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(p, q)-Chebyshev polynomials for the families of biunivalent function associating a new integral operator with (p, q)-Hurwitz zeta function

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Abstract: In the present article, making use of the (p, q)-Hurwitz zeta function, we provide and investigate a new integral operator. Also, we define two families $\mathcal{SM}_{p,q}(\xi, \zeta, \delta, u, \tau)$ and $\mathcal{SC}_{p,q}(\lambda, \zeta, \vartheta, u, \tau)$ of biunivalent and holomorphic functions in the unit disc connected with (p, q)-Chebyshev Polynomials. Then we find coefficient estimates $|a_2|$ and $|a_3|$. Finally, we obtain Fekete-Szegö inequalities for these families.

Key words: Biunivalent function, (p, q)-Chebyshev polynomial, (p, q)-Hurwitz zeta function, a new integral operator, coefficient estimates, and Fekete-Szegö inequality

1. Introduction

Let Ω be the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

where the class Ω is analytic in the open unit disk $\Sigma = \{z : |z| < 1\}$. We also define to $\mathcal{P}$ as the class of all Ω functions that are univalent in Σ.

By the *Koebe One Quarter Theorem* [17] it is clear that the function $f \in \mathcal{P}$ has an inverse function f^{-1}, which is defined by

$f^{-1}(f(z)) = z$ and $f\left(f^{-1}(\widetilde{\omega})\right) = \eta, \ \left(|\widetilde{\omega}| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$,

where

$$f^{-1}(\widetilde{\omega}) = \widetilde{\omega} + \sum_{n=2}^{\infty} g_n \widetilde{\omega}^n.$$

It is clear that

$$\widetilde{\omega} = f\left(f^{-1}(\widetilde{\omega})\right) = \widetilde{\omega} + (g_2 + a_2)\widetilde{\omega}^2 + \left(g_3 - 2a_2^2 + a_3\right)\widetilde{\omega}^3 + \left(g_4 + 5a_2^3 - 5a_2a_3 + a_4\right)\widetilde{\omega}^4 + \cdots.$$

Assume that $g_2 = -a_2, g_3 = 2a_2^2 - a_3$ and $g_4 = -5a_2^3 + 5a_2a_3 - a_4$, we find that

$$f^{-1}(\widetilde{\omega}) = \widetilde{\omega} - a_2\widetilde{\omega}^2 + \left(2a_2^2 - a_3\right)\widetilde{\omega}^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\widetilde{\omega}^4 + \cdots. \tag{1.2}$$

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If both f and f^{-1} are univalent in Ω, a function $f \in \Omega$ is said to be biunivalent in Ω. We denote the biunivalent function by Ξ. After that, many mathematicians have been interested in studying biunivalent functions and they have provided and investigated the bounds for the coefficients $|a_n|$ (see, [10, 11], [20, 21], [25], and [32]).
The Fekete-Szegö inequality (problem), related to the Bieberbach conjecture, is an inequality for the coefficients of univalent analytic functions $f(z)$ in (1.1) discovered by Fekete and Szegö [19]. The Fekete-Szegö problem is the search for the estimate of the maximum value of the coefficient functional $\left|a_3 - \gamma a_2^2\right|$. Many authors have presented many subclasses of biunivalent analytic functions and examined applications of Fekete-Szegö inequality, such as Amourah et al. [4], Arikan et al. [6], Darus and Thomas [13], Deniz and Orhan [16], Yousef et al. ([43] and [44]), and Srivastava et al. ([34]–[38]).
The function $f(z)$ is called subordinate to $\hbar(z)$, denoted by $f(z) \prec \hbar(z)$; if $f(z)$ and $\hbar(z)$ are two analytic functions in Ω and there is a Schwarz function φ with $\varphi(z) = 0$, $|\varphi(z)| < 1$ and $f(z) = \hbar(\varphi(z))$. In addition, we get the following equivalence if the function $\hbar$ is univalent in Ω

$$f(z) \prec \hbar(z) \Leftrightarrow f(0) = \hbar(0) \text{ and } f(\Omega) \subset \hbar(\Omega).$$

For any $j \geq 2$ and $0 < q < p \leq 1$, The second type of (p,q)-Chebyshev polynomial is known as the following relationship:

$$\mathcal{F}_j(x,\alpha,p,q) = \left(p^j + q^j\right) x \mathcal{F}_{j-1}(x,\alpha,p,q) + (pq)^{j-1} \alpha \mathcal{F}_{j-2}(x,\alpha,p,q) \tag{1.3}$$

with α is a variable and the initial values $\mathcal{F}_0(x,\alpha,p,q) = 1$ and $\mathcal{F}_1(x,\alpha,p,q) = (p+q)x$.
Depending on the values of $p, q, s,$ and x, we get special polynomials from (p,q)-Chebyshev polynomial such as (Fibonacci polynomials, Pell polynomials, Jacobsthal polynomials, and Chebyshev polynomial of the second type), it was presented by many researchers (see, [1], [3, 4], [10, 11], and [18]).
In 2019, Kizilates et al. [27] provided and investigated the first and second types of (p,q)- Chebyshev polynomial and obtained derived formulas, generating functions, and some important results of this polynomial.
The generating function of the second type is defined as follows.

$$\mathcal{C}_{p,q}(z) = \frac{1}{1 - xpz\eta_p - xqz\eta_q - \alpha pqz^2\eta_{p,q}} = \sum_{j=0}^{\infty} \mathcal{F}_j(x,\alpha,p,q) z^j \; (z \in \Sigma). \tag{1.4}$$

Let $\aleph(z)$ and $\varphi(\widetilde{\omega})$ be two analytic functions in the unit disk Ω with $\aleph(0) = \varphi(0) = 0$, $|\aleph(z)| < 1, |\varphi(\widetilde{\omega})| < 1$, and

$$\aleph(z) = \sigma_1 z + \sigma_2 z^2 + \sigma_3 z^3 + \cdots, \text{ and } \varphi(\widetilde{\omega}) = l_1\widetilde{\omega} + l_2\widetilde{\omega}^2 + l_3\widetilde{\omega}^3 + \cdots \quad (z, \widetilde{\omega} \in \Sigma). \tag{1.5}$$

The (p,q)-calculus denotes the possibility of extending the q-calculus to postquantum calculus. Chakrabarti and Jagannathan [12] proposed the (p,q)-calculus in quantum algebras to generalize the q-series, which has numerous applications in science and engineering. After that, many articles provided and investigated the application of (p,q)-calculus (see; [2], [7], [22], [23], [29], and [33]).
The (p,q)-derivative operator is a kind of derivative operator defined by

$$d_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p-q)z} \left(z \neq 0, \; 0 < q < p < 1, \; and \; p \neq q\right).$$

It is clear that for $n \in \mathbb{N}=\{1,2,3,\dots\}$ and $z \in \Sigma$

$$d_{pq}\left\{\sum_{j=1}^{\infty} a_j z^j\right\} = \sum_{j=1}^{\infty} [j]_{pq} z^{j-1},$$

where $[j]_{p,q} = \frac{p^j-q^j}{p-q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + \cdots + pq^{j-2} + q^{j-1}$ and $[0]_{p,q} = 0$.
This is a natural extension of the q-number ([24]),

$$\lim_{p\to 1} [j]_{p,q} = [j]_q = \frac{1-q^j}{1-q}, q \neq 1.$$

We introduce the p,q-Hurwitz zeta function, which is a generalization of q–Hurwitz zeta function (see, [30], [31], and [40]), motivated by the work cited above.
In the beginning, we define the (p,q)-Hurwitz zeta function $\zeta_{p,q}(u,\tau;z)$ by the following form

$$\zeta_{p,q}(u,\tau;z) = \sum_{j=0}^{\infty} \frac{z^j}{[j+u]_{p,q}^{\tau}},$$

where $u \in \mathbb{C}\backslash\mathbb{Z}_0^-, \tau \in \mathbb{C},$ when $|z| < 1,$ and $\mathrm{R}(\tau) > 1$ when $|z| = 1$.
Now, by the functions $f(z)$ in (1.1), we present the (p,q)-Srivastava-Attiya operator $\mathcal{J}_{p,q,\tau}^{u} f(z) : \Omega \to \Omega$ as follows

$$\mathcal{J}_{p,q,\tau}^{u} f(z) = \left(\Phi_{\tau}^{u}(p,q;z) * f(z)\right)\left(z \in \Sigma, u \in \mathbb{C}\backslash\mathbb{Z}_0^-; \tau \in \mathbb{C}\right), \tag{1.6}$$

where

$$\Phi_{\tau}^{u}(p,q;z) = [1+u]_{pq}^{\tau}\left[\zeta_{p,q}(u,\tau;z) - (u)_{pq}^{-\tau}\right]. \tag{1.7}$$

From (1.6) and (1.7), we note that

$$\mathcal{J}_{p,q,\tau}^{u} f(z) = z + \sum_{j=2}^{\infty}\left(\frac{[1+u]_{pq}}{[j+u]_{pq}}\right)^{\tau} a_j z^j. \tag{1.8}$$

From (1.8), we observe that
1. When $p = 1$, we get a q-Srivastava-Attiya operator [39].

$$\mathcal{J}_{q,\tau}^{b} f(z) = z + \sum_{n=2}^{\infty}\left(\frac{[1+u]_q}{[n+u]_q}\right)^{\tau} a_n z^n.$$

2. When $p = 1$ and $q \to 1$, we get a Srivastava-Attiya operator [34].

$$\mathcal{J}_{1s,\tau}^{u} f(z) = z + \sum_{j=2}^{\infty}\left(\frac{1+u}{j+u}\right)^{\tau} a_j z^j.$$

The aim of this work is to define a (p,q)-integral operator by using (p,q)-Hurwitz zeta function, which is a generalization of q-Srivastava-Attiya integral operator. After that, we determine initial coefficient bounds for some classes of analytic functions defined by subordination. Finally, our results deal with the Fekete-Szegö problem for (p,q)-Chebyshev polynomials.

2. New concepts and results

At he beginning, we introduce the concept of new families $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$ and $\mathcal{SC}_{p,q}(\lambda,\zeta,\vartheta,u,\tau)$.

Definition 2.1 *For $0 \le \xi \le 1, \zeta \ge 0$ and $0 \le \delta \le 1$. A function $f \in \Xi$ is said to be in the family $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,\ u,\tau)$ if the following subordinate conditions are satisfied:*

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f(z)\right]^{1-\zeta}}+\xi\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f(z)}{z}\right)^{\delta}\prec\mathcal{C}_{p,q}(z) \tag{2.1}$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}}+\xi\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta}\prec\mathcal{C}_{p,q}(\widetilde{\omega}). \tag{2.2}$$

Remark 2.2 *It is worth noting that the family $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$ contains many subfamilies, we mention them as follows:*

1- We get the subfamily $\mathcal{SM}_{p,q}(\xi,\zeta,u,\tau)$, when $\delta=0$; define as below

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f(z)\right]^{1-\zeta}}+\xi\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'\prec\mathcal{C}_{p,q}(z),$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}}+\xi\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'\prec\mathcal{C}_{p,q}(\widetilde{\omega}).$$

2- We get the subfamily $\mathcal{SM}_{p,q}(\xi,\delta,u,\tau)$, when $\zeta=1$; define as below

$$(1-\xi)\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'+\xi\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f(z)}{z}\right)^{\delta}\prec\mathcal{C}_{p,q}(z),$$

and

$$(1-\xi)\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'+\xi\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta}\prec\mathcal{C}_{p,q}(\widetilde{\omega}).$$

3- We get the subfamily $\mathcal{SM}_{p,q}(\xi,\zeta,u,\tau)$, when $\xi=0$; define as below

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f(z)\right]^{1-\zeta}}\prec\mathcal{C}_{p,q}(z),$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}}\prec\mathcal{C}_{p,q}(\widetilde{\omega}).$$

Definition 2.3 *For $\lambda \in \mathbb{C}\backslash\{0\}$, $\zeta \ge 0$ and $\vartheta \ge 0$. A function $f \in \Xi$ is said to be in the family $\mathcal{SC}_{p,q}(\lambda,\zeta,\vartheta,u,\tau)$ if satisfies the following subordinate conditions:*

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f(z)+\vartheta z^{2}(\mathcal{J}_{p,q,\tau}^{u}f(z))''}{z}+(\zeta-2\vartheta)\big(\mathcal{J}_{p,q,\tau}^{u}f(z)\big)'-1\right)\prec\mathcal{C}_{p,q}(z), \tag{2.3}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})+\vartheta\widetilde{\omega}^{2}\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\big)''}{\widetilde{\omega}}+(\zeta-2\vartheta)\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\big)'-1\right)\prec\mathcal{C}_{p,q}(\widetilde{\omega}). \tag{2.4}$$

Remark 2.4 *It is worth noting that the family $\mathcal{SC}_{p,q}(\lambda,\zeta,\vartheta,u,\tau)$ contains many subfamilies, we mention them as follows:*

1- We get the subfamily $\mathcal{SC}_{p,q}(\lambda,\vartheta,u,\tau)$, when $\zeta=1+2\vartheta$;

$$\frac{1}{\lambda}\left(\vartheta z(\mathcal{J}_{p,q,\tau}^{u}f(z))''+\big(\mathcal{J}_{p,q,\tau}^{u}f(z)\big)'-1\right)\prec\mathcal{C}_{p,q}(z),$$

and

$$\frac{1}{\lambda}\left(\vartheta\widetilde{\omega}\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\big)''+\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\big)'-1\right)\prec\mathcal{C}_{p,q}(\widetilde{\omega}).$$

2- We get the subfamily $\mathcal{SC}_{p,q}(\lambda,\zeta,u,\tau)$, when $\vartheta=0$;

$$\frac{1}{\lambda}\left((1-\zeta)\frac{\mathcal{J}_{p,q,\tau}^{u}f(z)}{z}+\zeta\big(\mathcal{J}_{p,q,\tau}^{u}f(z)\big)'-1\right)\prec\mathcal{C}_{p,q}(z),$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}+\zeta\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\big)'-1\right)\prec\mathcal{C}_{p,q}(\widetilde{\omega}).$$

3- We get the subfamily $\mathcal{SC}_{p,q}(\lambda,u,\tau)$, when $\zeta=1$ and $\vartheta=0$;

$$\frac{1}{\lambda}\left(\big(\mathcal{J}_{p,q,\tau}^{u}f(z)\big)'-1\right)\prec\mathcal{C}_{p,q}(z),$$

and

$$\frac{1}{\lambda}\left(\big(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\big)'-1\right)\prec\mathcal{C}_{p,q}(\widetilde{\omega}).$$

Theorem 2.5 *Let $f(z)\in\Omega$ be in the family $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$, then*

$$|a_2|\leq\frac{|(p+q)x|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\mathcal{H}_{p,q,x}(\xi,\zeta,\delta,u,\tau)+\mathcal{N}_{p,q,x}(\xi,\zeta,\delta,u,\tau)-\mathcal{Q}_{p,q,x}(\xi,\zeta,\delta,u,\tau)\right]}}, \tag{2.5}$$

and

$$|a_3|\leq\frac{|(p+q)x|\,(3+u)_{pq}^{\tau}}{[(1-\xi)(\zeta+1)+\xi\delta+2]\,(1+u)_{pq}^{\tau}}+\frac{(p+q)^{2}x^{2}(2+u)_{pq}^{2\tau}}{[(1-\xi)(\zeta+1)+\xi\delta+1]^{2}(1+u)_{pq}^{2\tau}}, \tag{2.6}$$

where $\mathcal{H}_{p,q,x}(\xi,\zeta,\delta,u,\tau)=(p+q)^{2}x^{2}\left((1-\xi)(\zeta+1)+\xi\delta+2\right)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,

$\mathcal{N}_{p,q,x}(\xi,\zeta,\delta,u,\tau)=(p+q)^{2}x^{2}\left(\frac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,

and $\mathcal{Q}_{p,q,x}(\xi,\zeta,\delta,u,\tau)=\left(p^{2}+q^{2}\right)(p+q)\,x^{2}[(1-\xi)(\zeta+1)+\xi\delta+1]^{2}(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

Proof Suppose that $f(z) \in \mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$, there are two analytic functions $\aleph, \varphi$ as defined in (1.5), such that

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f(z)\right]^{1-\zeta}} + \xi\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f(z)}{z}\right)^{\delta} \prec \mathcal{C}_{p,q}(\aleph(z)), \tag{2.7}$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}} + \xi\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta} \prec \mathcal{C}_{p,q}(\varphi(\widetilde{\omega})). \tag{2.8}$$

From (2.7) and (2.8), we have

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f(z)\right]^{1-\zeta}} + \xi\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f(z)}{z}\right)^{\delta} = 1+\mathcal{F}_1(x,\alpha,p,q)\,\aleph(z)+\ \mathcal{F}_2(x,\alpha,p,q)\,(\aleph(z))^2+\dots \tag{2.9}$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}} + \xi\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta}$$

$$= 1+\mathcal{F}_1(x,\alpha,p,q)\,\varphi(\widetilde{\omega})+\mathcal{F}_2(x,\alpha,p,q)\,(\varphi(\widetilde{\omega}))^2+\cdots. \tag{2.10}$$

Comparing Equations (1.5) and (2.7)–(2.10), we get the following relationship

$$(1-\xi)\frac{z^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f(z)\right]^{1-\zeta}} + \xi\left(\mathcal{J}^u_{p,q,\tau}f(z)\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f(z)}{z}\right)^{\delta}$$

$$= 1+\mathcal{F}_1(x,\alpha,p,q)\,\sigma_1 z+\left[\mathcal{F}_1(x,s,p,q)\,\sigma_2+\ \mathcal{F}_2(x,s,p,q)\sigma_1^2\right]z^2+\cdots. \tag{2.11}$$

and

$$(1-\xi)\frac{\widetilde{\omega}^{1-\zeta}\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'}{\left[\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right]^{1-\zeta}} + \xi\left(\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})\right)'\left(\frac{\mathcal{J}^u_{p,q,\tau}f^{-1}(\widetilde{\omega})}{\widetilde{\omega}}\right)^{\delta}$$

$$= 1+\mathcal{F}_1(x,\alpha,p,q)\,l_1\widetilde{\omega}+\left[\mathcal{F}_1(x,s,p,q)l_2+\mathcal{F}_2(x,s,p,q)l_1^2\right]\widetilde{\omega}^2+\cdots. \tag{2.12}$$

Since $|\aleph(z)|<1, |\varphi(\widetilde{\omega})|<1,$ then

$$|\sigma_j|<1 \ and \ |l_j|<1 \ for \ all \ j\in\mathbb{N}. \tag{2.13}$$

By calculating the right-hand side of two conditions in (2.11) and (2.12), and comparing the coefficients, we get

$$[(1-\xi)(\zeta+1)+\xi\delta+1]\frac{(1+u)^{\tau}_{pq}}{(2+u)^{\tau}_{pq}}a_2 = \mathcal{F}_1(x,\alpha,p,q)\,\sigma_1 \tag{2.14}$$

$$[(1-\xi)(\zeta+1)+\xi\delta+2]\frac{(1+u)^{\tau}_{pq}}{(3+u)^{\tau}_{pq}}a_3+\frac{[(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)](1+u)^{2\tau}_{pq}}{2(2+u)^{2\tau}_{pq}}a_2^2$$

$$= \mathcal{F}_1(x,s,p,q)\,\sigma_2+\mathcal{F}_2(x,s,p,q)\,\sigma_1^2 \tag{2.15}$$

$$-[(1-\xi)(\zeta+1)+\xi\delta+1]\frac{(1+u)_{pq}^{\tau}}{(2+u)_{pq}^{\tau}}a_2=\mathcal{F}_1(x,\alpha,p,q)\,l_1 \tag{2.16}$$

and

$$[(1-\xi)(\zeta+1)+\xi\delta+2]\,\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}(2a_2^2-a_3)+\frac{[(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)](1+u)_{pq}^{2\tau}}{2(2+u)_{pq}^{2\tau}}a_2^2$$

$$=\mathcal{F}_1(x,s,p,q)l_2+\mathcal{F}_2(x,s,p,q)l_1^2. \tag{2.17}$$

From (2.14) and (2.16), we have

$$\sigma_1=-l_1 \tag{2.18}$$

and

$$\frac{2[(1-\xi)(\zeta+1)+\xi\delta+1]^2(1+u)_{pq}^{2\tau}}{(2+u)_{pq}^{2\tau}}a_2^2=\mathcal{F}_1^2(x,\alpha,p,q)\left(\sigma_1^2+l_1^2\right). \tag{2.19}$$

Now, by adding (2.15) to (2.17), we obtain

$$[2((1-\xi)(\zeta+1)+\xi\delta+2)\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}+\frac{2[\frac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)](1+u)_{pq}^{2\tau}}{(2+u)_{pq}^{2\tau}}]a_2^2$$

$$=\mathcal{F}_1(x,s,p,q)(\sigma_2+l_2)+\mathcal{F}_2(x,s,p,q)(\sigma_1^2+l_1^2). \tag{2.20}$$

By (2.19) we substitute the value of $\left(\sigma_1^2+l_1^2\right)$ into Equation (2.20), we get

$$\begin{aligned}&[2\mathcal{F}_1^2(x,\alpha,p,q)((1-\xi)(\zeta+1)+\xi\delta+2)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}\\&+2\mathcal{F}_1^2(x,\alpha,p,q)\left(\tfrac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2)\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}\\&-2\mathcal{F}_2(x,s,p,q)[(1-\xi)(\zeta+1)+\xi\delta+1]^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}]a_2^2\end{aligned}$$

$$=\mathcal{F}_1^3(x,\alpha,p,q)(2k+u)^{2\tau}(3k+u)^{\tau}(\sigma_2+l_2). \tag{2.21}$$

From (1.3), (2.13), and (2.21), we have

$$|a_2|\leq\frac{|(p+q)x|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\mathcal{H}_{p,q,x}(\xi,\zeta,\delta,u,\tau)+\mathcal{N}_{p,q,x}(\xi,\zeta,\delta,u,\tau)-\mathcal{Q}_{p,q,x}(\xi,\zeta,\delta,u,\tau)\right]}}.$$

Now, to get the bound on $|a_3|$, using (2.18) and subtracting (2.17) from (2.15)

$$2\left[(1-\xi)(\zeta+1)+\xi\delta+2\right]\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}(a_3-a_2^2)=\mathcal{F}_1(x,s,p,q)(\sigma_2-l_2)+\mathcal{F}_2(x,s,p,q)(\sigma_1^2-l_1^2). \tag{2.22}$$

According to (2.18) and (2.22), we get

$$a_3=\frac{\mathcal{F}_1(x,s,p,q)(3+u)_{pq}^{\tau}(\sigma_2-l_2)}{2\left[(1-\xi)(\zeta+1)+\xi\delta+2\right](1+u)_{pq}^{\tau}}+\frac{\mathcal{F}_1^2(x,\alpha,p,q)(2+u)_{pq}^{2\tau}\left(\sigma_1^2+l_1^2\right)}{2[(1-\xi)(\zeta+1)+\xi\delta+1]^2(1+u)_{pq}^{2\tau}}.$$

Then

$$|a_3| \leq \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{[(1-\xi)\,(\zeta+1)+\xi\delta+2]\,(1+u)_{pq}^{\tau}} + \frac{(p+q)^2 x^2 (2+u)_{pq}^{2\tau}}{[(1-\xi)\,(\zeta+1)+\xi\delta+1]^2 (1+u)_{pq}^{2\tau}}.$$

The proof is complete. □

For the special case $\delta = 0$, Theorem 2.5 becomes:

Corollary 2.6 *Let $f(z) \in \Omega$ be in the family $\mathcal{SM}_{p,q}(\xi, \zeta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)\,x|\,(2+u)_{pq}^{\tau} \sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{N}_{p,q,x}(\xi,\zeta,u,\tau) + \tilde{K}_{p,q,x}(\xi,\zeta,u,\tau) - \tilde{L}_{p,q,x}(\xi,\zeta,u,\tau)\right]}},$$

$$|a_3| \leq \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{[\zeta\,(1-\xi)+\xi+3]\,(1+u)_{pq}^{\tau}} + \frac{(p+q)^2 x^2 (2+u)_{pq}^{2\tau}}{[\zeta\,(1-\xi)+\xi+2]^2 (1+u)_{pq}^{2\tau}},$$

where $\tilde{N}_{p,q,x}(\xi,\zeta,u,\tau) = (p+q)^2 x^2\,(\zeta\,(1-\xi)+\xi+3)\,(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,
$\tilde{K}_{p,q,x}(\xi,\zeta,u,\tau) = (p+q)^2 x^2 \left(\frac{1}{2}\,(\zeta-\xi\zeta)\,(\zeta+3)+4\xi\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$, *and*
$\tilde{L}_{p,q,x}(\xi,\zeta,u,\tau) = \left(p^2+q^2\right)(p+q)\,x^2(\zeta\,(1-\xi)+\xi+2)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

For the special case $\zeta = 1$, Theorem 2.5 becomes:

Corollary 2.7 *Let $f(z) \in \Omega$ be in the family $\mathcal{SM}_{p,q}(\xi, \delta, u, \tau)$, then*

$$|a_2| \leq \frac{|(p+q)\,x|\,(2+u)_{pq}^{\tau} \sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{V}_{p,q,x}(\xi,\delta,u,\tau) + \tilde{U}_{p,q,x}(\xi,\delta,u,\tau) - \tilde{O}_{p,q,x}(\xi,\delta,u,\tau)\right]}},$$

$$|a_3| \leq \frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{(\xi(1-\delta)+3)\,(1+u)_{pq}^{\tau}} + \frac{(p+q)^2 x^2 (2+u)_{pq}^{2\tau}}{(\xi(\delta-1)+2)^2 (1+u)_{pq}^{2\tau}},$$

where $\tilde{V}_{p,q,x}(\xi,\delta,u,\tau) = (p+q)^2 x^2\,(\xi(1-\delta)+3)\,(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,
$\tilde{U}_{p,q,x}(\xi,\delta,u,\tau) = (p+q)^2 x^2 \left(\frac{1}{2}(\xi(\delta\,(\delta+1)-2)\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$, *and*
$\tilde{O}_{p,q,x}(\xi,\delta,u,\tau) = \left(p^2+q^2\right)(p+q)\,x^2(\xi(\delta-1)+2)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

Theorem 2.8 *Let $f(z) \in \Omega$ be in the family $\mathcal{SC}_{p,q}(\lambda,\zeta,\vartheta,u,\tau)$, then*

$$|a_2| \leq \frac{|(p+q)\,x\lambda|\,(2+u)_{pq}^{\tau} \sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\,[Q_{p,q,x}(\lambda,\zeta,\vartheta,k,u,\tau) - M_{p,q,x}(\lambda,\zeta,\vartheta,k,u,\tau)]}},$$

and

$$|a_3| \leq |a_2|^2 + \frac{(p+q)\,x(3+u)_{pq}^{\tau}}{\lambda\,(1+2\zeta+2\vartheta)\,(1+u)_{pq}^{\tau}},$$

where $Q_{p,q,x}(\lambda,\zeta,\vartheta,u,\tau)=\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(p+q)^2x^2$

and $M_{p,q,x}(\lambda,\zeta,\vartheta,u,\tau)=\left(p^2+q^2\right)(p+q)\,x^2(1+\zeta)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

Proof Suppose that $f(z)\in\mathcal{SC}_{p,q}(\lambda,\zeta,\vartheta,u,\tau)$. There are two analytic functions $\aleph,\varphi$ as defined in (1.5), such that

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f(z)+\vartheta z^2(\mathcal{J}_{p,q,\tau}^{u}f(z))''}{z}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'-1\right)\prec\mathcal{C}_{p,q}(\aleph(z)),\tag{2.23}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})+\vartheta\widetilde{\omega}^2(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega}))''}{\widetilde{\omega}}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'-1\right)\prec\mathcal{C}_{p,q}(\varphi(\widetilde{\omega})).\tag{2.24}$$

From (2.23) and (2.24), we have

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f(z)+\vartheta z^2(\mathcal{J}_{p,q,\tau}^{u}f(z))''}{z}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'-1\right)$$
$$=1+\mathcal{F}_1(x,\alpha,p,q)\,\aleph(z)+\ \mathcal{F}_2(x,\alpha,p,q)\,(\aleph(z))^2+\cdots.\tag{2.25}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})+\vartheta\widetilde{\omega}^2(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega}))''}{\widetilde{\omega}}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'-1\right)$$
$$=1+\mathcal{F}_1(x,\alpha,p,q)\,\varphi(\widetilde{\omega})+\ \mathcal{F}_2(x,\alpha,p,q)\,(\varphi(\widetilde{\omega}))^2+\cdots.\tag{2.26}$$

Comparing Equations (1.5) and (2.23)–(2.26), we get the following relationship

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f(z)+\vartheta z^2(\mathcal{J}_{p,q,\tau}^{u}f(z))''}{z}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f(z)\right)'-1\right)$$
$$=1+\mathcal{F}_1(x,\alpha,p,q)\,\sigma_1 z+\left[\mathcal{F}_1(x,s,p,q)\sigma_2+\mathcal{F}_2(x,s,p,q)\sigma_1^2\right]z^2+\cdots\tag{2.27}$$

and

$$\frac{1}{\lambda}\left(\frac{(1-\zeta+2\vartheta)\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})+\vartheta\widetilde{\omega}^2(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega}))''}{\widetilde{\omega}}+(\zeta-2\vartheta)\left(\mathcal{J}_{p,q,\tau}^{u}f^{-1}(\widetilde{\omega})\right)'-1\right)$$
$$=1+\mathcal{F}_1(x,\alpha,p,q)\,l_1\widetilde{\omega}+\left[\mathcal{F}_1(x,s,p,q)l_2+\mathcal{F}_2(x,s,p,q)l_1^2\right]\widetilde{\omega}^2+\cdots.\tag{2.28}$$

Since $|\aleph(z)|<1, |\varphi(\widetilde{\omega})|<1,$ then

$$|\sigma_j|<1\ and\ |l_j|<1\ for\ all\ j\in\mathbb{N}.\tag{2.29}$$

By calculating the right-hand side of two conditions in (2.27) and (2.27), and comparing the coefficients, we get

$$\frac{1}{\lambda}(1+\zeta)\frac{(1+u)_{pq}^{\tau}}{(2+u)_{pq}^{\tau}}a_2=\mathcal{F}_1(x,\alpha,p,q)\,\sigma_1,\tag{2.30}$$

$$\frac{1}{\lambda}(1+2\zeta+2\vartheta)\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}a_3=\mathcal{F}_1(x,s,p,q)\sigma_2+\mathcal{F}_2(x,s,p,q)\sigma_1^2, \tag{2.31}$$

$$-\frac{1}{\lambda}(1+\zeta)\frac{(1+u)_{pq}^{\tau}}{(2+u)_{pq}^{\tau}}a_2=\mathcal{F}_1(x,\alpha,p,q)l_1, \tag{2.32}$$

and

$$\frac{1}{\lambda}(1+2\zeta+2\vartheta)\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}(2a_2^2-a_3)=\mathcal{F}_1(x,s,p,q)l_2+\mathcal{F}_2(x,s,p,q)l_1^2. \tag{2.33}$$

From (2.30) and (2.32), we have

$$\sigma_1=-l_1 \tag{2.34}$$

and

$$\frac{2}{\lambda^2}(1+\zeta)^2\frac{(1+u)_{pq}^{2\tau}}{(2+u)_{pq}^{2\tau}}a_2^2=\mathcal{F}_1^2(x,\alpha,p,q)\left(\sigma_1^2+l_1^2\right). \tag{2.35}$$

Now, by adding (2.31) to (2.33), we obtain

$$\frac{2}{\lambda}(1+2\zeta+2\vartheta)\frac{(1+u)_{pq}^{\tau}}{(3+u)_{pq}^{\tau}}a_2^2=\mathcal{F}_1(x,s,p,q)(\sigma_2+l_2)+\mathcal{F}_2(x,s,p,q)(\sigma_1^2+l_1^2). \tag{2.36}$$

By (2.35) we substitute the value of $\left(\sigma_1^2+l_1^2\right)$ into Equation (2.36), we get

$$2\left[\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}\mathcal{F}_1^2(x,\alpha,p,q)-\mathcal{F}_2(x,s,p,q)(1+\zeta)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}\right]a_2^2$$
$$=\mathcal{F}_1^3(x,\alpha,p,q)\lambda^2(2k+u)^{2\tau}(3k+u)^{\tau}(\sigma_2+l_2). \tag{2.37}$$

From (1.3), (2.29) and (2.37), we have

$$|a_2|\leq\frac{|(p+q)x\lambda|(2+u)_{pq}^{\tau}\sqrt{(p+q)x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[Q_{p,q,x}(\lambda,\zeta,\vartheta,u,\tau)-M_{p,q,x}(\lambda,\zeta,\vartheta,u,\tau)\right]}}.$$

Now, to get the bound on $|a_3|$, using (2.34) and subtracting (2.33) from (2.31)

$$\frac{2}{\lambda}(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}a_3=\frac{2}{\lambda}(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}a_2^2+(3+u)_{pq}^{\tau}\mathcal{F}_1(x,s,p,q)(\sigma_2-l_2). \tag{2.38}$$

According to (2.34) and (2.38), we get

$$\frac{2}{\lambda}(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}|a_3|\leq\frac{2}{\lambda}(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}|a_2|^2+2(3+u)_{pq}^{\tau}\mathcal{F}_1(x,s,p,q).$$

Then

$$|a_3|\leq|a_2|^2+\frac{|(p+q)x|(3+u)_{pq}^{\tau}}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}}.$$

Here, the proof is complete. □

For the special case $\zeta=1+2\vartheta$, Theorem 2.8 becomes:

Corollary 2.9 *Let* $f(z)\in\Omega$ *be in the family* $\mathcal{SC}_{p,q}(\lambda,\vartheta,u,\tau)$, *then*

$$|a_2|\le\frac{|(p+q)\,x\lambda|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{Q}_{p,q,x}(\lambda,\vartheta,u,\tau)-\tilde{M}_{p,q,x}(\lambda,\vartheta,u,\tau)\right]}},$$

$$|a_3|\le|a_2|^2+\frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{\lambda\,(1+2\zeta)\,(1+u)_{pq}^{\tau}},$$

where $\tilde{Q}_{p,q,x}(\lambda,\vartheta,u,\tau)=3\lambda\,(1+2\vartheta)\,(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(p+q)^2x^2$
and $\tilde{M}_{p,q,x}(\lambda,\vartheta,u,\tau)=2\left(p^2+q^2\right)(p+q)\,x^2(1+\zeta)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

For the special case $\vartheta=0$, Theorem 2.8 becomes:

Corollary 2.10 *Let* $f(z)\in\Omega$ *be in the family* $\mathcal{SC}_{p,q}(\lambda,\zeta,u,\tau)$, *then*

$$|a_2|\le\frac{|(p+q)\,x\lambda|\,(2+u)_{pq}^{\tau}\sqrt{(p+q)\,x(3+u)_{pq}^{\tau}}}{\sqrt{2\left[\tilde{D}_{p,q,x}(\lambda,\zeta,u,\tau)-\tilde{F}_{p,q,x}(\lambda,\zeta,u,\tau)\right]}},$$

$$|a_3|\le|a_2|^2+\frac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{\xi\,(1+2\zeta)\,(1+u)_{pq}^{\tau}},$$

where $\tilde{D}_{p,q,x}(\lambda,\zeta,u,\tau)=\lambda\,(1+2\zeta)\,(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(p+q)^2x^2$
and $\tilde{F}_{p,q,x}(\lambda,\zeta,u,\tau)=\left(p^2+q^2\right)(p+q)\,x^2(1+\zeta)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$.

3. Fekete-Szegö inequality

Theorem 3.1 *Let* $f(z)$ *be in the family* $\mathcal{SM}_{p,q}(\xi,\zeta,\delta,u,\tau)$ *and* $\gamma\in\mathbb{R}$, *then*

$$\left|a_3-\gamma a_2^2\right|\le\begin{cases}\dfrac{|(p+q)\,x|\,(3+u)_{pq}^{\tau}}{[(1-\xi)(\zeta+1)+\xi\delta+2]\,(1+u)_{pq}^{\tau}}, & \text{if}\quad|\gamma-1|\le\chi_1,\\[2ex] \dfrac{\left|(p+q)^3x^3\right|(2+u)_{pq}^{2\tau}(3+u)_{pq}^{\tau}\,|1-\gamma|}{\left|[(p+q)\,(\mathcal{A}(\xi,\zeta,\delta,u,\tau)+\mathcal{B}(\xi,\zeta,\delta,u,\tau))-(p^2+q^2)\,\mathcal{G}(\xi,\zeta,\delta,u,\tau)]\,(p+q)\,x^2-pq\alpha\mathcal{G}(\xi,\zeta,\delta,u,\tau)\right|}, & \\ \text{if}\quad|\gamma-1|\ge\chi_1\end{cases}$$

where $\mathcal{A}(\xi,\zeta,\delta,u,\tau)=((1-\xi)(\zeta+1)+\xi\delta+2)\,(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,
$\mathcal{B}(\xi,\zeta,\delta,u,\tau)=\left(\frac{1}{2}(\zeta-\xi\zeta)(\zeta+3)+\xi(\delta(\delta+1)-2\right)(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,
$\mathcal{G}(\xi,\zeta,\delta,u,\tau)=[(1-\xi)(\zeta+1)+\xi\delta+1]^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,
and $\chi_1=\frac{[(p+q)(\mathcal{A}(\xi,\zeta,\delta,u,\tau)+\mathcal{B}(\xi,\zeta,\delta,u,\tau))-\left(p^2+q^2\right)\mathcal{G}(\xi,\zeta,\delta,u,\tau)](p+q)x^2-pq\alpha\mathcal{G}(\xi,\zeta,\delta,u,\tau)}{(p+q)^2x^2[(1-\xi)(\zeta+1)+\xi\delta+2](2+u)_{pq}^{2\tau}(1+u)_{pq}^{\tau}}$.

Proof By Equations (2.21) and (2.22), it follows that

$$a_3-\gamma a_2^2=\frac{\mathcal{F}_1\left(x,s,p,q\right)\left(3+u\right)_{pq}^{\tau}\left(\sigma_2-l_2\right)}{2\left[\left(1-\xi\right)\left(\zeta+1\right)+\xi\delta+2\right]\left(1+u\right)_{pq}^{\tau}}+\left(1-\gamma\right)a_2^2$$

$$=\frac{\mathcal{F}_1\left(x,s,p,q\right)\left(3+u\right)_{pq}^{\tau}\left(\sigma_2-l_2\right)}{2\left[\left(1-\xi\right)\left(\zeta+1\right)+\xi\delta+2\right]\left(1+u\right)_{pq}^{\tau}}+$$

$$\frac{\mathcal{F}_1^3\left(x,\alpha,p,q\right)\left(3+u\right)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(\sigma_2+l_2)\left(1-\gamma\right)}{\mathcal{F}_1^2\left(x,\alpha,p,q\right)\left(\mathcal{A}\left(\xi,\zeta,\delta,u,\tau\right)+\mathcal{B}\left(\xi,\zeta,\delta,u,\tau\right)\right)-\mathcal{F}_2\left(x,s,p,q\right)\mathcal{G}\left(\xi,\zeta,\delta,u,\tau\right)}$$

$$=\mathcal{F}_1\left(x,\alpha,p,q\right)[\left(\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)+\frac{(3+u)_{pq}^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\right)\sigma_2+\left(\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)-\frac{(3+u)_{pq}^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\right)l_2],$$

where $\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)=\frac{\mathcal{F}_1^2(x,\alpha,p,q)(3+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(1-\gamma)}{\mathcal{F}_1^2(x,\alpha,p,q)(\mathcal{A}(\xi,\zeta,\delta,u,\tau)+\mathcal{B}(\xi,\zeta,\delta,u,\tau))-\mathcal{F}_2(x,s,p,q)\mathcal{G}(\xi,\zeta,\delta,u,\tau)}$.

Thus, according to (p,q)-Chebyshev polynomials in (1.3), we conclude that

$$\left|a_3-\gamma a_2^2\right|\leq\begin{cases}\frac{(p+q)x(3+u)_{pq}^{\tau}}{[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}, & 0\leq|\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)|\leq\frac{(3k+u)^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\\ 2\left|\left(p+q\right)x\right|\left|\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta\right)\right|, & for\ |\widetilde{\mathcal{G}}\left(\xi,\zeta,\delta,\gamma\right)|\geq\frac{(3+u)_{pq}^{\tau}}{2[(1-\xi)(\zeta+1)+\xi\delta+2](1+u)_{pq}^{\tau}}\end{cases}$$

After making several simplifications, the proof of Theorem 3.1 is complete. □

Theorem 3.2 *Let $f(z)$ be in the family $\mathcal{SC}_{p,q}\left(\lambda,\zeta,\vartheta,u,\tau\right)$ and $\gamma\in\mathbb{R}$, then*

$$\left|a_3-\gamma a_2^2\right|\leq\begin{cases}\dfrac{\left|\left(p+q\right)x\right|\left(3+u\right)_{pq}^{\tau}}{\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}} \quad if\quad |\gamma-1|\leq\chi_2,\\ \dfrac{\left|\left(p+q\right)^3x^3\right|\lambda^2(3+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}\left|1-\gamma\right|}{\left|[\left(p+q\right)\widetilde{\mathcal{A}}\left(\lambda,\zeta,\delta,u,\tau\right)-\left(p^2+q^2\right)\widetilde{\mathcal{B}}\left(\lambda,\zeta,\delta,u,\tau\right)](p+q)\,x^2-pq\alpha\widetilde{\mathcal{B}}\left(\lambda,\zeta,\delta,u,\tau\right)\right|}, \\ \quad if\quad |\gamma-1|\geq\chi_2\end{cases}$$

where $\widetilde{\mathcal{A}}\left(\lambda,\zeta,\delta,u,\tau\right)=\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}(2+u)_{pq}^{2\tau}$,

$\widetilde{\mathcal{B}}\left(\lambda,\zeta,\delta,u,\tau\right)=\left(1+\zeta\right)^2(3+u)_{pq}^{\tau}(1+u)_{pq}^{2\tau}$,

and $\chi_2=\frac{[(p+q)\widetilde{\mathcal{A}}(\lambda,\zeta,\delta,u,\tau)-\left(p^2+q^2\right)\widetilde{\mathcal{B}}(\lambda,\zeta,\delta,u,\tau)](p+q)x^2-pq\alpha\widetilde{\mathcal{B}}(\lambda,\zeta,\delta,u,\tau)}{(p+q)^2x^2\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}}$.

Proof By Equations (2.37) and (2.38), it follows that

$$a_3-\gamma a_2^2=\frac{(3+u)_{pq}^{\tau}\mathcal{F}_1\left(x,s,p,q\right)\left(\sigma_2-l_2\right)}{2\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}}+(1-\gamma)a_2^2$$

$$=\frac{\mathcal{F}_1\left(x,\alpha,p,q\right)}{2}\,[\left(\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)+\frac{\left(3+u\right)_{pq}^{\tau}}{\lambda\left(1+2\zeta+2\vartheta\right)\left(1+u\right)_{pq}^{\tau}}\right)\sigma_2$$

$$+\left(\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)-\frac{(3+u)_{pq}^{\tau}}{\lambda\left(1+2\zeta+2\vartheta\right)(1+u)_{pq}^{\tau}}\right)l_2]$$

where

$$\Upsilon\left(\gamma,\zeta,\xi,\vartheta\right)=\frac{\mathcal{F}_1^2\left(x,\alpha,p,q\right)\xi^2(3+u)_{pq}^{\tau}(2+u)_{pq}^{2\tau}(1-\gamma)}{\mathcal{F}_1^2\left(x,\alpha,p,q\right)\widetilde{\mathcal{A}}\left(\xi,\zeta,\delta,u,\tau\right)-\mathcal{F}_2(x,s,p,q)\widetilde{\mathcal{B}}\left(\xi,\zeta,\delta,u,\tau\right)}.$$

Thus, according to (p,q)-Chebyshev polynomials in (1.3), we conclude that

$$\left|a_3-\gamma a_2^2\right|\leq\begin{cases}\frac{|(p+q)x|(3+u)_{pq}^{\tau}}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}}, & 0\leq\left|\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)\right|\leq\frac{(3+u)_{pq}^{\tau}}{\xi(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}},\\ 2\left(p+q\right)x\left|\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)\right|, & for\ \ \left|\Upsilon\left(\gamma,\zeta,\lambda,\vartheta\right)\right|\geq\frac{(3+u)_{pq}^{\tau}}{\lambda(1+2\zeta+2\vartheta)(1+u)_{pq}^{\tau}}\end{cases}$$

After making several simplifications, the proof of Theorem 3.2 is complete. □

4. Conclusion

In this article, we introduced a new integral operator defined by (p,q)-Hurwitz zeta function, which is a generalization of the q-Srivastava-Attiya operator. We also provided two families $\mathcal{SM}_{p,q}\left(\xi,\zeta,\delta,u,\tau\right)$ and $\mathcal{SC}_{p,q}\left(\lambda,\zeta,\vartheta,u,\tau\right)$ of biunivalent and holomorphic functions in the unit disk, which is defined by (p,q)-Chebyshev polynomials, and we obtained Fekete-Szegö inequalities for these families. We also think that this construction has many applications.

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