Turkish Journal of Mathematics

Volume 46 | Number 6 Article 17

1-1-2022

On minimal absolutely pure domain of RD-flat modules

YUSUF ALAGÖZ

Follow this and additional works at: https://journals.tubitak.gov.tr/math

Part of the Mathematics Commons

Recommended Citation
ALAGÖZ, YUSUF (2022) "On minimal absolutely pure domain of RD-flat modules," *Turkish Journal of Mathematics*: Vol. 46: No. 6, Article 17. https://doi.org/10.55730/1300-0098.3269
Available at: https://journals.tubitak.gov.tr/math/vol46/iss6/17

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2292 – 2303
© TÜBİTAK
doi:10.55730/1300-0098.3269

On minimal absolutely pure domain of RD-flat modules

Yusuf ALAGÖZ*
Department of Mathematics, Faculty of Science, Siirt University, Siirt, Turkey

Received: 05.09.2021 • **Accepted/Published Online:** 06.05.2022 • **Final Version:** 04.07.2022

Abstract: Given modules A_R and ${}_RB$, ${}_RB$ is called absolutely A_R-pure if for every extension ${}_RC$ of ${}_RB$, $A\otimes B\to A\otimes C$ is a monomorphism. The class $\underline{\mathfrak{Fl}}^{-1}(A_R) = \{{}_RB : {}_RB$ is absolutely A_R-pure$\}$ is called the absolutely pure domain of a module A_R. If ${}_RB$ is divisible, then all short exact sequences starting with B is RD-pure, whence B is absolutey A-pure for every RD-flat module A_R. Thus the class of divisible modules is the smallest possible absolutely pure domain of an RD-flat module. In this paper, we consider RD-flat modules whose absolutely pure domains contain only divisible modules, and we referred to these RD-flat modules as rd-indigent. Properties of absolutely pure domains of RD-flat modules and of rd-indigent modules are studied. We prove that every ring has an rd-indigent module, and characterize rd-indigent abelian groups. Furthermore, over (commutative) SRDP rings, we give some characterizations of the rings whose nonprojective simple modules are rd-indigent.

Key words: RD-flat modules, absolutely pure domains, rd-indigent modules, QF-rings

1. Introduction and preliminaries

Throughout, R will denote an associative ring with identity, and modules will be unital right R-modules, unless otherwise stated. For any ring R, we denote by R-Mod, the category of left R-modules. For a module A, the module $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$ is called the character module of A and denoted by A^+.

Some recent works on module theory have focused on relative injectivity rather than considering classical injectivity (see [2, 4]). Given modules A_R and B_R, A is called B-subinjective if all homomorphisms $B\to A$ extends to some $E(B)\to A$, where $E(B)$ is the injective envelope of B. The subinjectivity domain $\underline{\mathfrak{In}}^{-1}(A)$ of A contains exactly all modules B such that A is B-subinjective. It is clear that if a module B is injective, then A is B-subinjective and so, $\underline{\mathfrak{In}}^{-1}(A)$ contains all injective modules. Hence, the modules whose subinjectivity domains contain only injective modules are defined to be indigent in [4]. Presently, it is not known whether indigent module exists for an arbitrary ring, but an affirmative answer is known for Noetherian rings ([13, Proposition 3.4]). Following ideas on subinjectivity domains, in [14], the pure-injective modules whose subinjectivity domains contain only absolutely pure modules are defined to be pi-indigent. In contrast to indigent modules, such pure-injective modules exist over any ring. The dual concepts of these units were studied in [12, 18].

In [13], Durğun is interested in the flat analog of these notions. Namely, given modules A_R and ${}_RB$, ${}_RB$ is called absolutely A_R-pure if for every extension ${}_RC$ of ${}_RB$, $A\otimes B\to A\otimes C$ is a monomorphism. The class

*Correspondence: yusuf.alagoz@siirt.edu.tr
2010 *AMS Mathematics Subject Classification:* 16D40, 16D60, 16L60, 18G25

This work is licensed under a Creative Commons Attribution 4.0 International License.

$\underline{\mathfrak{Fl}}^{-1}(A_R) = \{ {}_RB : {}_RB$ is absolutely A_R-pure$\}$ is called the absolutely pure domain of a module A_R (see [13]). It is clear that a module A_R is flat if and only if $\underline{\mathfrak{Fl}}^{-1}(A_R) = R$-Mod. As absolutely pure domains contains all absolutely pure modules, the authors in [13] considered f-indigent modules as modules whose absolutely pure domain consists of entire class of absolutely pure modules.

A submodule ${}_RA$ of ${}_RB$ is called RD-pure if for all $a \in R$, the induced map $\alpha : (R/aR) \otimes A \to (R/aR) \otimes B$ is a monomorphism, equivalently $\beta : \mathrm{Hom}(R/Ra, B) \to \mathrm{Hom}(R/Ra, B/A)$ is an epimorphism. An R-module ${}_RN$ is said to be RD-projective (resp. RD-injective) if it is projective (resp. injective) with respect to every RD-pure exact sequence ([20, 26]). An R-module F_R is called RD-flat if the map $\alpha : F \otimes A \to F \otimes B$ is monic for all modules ${}_RA$ and ${}_RB$ with ${}_RA$ is RD-pure in ${}_RB$ ([8]). According to Warfields criterions [8, 26], F_R is RD-projective if and only if F_R is RD-flat and pure-projective.

RD-purity is an important example of relative purity. It is the first notion of purity that appeared in the literature. Moreover, RD-purity coincides with purity for some classes of rings not necessarily commutative. On the other side, the RD-flat modules form an example of additive accessible category by [8, Proposition 1.1]. In [20, Proposition 2.3], Mao observed that a left R-module B is divisible if every short exact sequence starting with B is RD-pure.

In this paper, our aim is to reveal the links between the last trends mentioned above by considering questions similar to the seminal work on RD-purity and RD-flatness. Along the way, an easy observation shows that if a module ${}_RB$ is divisible, then ${}_RB$ is absolutey A_R-pure for any RD-flat module A_R. Thus the smallest possible absolutely pure domain of an RD-flat module is the class of divisible modules. We consider in this paper, the RD-flat modules whose absolutely pure domains contain only divisible modules, and we referred to these RD-flat modules as rd-indigent.

In Section 2, we study some properties of rd-indigent modules. We also establish connections between rd-indigent and f-indigent modules. We show that rd-indigent module exists over any ring (Proposition 2.3). For an abelian group H, we prove that H is rd-indigent if and only if $T(H) \neq pT(H)$ for every prime integer p and the torsion submodule $T(H)$ of H. A commutative domain R is shown to be Prüfer if and only if rd-indigent modules coincide with f-indigent modules. Furthermore, we prove that a ring R is left PP if and only if absolutely pure domain of any RD-flat right R-module is closed under quotients (Proposition 2.8). Moreover, a ring R is (von Neumann) regular if and only if there exists a flat rd-indigent right module (Proposition 2.9). Over a left P-coherent ring, R is left divisible if and only if there exists an rd-indigent RD-flat right module which embeds in a flat module (Proposition 2.11).

In Section 3, we address some questions studied and partially answered in [13] and [15]. The first question under consideration here is to give a characterization of a ring over which every simple module is rd-indigent or flat. Over a left Noetherian right C and a right SRDP ring R, we show that every simple right R-module is f-indigent or projective if and only if there exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either (a) right finitely $\sum$-extending right hereditary ring that contains a unique singular simple right R_2-module (up to isomorphism), or (b) n-saturated indecomposable matrix ring over a QF local ring (Theorem 3.6). For a commutative SRDP ring R, we prove that R is a C-ring and every simple R-module is rd-indigent or projective if and only if there exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either (a) finitely $\sum$-extending Noetherian hereditary ring that contains a unique singular simple module (up to isomorphism), or (b) matrix ring over a local QF ring (Theorem 3.8).

2. The absolutely pure domain of an RD-flat module

Following [19], an R-module ${}_RB$ is called divisible if $\mathrm{Ext}^1(R/Ra, B) = 0$ for all $a \in R$. An R-module A_R is said to be torsion-free if $\mathrm{Tor}_1(A, R/Ra) = 0$ for all $a \in R$. Furthermore, a module B is divisible (torsion-free) if and only if any exact sequences starting (ending) with B is an RD-pure exact ([20]). It is clear that a right R-module A is torsion-free if and only if A^+ is divisible. We refer the reader to [20, 21, 24], for more details about torsion-free and divisible modules. A cyclic right R-module $C \cong R/I$ is said to be cyclically presented if $I = aR$ for some $a \in R$. Thus, it is clear by the definitions that every cyclically presented right R-module is RD-flat.

Let A be an RD-flat right R-module. It is clear that A is flat if and only if $= \underline{\mathfrak{Fl}}^{-1}(A_R) = R$-Mod. If a left R-module B is divisible, every short exact sequence starting with B is RD-pure, whence B is absolutey A-pure. Thus the smallest possible absolutely pure domain of an RD-flat module is the class of divisible modules. The next result asserts that absolutely pure domain $\underline{\mathfrak{Fl}}^{-1}(A)$ of an RD-flat right R-module A how small can be. It should contain the divisible modules at least.

Proposition 2.1 $\bigcap_{A \in \mathscr{RF}} \underline{\mathfrak{Fl}}^{-1}(A) = \{B \in R\text{-}Mod \mid B \text{ is divisible}\}$, *where* $\mathscr{RF}$ *is the class of all* RD*-flat right* R*-modules.*

Proof Let $B \in \underline{\mathfrak{Fl}}^{-1}(A)$ for any $A \in \mathscr{RF}$. Since every cyclically presented right R-module is RD-flat, for each cyclically presented right R-module A, B is absolutely A-pure, which means that $\alpha : A \otimes B \to A \otimes E(B)$ is monic. Thus, B is divisible by [20, Proposition 2.3]. The converse is straightforward. □

In particular, if A is an RD-flat right R-module, then we have the following relations:

$\{$Absolutely pure left modules$\} \subseteq \{$Divisible left modules$\} \subseteq \underline{\mathfrak{Fl}}^{-1}(A) \subseteq R$-mod.

Therefore we wonder about the RD-flat modules whose absolutely pure domain contains only divisible modules.

Definition 2.2 *An* RD*-flat right* R*-module* A *is called* rd-indigent *if* $\underline{\mathfrak{Fl}}^{-1}(A) = \{$*Divisible left* R*-modules*$\}$.

In what follows, let $\mathfrak{F} := \bigoplus_{a \in R} R/aR$ for every $a \in R$. Since RD-flat right R-modules are closed under direct sums, $\mathfrak{F}$ is an RD-flat R-module.

The first problem that comes to mind is whether rd-indigent modules exist over all rings. A positive answer to that problem can be given by the following result.

Proposition 2.3 *The module* $\mathfrak{F}$ *is* rd*-indigent.*

Proof Recall that a left R-module B is divisible if and only if B is absolutely R/aR-pure for all $a \in R$. Thus Proposition 2.1 and [13, Proposition 2.4] implies that $\underline{\mathfrak{Fl}}^{-1}(\bigoplus_{a \in R} R/aR) = \bigcap_{a \in R} \underline{\mathfrak{Fl}}^{-1}(R/aR) = \{$Divisible left modules$\}$. Hence $\mathfrak{F}$ is rd-indigent. □

Recall that a ring R is said to be RD-ring if RD-pure exact sequence of R-modules is pure exact (see [8]). Serial rings, Dedekind prime rings and two-sided Warfield rings are examples of RD rings (see, [23]). Over an RD-ring R, every divisible left R-module is absolutely pure by [20, Proposition 2.15], but not conversely. Furthermore, if R is two-sided semihereditary such that maximal left and right quotient rings of R

are semisimple, then every torsion-free right R-module is flat by [3, Theorem 5.2]. In this case, every divisible left R-module is absolutely pure.

Corollary 2.4 *Over the following rings, $\mathfrak{F}$ is f-indigent.*

(1) RD-rings.

(2) Two-sided semihereditary such that maximal left and right quotient rings of R are semisimple.

Corollary 2.5 *Let R be a commutative semihereditary ring. The following are equivalent for a module A.*

1. *A is rd-indigent.*
2. *$Z(A)$ is rd-indigent, where $Z(A)$ is the singular submodue of A.*
3. *A is f-indigent.*

Proof (1) $\Leftrightarrow$ (3) Over a commutative semihereditary ring R, every pure projective R-module is RD-projective by [9, Corollary 2.11]. Thus, every R-module is RD-flat by [8, Theorem 1.4] and every divisible R-module is absolutely pure by [20, Proposition 2.15]. Thus (1) $\Leftrightarrow$ (3) follows.

(1) $\Leftrightarrow$ (2) follows by [13, Proposition 5.1]. □

There are several characterizations of Prüfer domains in the literature. A Prüfer domain is exactly a semihereditary integral domain. Over a commutative domain, the ring R is Prüfer if and only if divisible modules are absolutely pure (see [22]). Now, the following is easy by considering Corollary 2.5.

Corollary 2.6 *The following are equivalent for a commutative domain R.*

1. *R is Prüfer.*
2. *rd-indigent modules coincide with f-indigent modules.*
3. *$\mathfrak{F}$ is f-indigent.*

The f-indigent abelian groups are completely characterized in [13]. Using Corollary 2.5, rd-indigent abelian grups coincide with f-indigent groups. As a result of [13, Theorem 5.1] and [13, Corollary 5.1] we can give the characterization of rd-indigent groups.

Corollary 2.7 *The following are equivalent for an abelian group H with the torsion submodule $T(H)$:*

(1) H is rd-indigent.

(2) For every prime integer p, $T(H) \neq pT(H)$.

(3) $T(H) \otimes_R S \neq 0$ for all singular simple modules S.

(4) $\mathrm{Hom}(T(H), S) \neq 0$ for all singular simple modules S.

Recall that, a ring R is left PP if every principal left ideal of R is projective. A ring R is left PP if and only if every quotient of divisible left R-module is divisible (see, [20, Corollary 2.13]). The absolutely pure domain of an RD-flat module needs not be closed under quotients. For example, if we assume that R is not a left PP ring, by Proposition 2.3 we conclude that the absolutely pure domain of $\mathfrak{F}$ is not closed under quotients.

Proposition 2.8 *A ring R is left PP if and only if absolutely pure domain of any RD-flat right R-module A is closed under quotients.*

Proof First suppose that R is left PP and A is an RD-flat right R-module. Let B be a left R-module such that B is absolutely A-pure. For any submodule C of B, we claim that B/C is absolutely A-pure. Consider the commutative diagram below:

$$\begin{CD}
0 @>>> C @>>> B @>>> B/C @>>> 0\\
@. @VVhV @VVgV @VVfV @.\\
0 @>>> C @>>> E(B) @>>> E(B)/C @>>> 0
\end{CD}$$

with h is an isomorphism. Applying $A\otimes -$ to the diagram above gives the following diagram:

$$\begin{CD}
0 @>>> A\otimes C @>>> A\otimes B @>>> A\otimes B/C @>>> 0\\
@. @VVh^*V @VVg^*V @VVf^*V @.\\
0 @>>> A\otimes C @>>> A\otimes E(B) @>>> A\otimes E(B)/C @>>> 0
\end{CD}$$

Since h^* and g^* are monomorphisms, f^* is a monomorphism by the Five Lemma. Furthermore, PP condition on R gives that $E(B)/C$ is divisible (see [20, Corollary 2.13]), and so $E(B)/C \in \underline{\mathfrak{Fl}}^{-1}(A)$. On the other hand, consider the following diagram induced by the inclusions $\alpha : B/C \to E(B/C)$ and $\beta : E(B)/C \to E(E(B)/C)$.

$$\begin{CD}
B/C @>f>> E(B)/C\\
@VV\alpha V @VV\beta V\\
E(B/C) @. E(E(B)/C)
\end{CD}$$

Since $E(E(B)/C)$ is injective, there exists a homomorphism $\chi : E(B/C) \to E(E(B)/C)$ such that $\chi\alpha = \beta f$. Now, applying $A\otimes -$ to the diagram above gives the following commutative diagram:

$$\begin{CD}
A\otimes B/C @>f^*>> A\otimes E(B)/C\\
@VV\alpha^*V @VV\beta^*V\\
A\otimes E(B/C) @>\chi^*>> A\otimes E(E(B)/C)
\end{CD}$$

Since $E(B)/C \in \underline{\mathfrak{Fl}}^{-1}(A)$, β^* is monic, and so $\beta^* f^* = \chi^*\alpha^*$ is monic. This means that α^* is monic, whence $B/C \in \underline{\mathfrak{Fl}}^{-1}(A)$ by [13, Proposition 2.2]. For the converse, the hypothesis implies that the absolutely pure domain of $\mathfrak{F}$ is closed under quotients. But $\mathfrak{F}$ is rd-indigent, and so any quotient of a divisible left R-module is divisible, whence R is left PP by [20, Corollary 2.13]. □

A ring R is said to be (von Neumann) regular provided that for any $r \in R$ satisfies $r \in rRr$, equivalently every left R-module is absolutely pure.

Proposition 2.9 *Let R be a ring. The following are equivalent:*

(1) R is (von Neumann) regular.

(2) All left R-modules are divisible.

(3) All RD-flat right R-modules are flat.

(4) All (nonzero) RD-flat right R-modules are rd-indigent.

(5) There exists a flat rd-indigent right R-module.

Proof $(1) \Leftrightarrow (3)$ by [20, Corollary 2.14]. $(2) \Leftrightarrow (3)$, $(2) \Leftrightarrow (4)$ and $(5) \Rightarrow (2)$ are clear.

$(2) \Rightarrow (5)$ By hypothesis, $\underline{\mathfrak{Fl}}^{-1}(\mathfrak{F}) = R$-Mod, and so by Proposition 2.3, $\mathfrak{F}$ is flat and rd-indigent. □

From now on, all rings are supposed to be non (von Neumann) regular, equivalently by Proposition 2.9, there does not exist an rd-indigent flat right R-module.

Proposition 2.10 *If R is a commutative noetherian hereditary ring, then $\bigoplus_{S_i \in \Lambda} S_i$ is both an RD-flat and f-indigent module, where Λ is the set of representatives of simple singular modules.*

Proof Since R is commutative hereditary noetherian, every simple R-module is RD-projective by [5, Theorem 2.14], and so any $S_i \in \Lambda$ is RD-flat and finitely presented by [20, Lemma 2.1]. Since direct sum of any RD-flat modules is again RD-flat, $\bigoplus_{S_i \in \Lambda} S_i$ is RD-flat. Now, let us say that B is absolutely $\bigoplus_{S_i \in \Lambda} S_i$-pure for any module B. By [13, Corollary 2.1], $\bigoplus_{S_i \in \Lambda} S_i$ is B^+-subprojective and so by [13, Proposition 5.2], B^+ is flat. Thus coherence of R gives that B is absolutely pure. □

Recall that a ring R is called left P-coherent if every principal left ideal of R is finitely presented ([21]).

Proposition 2.11 *The following are equivalent for a left P-coherent ring R:*

1. *R is left divisible.*
2. *There exists an rd-indigent RD-flat right module that is embedded in a flat module.*
3. *All flat left modules are divisible.*

Proof $(1) \Rightarrow (2)$ Consider by Proposition 2.3 that $\mathfrak{F} := \bigoplus_{a_i \in R} R/a_iR$ is rd-indigent and RD-flat. Since R is left divisible, by [21, Proposition 4.2], for any $a_i \in R$, every R/a_iR contained in a flat R-module P_i. Set $P = \bigoplus_{P_i \in \Omega} P_i$, for a set Ω of the flat modules P_i. Then $\mathfrak{F}$ is embedded in a flat R-module P.

$(2) \Rightarrow (3)$ Existence of an rd-indigent RD-flat right module C that is embedded in a flat module gives that all flat left modules are absolutely C-pure by [13, Lemma 3.1], whence are divisible by rd-indigence of C.

$(3) \Rightarrow (1)$ is clear by [21, Proposition 4.2]. □

3. Rings whose nonprojective simples are rd-indigent

In this section the rings whose simple modules are rd-indigent or torsion-free are considered. By [20, Corollary 2.5], RD-flat torsion-free right modules are flat. By using this, the following is obvious.

Proposition 3.1 *The following are equivalent for a ring R:*

(1) All RD-flat right modules are rd-indigent or torsion-free.

(2) All RD-flat right modules are rd-indigent or flat.

Following [18], given modules A_R and B_R, A is called B-subprojective if for any epimorphisms $f: C \to B$ and any homomorphism $g: A \to B$, there exists a homomorphism $h: A \to C$ such that $fh = g$. The subprojectivity domain $\underline{\mathfrak{Pr}}^{-1}(A)$ of A contains exactly all modules B such that A is B-subprojective. Clearly, if B is projective, then A is vacuously B-subprojective, and so $\underline{\mathfrak{Pr}}^{-1}(A)$ contains all projective modules. Thus, the modules whose subprojectivity domains contains only projective modules is defined to be p-indigent in [18].

For an RD-projective right R-module A, we have the following relations:

$\{$Flat right modules$\} \subseteq \{$Torsion-free right modules$\} \subseteq \underline{\mathfrak{Pr}}^{-1}(A) \subseteq$ Mod-R.

Recall that, RD-projective modules whose subprojectivity domains contains only torsion-free (flat) modules is defined to be $(s)rdp$*-indigent* in [15].

Lemma 3.2 *Let R be a ring and A a finitely presented RD-flat right R-module. A is rd-indigent if and only if A is rdp-indigent and R is left P-coherent.*

Proof Let A be an rd-indigent right R-module such that A is B-subprojective for a right R-module B. So [13, Corollary 2.1] implies that B^+ is absolutely A-pure. Since A is rd-indigent, B^+ is divisible. Hence B is torsion-free. Also, since A is RD-projective by [20, Lemma 2.1(2)], A is rdp-indigent. Now, let C be a divisible left R-module. Since $C \in \underline{\mathfrak{Fl}}^{-1}(A)$, $C^+ \in \underline{\mathfrak{Pr}}^{-1}(A)$, and so C^+ is torsion-free, whence R is left P-coherent by [21, Theorem 2.7]. Conversely, let us say that B is absolutely A-pure for a left R-module B. Then A is B^+-subprojective by [13, Corollary 2.1] and the hypothesis implies that B^+ is torsion-free. Hence B is divisible by the P-coherence of R. □

Recall by [8, 26] that, a right R-module A is RD-projective if and only if A is RD-flat and pure-projective. A ring R is called right SRDP provided that every simple right R-module is RD-projective, equivalently every simple right R-module is RD-flat and finitely presented. A commutative ring R is SRDP if and only if every simple R-module is both finitely presented and RD-injective. PIR rings and commutative hereditary noetherian rings are SRDP (see [15]).

Corollary 3.3 *Let R be a right SRDP ring that is not left PP. Assume that every simple right R-module is either rd-indigent or projective. Then R has a unique nonprojective simple right module which satisfies the conditions given below:*

(1) R is left divisible.

(2) All flat left R-modules are divisible.

(3) All RD-projective right modules are embedded in a projective module.

(4) R is right Kasch.

Proof By using the Lemma 3.2 and [15, Lemma 3.7], the proof is easy. □

Corollary 3.4 *For a right SRDP ring R that is not left divisible, the following are equivalent.*

(1) All simple right R-modules are rd-indigent or projective.

(2) All simple right R-modules are rdp-indigent or projective.

(3) (i) Torsion-free right modules and the right modules with projective socle coincide.
(ii) R is a left PP ring with a unique nonprojective simple right R-module A (up to isomorphism).
Also, if R is right nonsingular, then the conditions given above are equivalent to:

(4) Singular finitely generated RD-projective right modules are rd-indigent.

Proof (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3) follows by Lemma 3.2 and [15, Proposition 3.6], respectively.

(3) $\Rightarrow$ (1) The hypothesis implies that nonprojective simple right R-modules are rdp-indigent by [15, Proposition 3.6]. Since R is a left PP ring, R is left P-coherent, whence S is rd-indigent by Lemma 3.2.

(4) $\Rightarrow$ (1) is clear.

(2) $\Rightarrow$ (4) let A be a singular RD-projective and finitely generated right module. So by [15, Proposition 3.2], $\underline{\mathfrak{Pr}}^{-1}(A) \subseteq \{\text{Torsion-free modules}\}$, whence A is rdp-indigent. Since R is left P-coherent, A is rd-indigent by Lemma 3.2.

□

Recall by [7, 10.10] that a ring R is called a right C-ring if for all essential right ideals I of R, $Soc(R/I) \neq 0$. Two-sided noetherian hereditary rings, left perfect rings and right semiartinian rings are trivial examples of such this rings.

Proposition 3.5 *Let R be a right C and right SRDP ring that is not left divisible. Assume that every simple right module is either rd-indigent or projective. Then R has a (up to isomorphism) unique nonprojective simple right module S and it satisfies the following conditions:*

(a) R is right Noetherian right hereditary.

(b) Classes of divisible right R-modules and injective right modules coincide.

(c) Classes of torsion-free right R-modules and nonsingular right modules coincide.

(d) Classes of torsion-free left R-modules and flat left modules coincide.

Proof Corollary 3.4 gives that R is a left PP ring which has a unique nonprojective simple right module S. By our hypothesis, S is rd-indigent, and by Lemma 3.2, S is rdp-indigent. Let A be a nonsingular right R-module. Since S is singular, S is A-subprojective, and so A is torsion-free. Let B be a divisible right module. Being R right SRDP implies that $\mathrm{Ext}(S, B) = 0$ by [20, Proposition 2.3]. By considering the epimorphism $E(B) \to E(B)/B \to 0$, we obtain the epimorphism $\mathrm{Hom}(S, E(B)) \to \mathrm{Hom}(S, E(B)/B) \to 0$. Hence by [7, 10.8 and 10.10], B is a closed submodule of $E(B)$, and so B is injective. In particular, every absolutely pure right module is injective whence R is right Noetherian. In this case, R has no infinite set of orthogonal nonzero idempotents and so R is right PP by [6, Lemma 8.4]. Being right PP implies by [21, Theorem 5.3] that quotients of injective right modules are injective, whence R is right hereditary. We claim that all torsion-free right R-modules are nonsingular. Let A be a torsion-free right R-module and $f : P \to A$ an epimorphism with P projective. Since R is a right C-ring and S is A-subprojective, $Ker(f)$ is closed in P by [16, Theorem 5]. Furthermore, since R is right nonsingular, the projective module P is nonsingular, whence A is nonsingular by [25, Lemma 2.3]. Now, we claim that all torsion-free left R-modules are flat. For a torsion-free left R-module T, T^+ is divisible, whence is injective by (b). Therefore T is flat. □

Recall from [10] that a right R-module A is called *extending* if every closed submodule of A is direct summand. A ring R is called *finitely* $\sum$*-extending* if every finite direct sums of copies of R_R is extending (see [10]).

Theorem 3.6 *For a left Noetherian right C and right SRDP ring R, the following are equivalent.*

(1) All simple right R-modules are f-indigent or projective.

(2) All simple right R-modules are srdp-indigent or projective.

(3) There exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either
(a) Right finitely $\sum$-extending right hereditary ring that contains a unique nonprojective simple right R_2-module (up to isomorphism), or
(b) n-saturated indecomposable matrix ring over a QF local ring.

Proof (1) ⇒ (2) Let S be a nonprojective simple right R-module and S a B-subprojective module for a right R-module B. By [13, Corollary 2.1], B^+ is absolutely S-pure, and by the f-indigence of S, B^+ is absolutely pure. Since R is right C and right SRDP, by the proof of Proposition 3.5, B^+ is injective. Thus, B is flat and S is *srdp*-indigent.

(2) ⇒ (1) Let S be a nonprojective simple right R-module and B an absolutely S-pure module for a right R-module B. This implies that S is B^+-subprojective by [13, Corollary 2.1], and by (2), B^+ is flat. Since B^{++} is injective and B is pure in B^{++}, B is absolutely pure. Thus, S is f-indigent.

(2) ⇒ (3) By using [15, Corollary 3.9], there is a unique nonprojective simple right module S and R is either left absolutely pure or $wD(R) \leq 1$. Since R is right C and right SRDP, every divisible (and so absolutely pure) right R-module is injective, whence R is right Noetherian. In the former case, by [15, Theorem 3.11], there is a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is an n-saturated indecomposable matrix ring over a QF local ring. The latter case implies R is not absolutely pure, and so by Lemma 3.2 and Proposition 3.5, R is right hereditary. Let I denote the sum of injective simple right ideals of R. Right Noetherianity of R implies that I is injective, whence R can be decomposed as $R = I \oplus J$ for some right ideal J of R such that $Soc(I) = I$ and J has no injective simple submodule. From this, by applying the same arguments as in [1, Theorem 1], if there is a nonzero homomorphism $\alpha : I \to J$, then $\alpha(Soc(I)) = \alpha(I) \subseteq Soc(J)$, where $\alpha(I)$ is injective since R is right hereditary. This means that $Soc(J)$ contains an injective simple submodule, a contradiction. Thus, I is also a left ideal by the fact that $\mathrm{Hom}(I, J) = 0$. For reverse order, if there is a nonzero homomorphism $\beta : J \to I$, then $J/Ker(\beta) \cong Im(\beta) \subseteq I$, where $J/Ker(\beta)$ is projective since R is right hereditary. Moreover, $J/Ker(\beta)$ is semisimple and injective as it is isomorphic to a submodule of I. By considering the split exact sequence $0 \to Ker(\beta) \to J \to J/Ker(\beta) \to 0$ we deduce that J contains a copy

of an injective simple R-module, a contradiction. Thus the fact that $\mathrm{Hom}(J, I) = 0$ implies J is an ideal, too. Consequently, we get a ring decomposition $R \cong I \times J$ where I is semisimple and J is right hereditary and right Noetherian. Now, we want to prove that J is right finitely $\sum$-extending. Let A be a nonsingular finitely generated right J-module. Since A is a nonsingular right R-module by [17, Proposition 1.28], any epimorphism $P \to A$ is closed exact by [25, Lemma 2.3]. Thus all simple right R-modules are projective relative to epimorphism $P \to A$ by [16, Theorem 5], and so S is A-subprojective. The fact that S is srdp-indigent implies that A is flat, and so A^+ is injective left R-module. This means that A^+ is an injective left J-module by [19, Example 3.11A]. So A is a flat and finitely generated right J-module, whence A is projective by noetherianity of J. Thus J is a right finitely $\sum$-extending ring by [10, Corollary 11.4]. Again by the uniqueness of a nonprojective simple right R-module S and by [17, Proposition 1.28], J has a unique nonprojective simple right J-module (up to isomorphism).

(3) ⇒ (2) Let $R = R_1 \times R_2$, where R_1 is semisimple and R_2 is either n-saturated indecomposable matrix ring over a QF local ring, or right finitely $\sum$-extending right hereditary ring with a unique singular simple right B-module (up to isomorphism). In the former case, (2) follows by [12, Theorem 3.1]. Assume the latter case, since R_2 is right hereditary right extending, R_2 cannot have an infinite set of orthogonal nonzero idempotents and so R_2 is right Noetherian by [11, Theorem 3.1]. Moreover, since R_2 is not right divisible, the rest follows by [15, Lemma 3.10]. □

In [15, Corollary 3.12], it is proven that a left Noetherian right PIR ring whose nonprojective simple right R-modules are *srdp*-indigent is a right C-ring. The following is now an easy result of Theorem 3.6 and [15, Corollary 3.12].

Corollary 3.7 *The following are equivalent for a left Noetherian right PIR ring R.*

(1) All simple right R-modules are f-indigent or projective.

(2) All simple right R-modules are srdp-indigent or projective.

(3) There exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either
(a) Right finitely $\sum$-extending right hereditary right C-ring that contains a unique nonprojective simple right B-module (up to isomorphism), or
(b) n-saturated indecomposable matrix ring over a QF local ring.

Recall that if E is an injective cogenerator in R-Mod over a commutative ring R, then for each simple module S, $\mathrm{Hom}(S, E) \cong S$, in particular $S \cong S^+$.

Theorem 3.8 *The following are equivalent for a commutative SRDP ring R.*

(1) R is a C-ring and every simple R-module is rd-indigent or projective.

(2) R is a C-ring and every simple R-module is rdp-indigent or projective.

(3) There exists a decomposition of rings $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is either
(a) Finitely $\sum$-extending hereditary Noetherian ring that contains a unique nonprojective simple module

(up to isomorphism), or

(b) matrix ring over a QF local ring.

Proof (1) ⇔ (2) is clear by the fact that SRDP C-rings are Noetherian and by Lemma 3.2.

(1) ⇒ (3) Let S be a nonprojective simple R-module. Then by the hypothesis, divisible modules contained in $\underline{\mathfrak{Fl}}^{-1}(S)$. Since R is SRDP and C-ring, by the proof of Proposition 3.5 all divisible modules are injective and so R is Noetherian, i.e. S is f-indigent. By [13, Theorem 5.2], $R \cong R_1 \times R_2$, where R_1 is semisimple and R_2 is a ring which is either matrix ring over a QF local ring, or noetherian hereditary ring with a unique singular simple module S'. In the later case, R_2 is not divisible, otherwise R_2 would be a semisimple ring. Since S' is rdp-indigent by Lemma 3.2, S' is srdp-indigent by Proposition 3.5(d). Thus, R_2 is finitely $\sum$-extending by Theorem 3.6.

(3) ⇒ (1) In either cases, R_1 and R_2 are C-rings, whence $R \cong R_1 \times R_2$ is a C-ring. Now the rest follows again by [13, Theorem 5.2]. □

Acknowledgment

The author would like to thank Professor Yılmaz Durğun for his valuable comments on an earlier version of the paper. The author is extremely grateful to the anonymous referees for their careful reading of the manuscript and for their valuable comments and suggestions.

References

[1] Alagöz Y, Büyükaşık E. Max-projective modules. Journal of Algebra and Its Applications 2021; 20 (6): 2150095. doi: 10.1142/S021949882150095X

[2] Alahmadi AN, Alkan M, López-Permouth SR. Poor modules: The opposite of injectivity. Glasgow Mathematical Journal 2010; 52 (A): 7-17. doi: 10.1017/S001708951000025X

[3] Albrecht U, Dauns J, Fuchs L. Torsion-freeness and non-singularity over right p.p.-rings. Journal of Algebra 2005; 285 (1): 98-119. doi: 10.1016/j.jalgebra.2004.10.020

[4] Aydoğdu P, López-Permouth SR. An alternative perspective on injectivity of modules. Journal of Algebra 2011; 338: 207-219. doi: 10.1016/j.jalgebra.2011.04.021

[5] Behboodi M, Ghorbani A, Moradzadeh-Dehkordi A, Shojaee SH. C-pure projective modules. Communications in Algebra 2013; 42 (12): 4559-4575. doi: 10.1080/00927872.2012.705934

[6] Chatters AW, Hajarnavis CR. Rings with chain conditions. Boston: Pitman, 1980.

[7] Clark J, Lomp C, Vanaja N, Wisbauer R. Lifting modules. Frontiers in Mathematics, Basel: Birkhäuser Verlag, 2006.

[8] Couchot F. RD-flatness and RD-injectivity. Communications in Algebra 2006; 34 (10): 3675-3689. doi: 10.1080/00927870600860817

[9] Divaani-Aazar K, Esmkhani MA, Tousi M. A criterion for rings which are locally valuation rings. Colloquium Mathematicum 2009; 116: 153-164. doi: 10.4064/cm116-2-2

[10] Dung NV, Huynh DV, Smith PF, Wisbauer R. Extending modules. Harlow: Longman Scientific and Technical, 1994.

[11] Dung NV. A note on hereditary rings or non-singular rings with chain condition. Mathematica Scandinavica 1990; 66: 301-306.

[12] Durğun Y. Rings whose modules have maximal or minimal subprojectivity domain. Journal of Algebra and Its Applications 2015; 14 (6): 1550083. doi: 10.1142/S0219498815500838

[13] Durğun Y. An alternative perspective on flatness of modules. Journal of Algebra and Its Applications 2016; 15 (8): 1650145. doi: 10.1142/S0219498816501450

[14] Durğun Y. On subinjectivity domains of pure-injective modules. Rocky Mountain Journal of Mathematics 2021; 51 (4): 1227-1238. doi: 10.1216/rmj.2021.51.1227

[15] Durğun Y. RD-projective module whose subprojectivity domain is minimal. Hacettepe Journal of Mathematics and Statistics 2022; 51 (2): 373-382. doi: 10.15672/hujms.776933

[16] Generalov AI. Weak and ω-high purities in the category of modules. Sbornik Mathematics 1978; 34 (3): 345-356. doi: 10.1070/SM1978v034n03ABEH001209

[17] Goodearl KR. Ring theory. New York: Marcel Dekker Inc., 1976.

[18] Holston C, López-Permouth SR, Mastromatteo J, Simental-Rodríguez JE. An alternative perspective on projectivity of modules. Glasgow Mathematical Journal 2015; 57 (1): 83-99. doi: 10.1017/S0017089514000135

[19] Lam TY. Lectures on modules and rings. New York: Springer, 1999.

[20] Mao L. Properties of RD-projective and RD-injective modules. Turkish Journal of Mathematics 2011; 35 (2): 187-205. doi: 10.3906/mat-0904-53

[21] Mao L, Ding N. On divisible and torsionfree modules. Communications in Algebra 2008; 36 (2): 708-731. doi: 10.1080/00927870701724201

[22] Megibben C. Absolutely pure modules. Proceedings of the American Mathematical Society 1970; 26 (4): 561-566. doi: 10.2307/2037108

[23] Puninski G, Prest M, Rothmaler P. Rings described by various purities. Communications in Algebra 1999; 27 (5): 2127-2162. doi: 10.1080/00927879908826554

[24] Rothmaler P. Torsion-free, divisible, and Mittag-Leffler modules. Communications in Algebra 2015; 43 (8): 3342-3364. doi: 10.1080/00927872.2014.918990

[25] Sandomierski FL. Nonsingular rings. Proceedings of the American Mathematical Society 1968; 19: 225-230. doi: 10.1090/S0002-9939-1968-0219568-5

[26] Warfield Jr. RB. Purity and algebraic compactness for modules. Pacific Journal of Mathematics 1969; 28 (3): 699-719.