{\beta\gamma_2}\right\}\right] \\
&+ \frac{\psi(d_0)\omega(z,\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\omega(\mathfrak{P}\mathscr{O},\epsilon) \\
&\leq \omega(g,\epsilon) + d_0\left[\frac{\omega_u(\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} + \frac{2\psi(d_0)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}\max\left\{\beta^{\gamma_1}\epsilon^{\gamma_1} + \beta^{\gamma_2}\epsilon^{\gamma_2}, \epsilon^{\beta\gamma_1} + \epsilon^{\beta\gamma_2}\right\}\right] \\
&+ \frac{\psi(d_0)\omega(\mathscr{O},\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}.
\end{aligned}
$$

Since g is continuous and u is uniformly continuous on $I \times I \times I \times I \times [0, d_0]$, as $\epsilon \to 0$, we have

$$
\omega(\mathfrak{P}\mathscr{O},\epsilon) \leq \frac{\psi(d_0)\omega(\mathscr{O},\epsilon)}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}.
$$

Thus, by assumption (3) and Theorem 2.4, we have that $\mathfrak{P}$ has at least one fixed point in $\mathscr{O} \subseteq B_{d_0}$. Hence, Equation (3.1) has at least one solution in $\mathscr{E}$. This completes the proof. □

Example 3.3 Consider the following equation

$$
z(x,y) = \frac{x^2y^2}{4} + \frac{z(x,y)}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}\int_0^x\int_0^y \frac{xyz(t,s)}{2s^{\frac{2}{3}}t^{\frac{2}{3}}\sqrt{\left(y^{\frac{1}{3}} - t^{\frac{1}{3}}\right)\left(x^{\frac{1}{3}} - s^{\frac{1}{3}}\right)}}dtds \tag{3.2}
$$

for $x, y \in [0,1] = I$.

Here we have

$$
g(x,y) = \frac{x^2y^2}{4},\ \| g \| = \frac{1}{4},\ \beta = \frac{1}{3},\ \gamma_1 = \gamma_2 = \frac{1}{2}, u(x,y,t,s,z) = \frac{xyz}{2}.
$$

The function u is continuous and $|u(x,y,t,s,z)| = \frac{|xyz|}{2} \leq \frac{|z|}{2} = \psi(|z|)$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(p)\frac{p}{2}$, $p \geq 0$ which is nondecreasing function.

Also, for $d_0 = 0.7$, both $\frac{1}{4} + \frac{d_0^2}{2\left(\Gamma(\frac{3}{2})\right)^2} \leq d_0$ and $d_0 \leq \left(\Gamma(\frac{3}{2})\right)^2$ are satisfied.

For $d_0 = 0.7$, we noticed that, the assumptions $(1)-(3)$ of Theorem 3.2 are satisfied. *Hence, by Theorem 3.2, Equation (3.2) has at least one solution in* $\mathscr{E}$.

4. An iterative algorithm by two-dimensional sinc interpolation

Finding the exact solution of (3.2) is difficult, so numerical methods are effective to approximate the solution. Some numerical techniques to solve integral equations are based on the collocation method can be seen in [26, 34, 40, 41]. Numerical methods such as expansion and projection methods are used in [10] and Galerkin multiwavelet bases to solve the integral equation with singular kernel have been used in [36]. Generally in the above methods, the nonlinear problems are discretized to an algebraic system with unknown coefficients which must be determined. Moreover the convergence rate of these methods are usually of polynomial order with respect to N, where N represents the number of terms of the expansion or the number of points of the quadrature formula.

But, we use two-dimensional sinc interpolation to make an iterative algorithm for finding an approximation of solution such that firstly, it does not need to convert the nonlinear problems to an algebraic system by expanding $u(t)$ in terms of Sinc functions with unknown coefficients. Secondly, this algorithm decrease computations and has exponential accuracy (in [39], it is shown that if we use the Sinc method, the convergence rate is $\exp(-c\sqrt{N})$ with some $c>0$, which convergence rate is much faster than that of polynomial order). Also to give an error bound for the Sinc approximate solution, we consider Sinc function and some of its properties [39]

$$\operatorname{sinc}(u) = \begin{cases} \dfrac{\sin(\pi u)}{\pi u}, & u \neq 0 \\ 1, & u = 0. \end{cases} \tag{4.1}$$

For $h > 0$ and integer k, the shifted sinc function named k'th sinc function with step size h is introduced as follows:

$$S(k,h)(u) = \operatorname{sinc}\left(\frac{u-kh}{h}\right). \tag{4.2}$$

We easily conclude that

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, k = j \\ 0, k \neq j. \end{cases} \tag{4.3}$$

Definition 4.1 Let $d > 0$, $\mathbb{C}$ be the complex plane and strip region $D_d = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < d\}$. For every positive α, define $L_\alpha(D_d) = \{f : f$ is an analytic function in $D_d\}$ and for some $c > 0$, and for all $z \in D_d$ function f satisfies the following inequality

$$|f(z)| \leq \frac{c\,|e^{\alpha z}|}{(1+|e^z|)^{2\alpha}}. \tag{4.4}$$

According to (4.4) a bounded error was introduced by Theorem 4.2 in [39].

Theorem 4.2 *Let $f \in L_\alpha(D_d)$ for $\alpha > 0$ and $h = \sqrt{\frac{\pi d}{\alpha N}}$, then*

$$\|f(.+iy) - \sum_{k=-\infty}^{\infty} u(kh)S(k,h)(.+iy)\|_\infty \le c_1 N^{\frac{1}{2}} \exp((-\frac{\pi\alpha}{d})^{\frac{1}{2}}(d-|y|)N^{\frac{1}{2}}).$$

In this article we discuss in real line therefore, we have

$$\|f(.) - \sum_{k=-\infty}^{\infty} u(kh)S(k,h)(.)\|_\infty \le c_1 N^{\frac{1}{2}} \exp(-c_2 N^{\frac{1}{2}}). \tag{4.5}$$

In the case of two dimensional the error bound is similar to (4.5) and we ignore it.

Definition 4.3 Let u be a function defined on real line; then for $h > 0$ the series,

$$C(u,h)(u) = \sum_{k=-\infty}^{\infty} u(kh)S(k,h)(u) \tag{4.6}$$

is called the Whittaker cardinal expansion of u, wherever this series converges (see [39]). Obviously, by (4.3)–(4.6) the cardinal function interpolates u at the points $\{kh\}_{k=-\infty}^{\infty}$.

The intervals of integrating in (3.2) are $[0,x]$ and $[0,y]$, where $x, y \in [0,1]$, so we introduce a conformal map as follows:

$$\begin{aligned} \varphi : [0,1] &\longrightarrow (-\infty, \infty) \\ t &\longrightarrow \ln(\tfrac{t}{1-t}). \end{aligned} \tag{4.7}$$

Clearly, $\lim_{t\to 0} \varphi(t) = -\infty$ and $\lim_{t\to 1} \varphi(t) = \infty$. By (4.2) and (4.7) combination of $S(k,h)$ and φ functions in the case of two dimensional is $S(k,h).\varphi(t)S(k',h).\varphi(s)$ function with $[0,1]\times[0,1]$ domain, thus the integrand function of (3.2) can be approximated by $S(k,h).\varphi(t)S(k',h).\varphi(s)$ interpolation. Let z be an integrand function, then by cardinal function (4.6), we have

$$z_n(t,s) = \sum_{k=-N}^{N} \sum_{k'=-N}^{N} z(kh, k'h)S(k,h).\varphi(t)S(k',h).\varphi(s). \tag{4.8}$$

Considering (4.8) and (4.3), if $\varphi(t) = kh$ and $\varphi(s) = k'h$ for $k, k' = -N, \dots, N$, then $z_n(kh, k'h) = z(kh, k'h)$. In other word, (4.8) is an interpolation of z such that the interpolating points can be given by

$$\begin{cases} t_k = \varphi^{-1}(kh) = \dfrac{e^{kh}}{1+e^{kh}}, \ k = -N+1, \dots, N, \ t_{-N} = 0 \\ s_{k'} = \varphi^{-1}(k'h) = \dfrac{e^{k'h}}{1+e^{k'h}}, \ k' = -N+1, \dots, N, \ s_{-N} = 0. \end{cases} \tag{4.9}$$

Using (4.6)–(4.9) and similar to [39], we compute the integral on $[0,x]\times[0,y]$ for $x, y \in [0,1]$ as follows:

$$\int_0^x \int_0^y z(t,s)dtds \approx h^2 \sum_{k=-N}^{N} \sum_{k'=-N}^{N} \frac{z(t_k, s_{k'})}{\varphi'(t_k)\varphi'(s_{k'})}, \text{ where, } \varphi'(\varsigma_k) = \frac{1}{\varsigma_k(1-\varsigma_k)}, \varsigma = t, s,$$

(4.10) is an extend results of [9, 35] to two-dimensions.

Now, with the help of (4.8)–(4.10), we can give an iterative algorithm to solve Eq. (3.2).

Algorithm

$$\begin{cases} z_0(x,y) = 0, \\ z_{n+1}(x,y) = \dfrac{x^2y^2}{4} + \dfrac{xyz_n(x,y)}{2\Gamma(\frac{1}{2})^2} h^2 \displaystyle\sum_{k=-N}^{N} \sum_{k'=-N}^{N} \dfrac{z_n(t_k, s_{k'}) t_k^{\frac{1}{3}} s_{k'}^{\frac{1}{3}} (1-t_k)(1-s_{k'})}{\sqrt{\left(y^{\frac{1}{3}} - t_k^{\frac{1}{3}}\right)\left(x^{\frac{1}{3}} - s_{k'}^{\frac{1}{3}}\right)}}, \\ n = 1, 2, 3, \ldots, \end{cases} \tag{4.10}$$

where collocation points t_k and $s_k^{'}$ for $k, k^{'} = -N, \ldots, N$ are given by (4.9).

For $N = 5, 10$ and $h = \frac{\pi}{\sqrt{2N}}$, we obtain approximate solutions $z_i(x,y)$ for $i = 0,1,2$ by algorithm (4.10). Then, substituting $z_2(x,y)$ for $N = 5, 10$ in (3.2) and comparing both sides, the absolute errors are shown in Tables 1 and 2.

Table 1. Absolute errors of $z_2(x,y)$ for $N = 5$.

$(t,\ s)$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0	0	0	0	0	0
0.2	0	4.4×10^{-7}	4.5×10^{-6}	1.6×10^{-5}	4.8×10^{-5}	1.2×10^{-4}
0.4	0	4.5×10^{-6}	4.9×10^{-5}	1.8×10^{-4}	5.3×10^{-4}	1.2×10^{-3}
0.6	0	1.6×10^{-5}	1.8×10^{-4}	7.0×10^{-4}	1.9×10^{-3}	3.7×10^{-3}
0.8	0	4.8×10^{-5}	5.3×10^{-4}	1.9×10^{-3}	4.9×10^{-3}	7.1×10^{-3}
1.0	0	1.2×10^{-4}	1.2×10^{-3}	3.7×10^{-3}	7.1×10^{-3}	1.6×10^{-2}

Table 2. Absolute errors of $z_2(x,y)$ for $N = 10$.

$(t,\ s)$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0	0	0	0	0	0
0.2	0	4.2×10^{-7}	4.3×10^{-6}	1.8×10^{-5}	1.0×10^{-4}	1.2×10^{-4}
0.4	0	4.3×10^{-6}	4.4×10^{-5}	1.9×10^{-4}	1.1×10^{-3}	1.2×10^{-3}
0.6	0	1.8×10^{-5}	1.9×10^{-4}	8.7×10^{-4}	5.09×10^{-3}	4.7×10^{-3}
0.8	0	1.0×10^{-4}	1.1×10^{-3}	5.0×10^{-3}	2.8×10^{-3}	2.8×10^{-2}
1.0	0	1.2×10^{-4}	1.2×10^{-3}	4.7×10^{-3}	2.8×10^{-2}	1.3×10^{-2}

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Conflict of interest

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References

[1] Abbas S, Benchohra M. Fractional order integral equations of two independent equations. Applied Mathematics and Computation. 2014; 227: 755-761.

[2] Abel NH. Oplosning af et par opgaver ved hjelp af bestemte integraler, Magazin for Naturvidenskaberne. Kristiania (Oslo). 1823; 55-68.

[3] Agarwal RP, O'Regan D. Fixed point theory and applications. Cambridge University Press, 2004.

[4] Aghajani A. Allahyari R, Mursaleen M. A generalization of Darbo's theorem with application to the solvability of systems of integral equations. Journal of Computational Applied Mathematics. 2014; 260 : 68-77.

[5] Aghajani A, Pourhadi E, Trujillo JJ. Application of measure of noncompactness to a Cauchy problem for fractional differential equation in Banach spaces. Fractional Calculus and Applied Analysis. 2003; 16 (4): 962-977.

[6] Alamo JA, Rodríguez J. Operational calculus for modified Eedélyi-Kober operators. Serdica : Bulgaricae mathematicae publicationes. 1994; 20 : 351-363.

[7] Altun I, Turkoglu D. A fixed point theorem for mapping satisfying a general contractive condition of operator type. Journal of Computational Analysis and Applications. 2007; 9 (1): 9-14.

[8] Arab R, Nashine HK, Can NH, Binh TT. Solvability of functional-integral equations (fractional order) using measure of noncompactness. Advances in Difference Equations. 2020; Article number: 12. doi.org/10.1186/s13662-019-2487-4

[9] Arab R, Rabbani M, Mollapourasl R. On solution of nonlinear integral equations with deviating argument based on fixed point technique. Applied and Computational Mathematics. 2015; 14 (1): 38-49.

[10] Atkinson K. The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press. 1997.

[11] Banaś J, Goebel K. Measure of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York, 1980.

[12] Banaś J, Rzepka B. Monotonic solutions of a quadratic integral equation of fractional order. Journal of Mathematical Analysis and Applications. 2007; 332: 1370-1378.

[13] Banaś J, Leszek O. Measure of noncompactness related to monotonicity. Commentationes Mathematicae. 2001; 41: 13-23.

[14] Chandrasekhar S. Radiative Transfer. Dover Publications, New York, 1960.

[15] Chang SS, Huang YJ. Coupled fixed point theorems with applications. Journal of the Korean Mathematical Society. 1996; 33 (3): 575-585.

[16] Darbo G. Punti uniti in trasformazioni a codominio non compatto (Italian). Rendiconti del Seminario Matematico della Università di Padova. 1955; 24: 84-92.

[17] Darwish MA, Sadarangani K. On a quadratic integral equation with supremum involving Erdélyi-Kober fractional order. Mathematische Nachrichten. 2015; 288 (5-6): 566-576.

[18] Darwish MA, Sadarangani K. On Erdélyi-Kober type quadratic integral equation with linear modification of the argument. Applied Mathematics and Computation. 2014; 238: 30-42.

[19] Geraghty M. On contractive mappings. Proceedings of the American Mathematical Society. 1973; 40: 604-608.

[20] Hazarika B, Arab R, Mursaleen M. Applications of Measure of Noncompactness and Operator Type Contraction for Existence of Solution of Functional Integral Equations. Complex Analysis and Operator Theory. 2019; 13: 3837-3851.

[21] Hazarika B, Arab R, Mursaleen M. Application Measure of Noncompactness and Operator Type Contraction for Solvability of an Infinite System of Differential Equations in ℓ_p-space. Filomat. 2019; 33 (7): 2181-2189.

[22] Hazarika B, Srivastava HM, Arab R, Rabbani M. Existence of solution for an infinite system of nonlinear integral equations via measure of noncompactness and homotopy perturbation method to solve it. Journal of Computational and Applied Mathematics. 2018; 343: 341-352.

[23] Hilfer R. Applications of Fractional Calculus in Physics. World Scientific, Singapore, 2000.

[24] Katugampola UN. A New Approach To Generalized Fractional Derivatives. Bulletin of Mathematical Analysis and Applications. 2014; 6 (4): 1-15.

[25] Kuratowski K. Sur les espaces completes. Fundamenta Mathematicae. 1930; 1 : 301-309.

[26] Marzban HR, Tabrizidooz HR, Razzaghi M. A composite collocation method for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equations. Communications in Nonlinear Science and Numerical Simulation. 2011; 16 (3): 1186-1194.

[27] Mursaleen M, Mohiuddine SA. Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces. Nonlinear Analysis: Theory Methods Applications. 2012; 75 : 2111-2115.

[28] Mursaleen M, Bilal B, Rizvi SMH. Applications of Measure of Noncompactness to Infinite System of Fractional Differential Equations. Filomat. 2017; 31 (11) : 3421-3432.

[29] Nashine HK, Arab R, Agarwal RP, Haghigh AS. Darbo type fixed and coupled fixed point results and its application to integral equation. Periodica Mathematica Hungarica. 2018; 77 : 94-107.

[30] Panda SK, Abdeljawad T, Ravichandran C. A complex valued approach to the solutions of Riemann-Liouville integral, Atangana-Baleanu integral operator and nonlinear Telegraph equation via fixed point method. Chaos, Solitons and Fractals. 2020; 130: 109439. doi.org/10.1016/j.chaos.2019.109439.

[31] Panda SK, Tassaddiq A, Agarwal RP. A New Approach to the Solution of nonlinear integral equations via various F_{B_e}-contractions. Symmetry. 2019; 11: Article no 206.

[32] Rabbani M, Arab R, Hazarika B. Solvability of nonlinear quadratic integral equation by using simulation type condensing operator and measure of noncompactness. Applied Mathematics and Computation. 2019; 349 (15) : 102-117.

[33] Rabbani M, Das A, Hazarika B, Arab R. Existence of solution for two dimensional nonlinear fractional integral equation by measure of noncompactness and iterative algorithm to solve it. Journal of Computational and Applied Mathematics. 2020; 370 : 1-17.

[34] Rabbani M, Kiasoltani SH. Solving of nonlinear system of Fredholm-Volterra integro-differential equations by using discrete collocation method. Journal of Mathematics and Computer Science. 2011; 3 (4): 382-389.

[35] Rabbani M, Mollapourasl R. Sinc function in strip for Fredholm integral equation of the second kind. Mathematical Sciences. 2008; 2 (3): 273–280.

[36] Rabbani M, Tabatabai-Adnani AA, Tamizkar M. Galerkin multi-wavelet bases and mesh points method to solve integral equation with singular logarithmic kernel. Mathematical Sciences. 2018; 12 (1): 55-60.

[37] Rabbani M, Das A, Hazarika B, Arab R. Measure of noncompactness of a new space of tempered sequences and its application on fractional differential equations. Chaos, Solitons and Fractals. 2020; 140: article no 110221, doi:10.1016/j.chaos.2020.110221.

[38] Ravichandran C, Logeswari K, Panda SK, Nisar KS. On new approach of fractional derivative by Mittag-Leffler kernel to neutral integro-differential systems with impulsive conditions. Chaos, Solitons and Fractals. 2020; 139: article no 110012.

[39] Stenger F. Numerical Methods Based on Sinc and Analytic Functions. Springer, NewYork, 1993.

[40] Wang KY, Wang QS. Lagrange collocation method for solving Volterra-Fredholm integral equations. Applied Mathematics and Computation. 2013; 219: 10434-10440.

[41] Youssri YH, Hafez RM. Chebyshev collocation treatment of Volterra-Fredholm integral equation with error analysis. Arabian Journal of Mathematics. 2019; 9: 471-480.