[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

[Volume 46](https://journals.tubitak.gov.tr/math/vol46) | [Number 6](https://journals.tubitak.gov.tr/math/vol46/iss6) Article 11

1-1-2022

Hermite-Hadamard-Mercer type inclusions for interval-valued functions via Riemann-Liouville fractional integrals

HASAN KARA

MUHAMMAD AAMIR ALI

HÜSEYİN BUDAK

Follow this and additional works at: https://journals.tubitak.gov.tr/math

Part of the Mathematics Commons

Recommended Citation

KARA, HASAN; ALI, MUHAMMAD AAMIR; and BUDAK, HÜSEYİN (2022) "Hermite-Hadamard-Mercer type inclusions for interval-valued functions via Riemann-Liouville fractional integrals," *Turkish Journal of Mathematics*: Vol. 46: No. 6, Article 11. https://doi.org/10.55730/1300-0098.3263
Available at: https://journals.tubitak.gov.tr/math/vol46/iss6/11

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2193 – 2207
© TÜBİTAK
doi:10.55730/1300-0098.3263

Hermite–Hadamard–Mercer type inclusions for interval-valued functions via Riemann–Liouville fractional integrals

Hasan KARA[1,*], **Muhammad Aamir ALI**[2], **Hüseyin BUDAK**[1]

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

[2]Jiangsu Key Laboratory of NSLSCS, School of Mathematical Sciences, Nanjing Normal University, China

Received: 31.08.2021 • **Accepted/Published Online:** 26.04.2022 • **Final Version:** 04.07.2022

Abstract: In this research, we first establish some inclusions of fractional Hermite–Hadamard–Mercer type for interval-valued functions. Moreover, by special cases of our main results, we show that our main results reduce several inclusions obtained in the earlier works.

Key words: Hermite–Hadamard–Mercer inequality, Riemann–Liouville fractional integrals, interval-valued function, convex function

1. Introduction

In literature, the well-known Jensen inequality [21] states that if $\mathcal{F}$ is a convex function on an interval contains the $\varkappa_k$, then

$$\mathcal{F}\left(\sum_{j=1}^{n}\lambda_j\varkappa_j\right)\leq\sum_{j=1}^{n}\lambda_j\mathcal{F}\left(\varkappa_j\right) \tag{1.1}$$

where $\sum\limits_{j=1}^{n}\lambda_j=1,\ \lambda_j\in[0,1]$ and $0<\varkappa_1\leq\varkappa_2\leq\varkappa_3\leq...\leq\varkappa_n$.

The Hermite–Hadamard (H–H) inequality, discovered by C. Hermite and J. Hadamard (see, also, [13], and [31, p.137]), is one of the most well-known inequalities in the theory of convex functions, with a geometrical interpretation and a wide range of applications. The H–H inequality is stated as:

$$\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right)\leq\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\mathcal{F}\left(\varkappa\right)d\varkappa\leq\frac{\mathcal{F}\left(\varpi_1\right)+\mathcal{F}\left(\varpi_2\right)}{2}, \tag{1.2}$$

where $\mathcal{F}:I\rightarrow\mathbb{R}$ is a convex function over I and $\varpi_1,\varpi_2\in I$ with $\varpi_1<\varpi_2$. In the case that $\mathcal{F}$ is concave mapping, the above inequality satisfies in reverse order. We should point out that H-H inequality is a refinement of the concept of convexity, and it follows obviously from Jensen's inequality. In recent years, the H–H inequality for convex functions has gotten a lot of attention, and a lot of refinements and generalizations have been studied.

The following variant of Jensen inequality, known as the Jensen–Mercer, was demonstrated by Mercer [20]:

*Correspondence: hasan64kara@gmail.com

2010 *AMS Mathematics Subject Classification:* 26D10, 26A51, 26D15

This work is licensed under a Creative Commons Attribution 4.0 International License.

Theorem 1.1 *If $\mathcal{F}$ is a convex function on $[\varpi_1, \varpi_2]$, then the following inequality is true:*

$$\mathcal{F}\left(\varpi_1+\varpi_2-\sum_{j=1}^{n}\lambda_j\varkappa_j\right) \leq \mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)-\sum_{j=1}^{n}\lambda_j\mathcal{F}(\varkappa_j), \tag{1.3}$$

where $\sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\in[0,1]$ *and* $\varkappa_j\in[\varpi_1,\varpi_2]$.

In [16], the idea of Jensen–Mercer inequality has been used by Kian and Moslehian, and the following H–H–Mercer inequalities were demonstrated:

$$\begin{aligned}\mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right) &\leq \mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)-\frac{1}{\upsilon-\varkappa}\int_{\varkappa}^{\upsilon}\mathcal{F}(\vartheta)\,d\vartheta \\ &\leq \mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)-\mathcal{F}\left(\frac{\varkappa+\upsilon}{2}\right)\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}\mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right) &\leq \frac{1}{\upsilon-\varkappa}\int_{\varkappa}^{\upsilon}\mathcal{F}(\varpi_1+\varpi_2-\vartheta)\,d\vartheta \\ &\leq \frac{\mathcal{F}(\varpi_1+\varpi_2-\varkappa)+\mathcal{F}(\varpi_1+\varpi_2-\upsilon)}{2} \\ &\leq \mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)-\frac{\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon)}{2},\end{aligned} \tag{1.5}$$

where $\mathcal{F}$ is convex function on $[\varpi_1, \varpi_2]$. For some recent studies linked to Jensen–Mercer inequality, one can consult [1, 2, 10, 25].

In contrast, interval analysis is a well-known example of set-valued analysis, which is the study of sets in the context of mathematical and general topology analysis. It was created as a solution to the interval instability of deterministic real-world phenomena that can be found in many mathematical or computer models. The technique of Archimede's, which is related to computing the diameter of a circle, is an old example of an interval enclosure. Moore, who is credited with being the first to use intervals in computational mathematics, published the first book on interval analysis in 1966 (see, [23]). Following the publication of his book, a number of scientists began to study the theory and applications of interval arithmetic.

In addition, several significant inequalities (H–H, Ostrowski, and others) for interval-valued functions have been studied in recent years. Chalco-Cano et al. obtained Ostrowski type inequalities for interval-valued functions in [8, 9] using the Hukuhara derivative for interval-valued functions. We refer readers to [7, 11, 12, 14, 15, 22, 26, 27, 32–34, 38, 39] for additional relevant results.

The aim of this paper is to generalize the Hermite–Hadamard–Mercer inequality for interval-valued functions in the case of Riemann–Liouville fractional integrals.

2. Interval calculus and inequalities

In this section, we provide notation and background information on interval analysis. The space of all closed intervals of $\mathbb{R}$ is denoted by I_c and Δ is a bounded element of I_c. We have the representation

$$\Delta=[\underline{\alpha},\overline{\alpha}]=\{\vartheta\in\mathbb{R}:\underline{\alpha}\leq\vartheta\leq\overline{\alpha}\},$$

where $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ and $\underline{\alpha} \leq \overline{\alpha}$. $L(\Delta) = \overline{\alpha} - \underline{\alpha}$ can be used to express the length of the interval $\Delta = [\underline{\alpha}, \overline{\alpha}]$. The left and right endpoints of interval Δ are denoted by the numbers $\underline{\alpha}$ and $\overline{\alpha}$, respectively. The interval Δ is said to be degenerate when $\underline{\alpha} = \overline{\alpha}$, and the form $\Delta = \alpha = [\alpha, \alpha]$ is used. Also, if $\underline{\alpha} > 0$, we can say Δ is positive, and if $\overline{\alpha} < 0$, we can say Δ is negative. I_c^+ and I_c^- denote the sets of all closed positive intervals and closed negative intervals of $\mathbb{R}$, respectively. Between the intervals Δ and Λ, the Pompeiu–Hausdorff distance is defined by

$$d_H(\Delta, \Lambda) = d_H\left([\underline{\alpha}, \overline{\alpha}], [\underline{\beta}, \overline{\beta}]\right) = \max\left\{\left|\underline{\alpha} - \underline{\beta}\right|, \left|\overline{\alpha} - \overline{\beta}\right|\right\}. \tag{2.1}$$

(I_c, d_H) is a complete metric space, as far as we know (see, [4]).

$|\Delta|$ denotes the absolute value of Δ, which is the maximum of the absolute values of its endpoints:

$$|\Delta| = \max\{|\underline{\alpha}|, |\overline{\alpha}|\}.$$

The following are the concepts for fundamental interval arithmetic operations for the intervals Δ and Λ:

$$\begin{aligned}
\Delta + \Lambda &= \left[\underline{\alpha} + \underline{\beta}, \overline{\alpha} + \overline{\beta}\right], \\
\Delta - \Lambda &= \left[\underline{\alpha} - \overline{\beta}, \overline{\alpha} - \underline{\beta}\right], \\
\Delta \cdot \Lambda &= [\min U, \max U] \text{ where } U = \left\{\underline{\alpha}\ \underline{\beta}, \underline{\alpha}\ \overline{\beta}\ \overline{\alpha}\underline{\beta}, \overline{\alpha}\ \overline{\beta}\right\}, \\
\Delta / \Lambda &= [\min V, \max V] \text{ where } V = \left\{\underline{\alpha}/\underline{\beta}, \underline{\alpha}/\overline{\beta}, \overline{\alpha}/\underline{\beta}, \overline{\alpha}/\overline{\beta}\right\} \text{ and } 0 \notin \Lambda.
\end{aligned}$$

The interval Δ's scalar multiplication is defined by

$$\mu\Delta = \mu[\underline{\alpha}, \overline{\alpha}] = \begin{cases} [\mu\underline{\alpha}, \mu\overline{\alpha}], & \mu > 0; \\ \{0\}, & \mu = 0; \\ [\mu\overline{\alpha}, \mu\underline{\alpha}], & \mu < 0, \end{cases}$$

where $\mu \in \mathbb{R}$.

The opposite of the interval Δ is

$$-\Delta := (-1)\Delta = [-\overline{\alpha}, -\underline{\alpha}],$$

where $\mu = -1$.

In general, $-\Delta$ is not additive inverse for Δ, i.e. $\Delta - \Delta \neq 0$.

Definition 2.1 *For some kind of the intervals Δ, $\Lambda \in I_c$, we denote the the H-difference of Δ and Λ as the $\Omega \in I_c$, we have*

$$\Delta \ominus_g \Lambda = \Omega \Leftrightarrow \begin{cases} (i)\ \Delta = \Lambda + \Omega \\ \quad or \\ (ii)\ \Lambda = \Delta + (-\Omega). \end{cases}$$

It seems beyond controversy that

$$\Delta \ominus_g \Lambda = \begin{cases} \left[\underline{\alpha} - \underline{\beta}, \overline{\alpha} - \overline{\beta}\right], \text{ if } L(\Delta) \geq L(\Lambda) \\ \left[\overline{\alpha} - \overline{\beta}, \underline{\alpha} - \underline{\beta}\right], \text{ if } L(\Delta) \leq L(\Lambda), \end{cases}$$

where $L(\Delta) = \overline{\alpha} - \underline{\alpha}$ *and* $L(\Lambda) = \overline{\beta} - \underline{\beta}$.

The definitions of operations generate a large number of algebraic properties, enabling I_c to be a quasilinear space (see, [19]). The following are some of these characteristics (see, [4, 18, 19, 23]):

(1) (Law of associative under $+$) $(\Delta + \Lambda) + \mathcal{C} = \Delta + (\Lambda + \mathcal{C})$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
(2) (Additivity element) $\Delta + 0 = 0 + \Delta = \Delta$ for all $\Delta \in I_c$,
(3) (Law of commutative under $+$) $\Delta + \Lambda = \Lambda + \Delta$ for all $\Delta, \Lambda \in I_c$,
(4) (Law of cancellation under $+$) $\Delta + \mathcal{C} = \Lambda + \mathcal{C} \Longrightarrow \Delta = \Lambda$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
(5) (Law of associative under $\times$) $(\Delta \cdot \Lambda) \cdot \mathcal{C} = \Delta \cdot (\Lambda \cdot \mathcal{C})$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
(6) (Law of commutative under $\times$) $\Delta \cdot \Lambda = \Lambda \cdot \Delta$ for all $\Delta, \Lambda \in I_c$,
(7) (Multiplicativity element) $\Delta \cdot 1 = 1 \cdot \Delta$ for all $\Delta \in I_c$,
(8) (The first law of distributivity) $\lambda(\Delta + \Lambda) = \lambda\Delta + \lambda\Lambda$ for all $\Delta, \Lambda \in I_c$ and all $\lambda \in \mathbb{R}$,
(9) (The second law of distributivity) $(\lambda + \mu)\Delta = \lambda\Delta + \mu\Delta$ for all $\Delta \in I_c$ and all $\lambda, \mu \in \mathbb{R}$.

Aside from any of these characteristics, the distributive law does not always apply to intervals. As an example, $\Delta = [1, 2], \Lambda = [2, 3]$ and $\mathcal{C} = [-2, -1]$.

$$\Delta \cdot (\Lambda + \mathcal{C}) = [0, 4],$$

whereas

$$\Delta \cdot \Lambda + \Delta \cdot \mathcal{C} = [-2, 5].$$

Another distinct feature is the inclusion $\subseteq$, which is described by

$$\Delta \subseteq \Lambda \Longleftrightarrow \underline{\alpha} \geq \underline{\beta} \text{ and } \overline{\alpha} \leq \overline{\beta}.$$

In [23], Moore gave the definition of the Riemann integral for functions of interval-valued. $IR_{([\varpi_1, \varpi_2])}$ and $R_{([\varpi_1, \varpi_2])}$ denote the set of all Riemann integrable interval-valued functions and real-valued functions on $[\varpi_1, \varpi_2]$, respectively. The following theorem defines a relationship between Riemann integrable (R-integrable) functions and $(\mathcal{IR})$-integrable functions (see, [24, pp. 131]):

Theorem 2.2 *For an interval-valued mapping* $\mathcal{F} : [\varpi_1, \varpi_2] \to I_c$ *with* $\mathcal{F}(\vartheta) = \left[\underline{\mathcal{F}}(\vartheta), \overline{\mathcal{F}}(\vartheta)\right]$. *The mapping* $\mathcal{F} \in IR_{([\varpi_1, \varpi_2])}$ *if and only if* $\underline{\mathcal{F}}(\vartheta)$, $\overline{\mathcal{F}}(\vartheta) \in R_{([\varpi_1, \varpi_2])}$ *and*

$$(\mathcal{IR}) \int_{\varpi_1}^{\varpi_2} \mathcal{F}(\vartheta) d\vartheta = \left[(R) \int_{\varpi_1}^{\varpi_2} \underline{\mathcal{F}}(\vartheta) d\vartheta, (R) \int_{\varpi_1}^{\varpi_2} \overline{\mathcal{F}}(\vartheta) d\vartheta \right].$$

Zhao et al. defined the following convex interval-valued function in [40, 42]:

Definition 2.3 *For all* $\varkappa, \upsilon \in [\varpi_1, \varpi_2]$ *and* $\vartheta \in (0, 1)$, *the* h*-convex mapping* $\mathcal{F} : [\varpi_1, \varpi_2] \to I_c^+$ *is stated as:*

$$h(\vartheta)\mathcal{F}(\varkappa) + h(1 - \vartheta)\mathcal{F}(\upsilon) \subseteq \mathcal{F}(\vartheta\varkappa + (1 - \vartheta)\upsilon). \tag{2.2}$$

Here, $h : [c, d] \to \mathbb{R}$ *is a nonnegative mapping,* $h \neq 0$, $(0, 1) \subseteq [c, d]$. *We will show the set of all* h*-convex interval-valued functions with* $SX(h, [\varpi_1, \varpi_2], I_c^+)$.

The standard definition of a convex interval-valued function is (2.2) with $h(\vartheta)=\vartheta$ (see, [34]). In addition, if we take $h(\vartheta)=\vartheta^{s}$ into (2.2), then Definition 2.3 gives the definition of s-convex interval-valued function (see, [6]).

In [40], Zhao et al. used the h-convexity of interval-valued functions and obtained the following H-H inclusion:

Theorem 2.4 *If $\mathcal{F}\in SX(h,[\varpi_1,\varpi_2],I_c^+)$ and $h\left(\frac{1}{2}\right)\neq 0$, then following inclusions are true:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right)\supseteq\frac{1}{\varpi_2-\varpi_1}(\mathcal{IR})\int\limits_{\varpi_1}^{\varpi_2}\mathcal{F}(\varkappa)d\varkappa\supseteq[\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)]\int\limits_{0}^{1}h(\vartheta)d\vartheta. \tag{2.3}$$

Remark 2.5 *(i) The Inclusions (2.3) becomes the following for $h(\vartheta)=\vartheta$:*

$$\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right)\supseteq\frac{1}{\varpi_2-\varpi_1}(\mathcal{IR})\int\limits_{\varpi_1}^{\varpi_2}\mathcal{F}(\varkappa)d\varkappa\supseteq\frac{\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)}{2}, \tag{2.4}$$

which Sadowska has discovered in [34].

(ii) The Inclusions (2.3) becomes the following for $h(\vartheta)=\vartheta^{s}$:

$$2^{s-1}\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right)\supseteq\frac{1}{\varpi_2-\varpi_1}(\mathcal{IR})\int\limits_{\varpi_1}^{\varpi_2}\mathcal{F}(\varkappa)d\varkappa\supseteq\frac{\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)}{s+1},$$

which Osuna-Gómez et al. have discovered in [29].

Definition 2.6 *[34] A function $\mathcal{F}:[\varpi_1,\varpi_2]\to I_c^+$ is said to be a convex interval-valued, if for all $\varkappa,\upsilon\in[\varpi_1,\varpi_2]$ and $\vartheta\in(0,1)$, we have*

$$\vartheta\mathcal{F}(\varkappa)+(1-\vartheta)\mathcal{F}(\upsilon)\subseteq\mathcal{F}(\vartheta\varkappa+(1-\vartheta)\upsilon).$$

Theorem 2.7 *[5, 34] A function $\mathcal{F}:[\varpi_1,\varpi_2]\to I_c^+$ is said to be a convex interval-valued if and only if $\underline{\mathcal{F}}$ is a convex function on $[\varpi_1,\varpi_2]$ and $\overline{\mathcal{F}}$ is a concave function on $[\varpi_1,\varpi_2]$.*

In [18] Lupulescu introduced the following left-sided Riemann–Liouville fractional integral based on interval-valued functions.

Definition 2.8 *Let $\mathcal{F}:[\varpi_1,\varpi_2]\to\mathbb{R}_{\mathcal{I}}$ be an interval-valued mapping such that $\mathcal{F}(\vartheta)=\left[\underline{\mathcal{F}}(\vartheta),\overline{\mathcal{F}}(\vartheta)\right]$ and let $\alpha>0$. The interval-valued left-sided Riemann–Liouville fractional integral of function $\mathcal{F}$ is defined by*

$$J_{\varpi_1+}^{\alpha}\mathcal{F}(\varkappa)=\frac{1}{\Gamma(\alpha)}(\mathcal{IR})\int\limits_{\varpi_1}^{\varkappa}(\varkappa-\vartheta)^{\alpha-1}\mathcal{F}(\vartheta)d\vartheta,\quad \varkappa>\varpi_1,$$

where Γ is Euler Gamma function.

On the other hand, Budak et al. in [7] gave the definition of interval-valued right-sided Riemann–Liouville fractional integral of function $\mathcal{F}$ *by*

$$J^{\alpha}_{\varpi_2-}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)}(\mathcal{IR})\int_{\varkappa}^{\varpi_2}(\vartheta-\varkappa)^{\alpha-1}\mathcal{F}(\vartheta)d\vartheta, \quad \varkappa<\varpi_2,$$

where Γ *is Euler Gamma function.*

In addition, the authors gave the following fractional Hermite–Hadamard inequality for interval-valued functions.

Theorem 2.9 *[7] Let* $F=[\underline{F}(\vartheta),\overline{F}(\vartheta)]:[\varpi_1,\varpi_2]\to\mathbb{R}^{+}_{\mathcal{I}}$ *be a convex interval-valued function and* $\alpha>0,$ *then we have the following inclusion*

$$F\left(\frac{\varpi_1+\varpi_2}{2}\right)\supseteq\frac{\Gamma(1+\alpha)}{2(\varpi_2-\varpi_1)^{\alpha}}\left[J^{\alpha}_{\varpi_1+}F(\varpi_2)+J^{\alpha}_{\varpi_2-}F(\varpi_1)\right]\supseteq\frac{F(\varpi_1)+F(\varpi_2)}{2}. \tag{2.5}$$

Many researchers in the literature have worked on generalizations and extensions of the inclusions (2.5). For example, Liu et al. introduced the Hermite–Hadamard type inequalities for interval-valued harmonically convex functions in [17]. Zhou et al. obtained Hermite–Hadamard type inequalities for interval-valued exponential type preinvex functions in Riemann-Liouville interval-valued fractional operator settings in [41]. Shi et al. presented some fractional Hermite–Hadamard-type inequalities via the product of two coordinated h-convex interval-valued mappings in [37]. Sharma et al. proved $(h1,h2)$-preinvex interval-valued function and obtained the Hermite–Hadamard inequality for preinvex interval-valued functions by utilizing interval-valued Riemann–Liouville fractional integrals in [36].

Theorem 2.10 (Jensen's Inclusion) *[35] Let* $0<\varkappa_1\leq\varkappa_2\leq\varkappa_3\leq\ldots\leq\varkappa_n$ *and* $\mathcal{F}$ *be a convex interval-valued function on an interval containing* $\varkappa_k$*, then following inclusion is true:*

$$\mathcal{F}\left(\sum_{j=1}^{n}\lambda_j\varkappa_j\right)\supseteq\sum_{j=1}^{n}\lambda_j\mathcal{F}(\varkappa_j), \tag{2.6}$$

where $\sum\limits_{j=1}^{n}\lambda_j=1,\ \lambda_j\in[0,1]$.

In [35], the authors extend the inequality (2.6) to the convex interval-valued functions as follows:

Theorem 2.11 (Jensen–Mercer inclusion) *[35] Let* $\mathcal{F}$ *be a convex interval-valued function on* $[\varpi_1,\varpi_2]$ *such that* $L(\varpi_2)\geq L(\varpi_0)$ *for all* $\varpi_0\in[\varpi_1,\varpi_2]$*, then following inclusion is true:*

$$\mathcal{F}\left(\varpi_1+\varpi_2-\sum_{j=1}^{n}\lambda_j\varkappa_j\right)\supseteq\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\ominus_g\sum_{j=1}^{n}\lambda_j\mathcal{F}(\varkappa_j) \tag{2.7}$$

where $\sum\limits_{j=1}^{n}\lambda_j=1,\ \lambda_j\in[0,1]$.

3. Main results

In this section, we prove Hermite–Hadamard–Mercer type inclusion for convex interval-valued function via Riemann–Liouville fractional integrals.

Theorem 3.1 *Let $\mathcal{F}:[\varpi_1,\varpi_2]\to I_c^+$ be an interval-valued convex function such that $\mathcal{F}(\vartheta)=\left[\underline{\mathcal{F}}(\vartheta),\overline{\mathcal{F}}(\vartheta)\right]$ and $L(\varpi_2)\geq L(\varpi_0)$ for all $\varpi_0\in[\varpi_1,\varpi_2]$. Then, we have the following inclusions:*

$$\begin{aligned} & \mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right) \qquad (3.1)\\ \supseteq\ & \frac{\Gamma(\alpha+1)}{2(\upsilon-\varkappa)^\alpha}\left[J^\alpha_{(\varpi_1+\varpi_2-\varkappa)-}\mathcal{F}(\varpi_1+\varpi_2-\upsilon)+\ J^\alpha_{(\varpi_1+\varpi_2-\upsilon)+}\mathcal{F}(\varpi_1+\varpi_2-\varkappa)\right]\\ \supseteq\ & \frac{\mathcal{F}(\varpi_1+\varpi_2-\varkappa)+\mathcal{F}(\varpi_1+\varpi_2-\upsilon)}{2}\supseteq\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\ominus_g\frac{\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon)}{2}. \end{aligned}$$

Proof Since $\mathcal{F}$ is an interval-valued convex function, then for all $u,v\in[\varpi_1,\varpi_2]$, we have

$$\begin{aligned} \mathcal{F}\left(\varpi_1+\varpi_2-\frac{u+v}{2}\right) &= \mathcal{F}\left(\frac{\varpi_1+\varpi_2-u+\varpi_1+\varpi_2-v}{2}\right) \qquad (3.2)\\ &\supseteq \frac{1}{2}\left[\mathcal{F}(\varpi_1+\varpi_2-u)+\mathcal{F}(\varpi_1+\varpi_2-v)\right]. \end{aligned}$$

Using $\varpi_1+\varpi_2-u=\vartheta(\varpi_1+\varpi_2-\varkappa)+(1-\vartheta)(\varpi_1+\varpi_2-\upsilon)$ and $\varpi_1+\varpi_2-v=\vartheta(\varpi_1+\varpi_2-\upsilon)+(1-\vartheta)(\varpi_1+\varpi_2-\varkappa)$ for $\varkappa,\upsilon\in[\varpi_1,\varpi_2]$ and $\vartheta\in[0,1]$, we get

$$\begin{aligned} & \mathcal{F}\left(\varpi_1+\varpi_2-\frac{u+v}{2}\right) \qquad (3.3)\\ \supseteq\ & \frac{1}{2}\left[\mathcal{F}(\vartheta(\varpi_1+\varpi_2-\varkappa)+(1-\vartheta)(\varpi_1+\varpi_2-\upsilon))\right.\\ & \left.+\mathcal{F}(\vartheta(\varpi_1+\varpi_2-\upsilon)+(1-\vartheta)(\varpi_1+\varpi_2-\varkappa))\right]. \end{aligned}$$

Now multiplying both sides of (3.3) by $\vartheta^{\alpha-1}$ and integrating the obtaining inclusion with respect to ϑ over $[0,1]$, we obtain

$$\begin{aligned} & \frac{1}{\alpha}\mathcal{F}\left(\varpi_1+\varpi_2-\frac{u+v}{2}\right)\\ \supseteq\ & \frac{1}{2}\left[\int_0^1\vartheta^{\alpha-1}\mathcal{F}(\vartheta(\varpi_1+\varpi_2-\varkappa)+(1-\vartheta)(\varpi_1+\varpi_2-\upsilon))\,d\vartheta\right.\\ & \left.+\int_0^1\vartheta^{\alpha-1}\mathcal{F}(\vartheta(\varpi_1+\varpi_2-\upsilon)+(1-\vartheta)(\varpi_1+\varpi_2-\varkappa))\,d\vartheta\right]\\ =\ & \frac{1}{2(\upsilon-\varkappa)^\alpha}\left[\int_{\varpi_1+\varpi_2-\upsilon}^{\varpi_1+\varpi_2-\varkappa}\left[w-(\varpi_1+\varpi_2-\upsilon)\right]^{\alpha-1}\mathcal{F}(w)\,dw\right.\\ & \left.+\int_{\varpi_1+\varpi_2-\upsilon}^{\varpi_1+\varpi_2-\varkappa}\left[(\varpi_1+\varpi_2-\varkappa)-w\right]^{\alpha-1}\mathcal{F}(w)\,dw\right]\\ =\ & \frac{\Gamma(\alpha)}{2(\upsilon-\varkappa)^\alpha}\left[J^\alpha_{(\varpi_1+\varpi_2-\upsilon)+}\mathcal{F}(\varpi_1+\varpi_2-\varkappa)+\ J^\alpha_{(\varpi_1+\varpi_2-\varkappa)-}\mathcal{F}(\varpi_1+\varpi_2-\upsilon)\right] \end{aligned}$$

and the proof of first inclusion in (3.1) is completed. To prove the second inclusion in (3.1), first we note that $\mathcal{F}$ is an interval-valued convex functions, we have

$$\begin{aligned}
&\mathcal{F}\left(\vartheta\left(\varpi_1+\varpi_2-\varkappa\right)+(1-\vartheta)\left(\varpi_1+\varpi_2-\upsilon\right)\right) \\
\supseteq\ &\vartheta\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)+(1-\vartheta)\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right)
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\mathcal{F}\left(\vartheta\left(\varpi_1+\varpi_2-\upsilon\right)+(1-\vartheta)\left(\varpi_1+\varpi_2-\varkappa\right)\right) \\
\supseteq\ &(1-\vartheta)\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)+\vartheta\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right).
\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), we obtain the following from Jensen–Mercer inclusion

$$\begin{aligned}
&\mathcal{F}\left(\vartheta\left(\varpi_1+\varpi_2-\varkappa\right)+(1-\vartheta)\left(\varpi_1+\varpi_2-\upsilon\right)\right) \\
&+\mathcal{F}\left(\vartheta\left(\varpi_1+\varpi_2-\upsilon\right)+(1-\vartheta)\left(\varpi_1+\varpi_2-\varkappa\right)\right) \\
\supseteq\ &\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)+\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right) \\
\supseteq\ &2\left[\mathcal{F}\left(\varpi_1\right)+\mathcal{F}\left(\varpi_2\right)\right]\ominus_g\left[\mathcal{F}\left(\varkappa\right)+\mathcal{F}\left(\upsilon\right)\right].
\end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by $\vartheta^{\alpha-1}$ and integrating the resultant one with respect to ϑ over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1\vartheta^{\alpha-1}\mathcal{F}\left(\vartheta\left(\varpi_1+\varpi_2-\varkappa\right)+(1-\vartheta)\left(\varpi_1+\varpi_2-\upsilon\right)\right)d\vartheta \\
&+\int_0^1\vartheta^{\alpha-1}\mathcal{F}\left(\vartheta\left(\varpi_1+\varpi_2-\upsilon\right)+(1-\vartheta)\left(\varpi_1+\varpi_2-\varkappa\right)\right)d\vartheta \\
\supseteq\ &\frac{1}{\alpha}\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)+\frac{1}{\alpha}\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right) \\
\supseteq\ &\frac{2}{\alpha}\left[\mathcal{F}\left(\varpi_1\right)+\mathcal{F}\left(\varpi_2\right)\right]\ominus_g\frac{1}{\alpha}\left[\mathcal{F}\left(\varkappa\right)+\mathcal{F}\left(\upsilon\right)\right].
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\upsilon-\varkappa)^{\alpha}}\left[J^{\alpha}_{(\varpi_1+\varpi_2-\upsilon)+}\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)+J^{\alpha}_{(\varpi_1+\varpi_2-\varkappa)-}\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right)\right] \\
\supseteq\ &\frac{1}{\alpha}\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)+\frac{1}{\alpha}\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right) \\
\supseteq\ &\frac{2}{\alpha}\left[\mathcal{F}\left(\varpi_1\right)+\mathcal{F}\left(\varpi_2\right)\right]\ominus_g\frac{1}{\alpha}\left[\mathcal{F}\left(\varkappa\right)+\mathcal{F}\left(\upsilon\right)\right].
\end{aligned}$$

We obtain second and third inclusions in (3.1). □

Remark 3.2 *Under the assumption of Theorem 3.1 with* $\underline{\mathcal{F}}(\vartheta)=\overline{\mathcal{F}}(\vartheta)$*, we have*

$$\begin{aligned}
\mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(\upsilon-\varkappa)^{\alpha}}\left[J^{\alpha}_{(\varpi_1+\varpi_2-\varkappa)-}\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right)+\ J^{\alpha}_{(\varpi_1+\varpi_2-\upsilon)+}\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)\right] \\
&\leq \frac{\mathcal{F}\left(\varpi_1+\varpi_2-\varkappa\right)+\mathcal{F}\left(\varpi_1+\varpi_2-\upsilon\right)}{2}\leq\mathcal{F}\left(\varpi_1\right)+\mathcal{F}\left(\varpi_2\right)-\frac{\mathcal{F}\left(\varkappa\right)+\mathcal{F}\left(\upsilon\right)}{2},
\end{aligned}$$

which is given by Öğülmüş and Sarikaya in [28].

Remark 3.3 *Under the assumption of Theorem 3.1 with $\varkappa = \varpi_1$ and $\upsilon = \varpi_2$, Theorem 3.1 reduces to [7, Theorem 3.4].*

Theorem 3.4 *Let $\mathcal{F} : [\varpi_1, \varpi_2] \to I_c^+$ be an interval-valued convex function such that $\mathcal{F}(\vartheta) = [\underline{\mathcal{F}}(\vartheta), \overline{\mathcal{F}}(\vartheta)]$ and $\mathcal{F}(\varpi_2) \geq \mathcal{F}(\kappa_0)$ for all $\kappa_0 \in [\varpi_1, \varpi_2]$. Then, we have the following inclusions for generalized fractional integrals:*

$$\begin{aligned} & \mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) \\ \supseteq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon - \varkappa)^{\alpha}} \left[J^{\alpha}_{\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)-} \mathcal{F}(\varpi_1 + \varpi_2 - \upsilon) + J^{\alpha}_{\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)+} \mathcal{F}(\varpi_1 + \varpi_2 - \varkappa) \right] \\ \supseteq\ & \mathcal{F}(\varpi_1) + \mathcal{F}(\varpi_2) \ominus_g \frac{\mathcal{F}(\varkappa) + \mathcal{F}(\upsilon)}{2}. \end{aligned} \tag{3.7}$$

Proof Since $\mathcal{F}$ is an interval-valued convex function, so we have

$$\begin{aligned} \mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{u + v}{2}\right) &= \mathcal{F}\left(\frac{\varpi_1 + \varpi_2 - u + \varpi_1 + \varpi_2 - v}{2}\right) \\ &\supseteq \frac{1}{2}\left[\mathcal{F}(\varpi_1 + \varpi_2 - u) + \mathcal{F}(\varpi_1 + \varpi_2 - v)\right]. \end{aligned} \tag{3.8}$$

By setting $u = \frac{\vartheta}{2}\varkappa + \frac{2-\vartheta}{2}\upsilon$, $v = \frac{2-\vartheta}{2}\varkappa + \frac{\vartheta}{2}\upsilon$ for all $\varkappa, \upsilon \in [\varpi_1, \varpi_2]$ and $\vartheta \in [0, 1]$, we obtain

$$\begin{aligned} & \mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) \\ \supseteq\ & \frac{1}{2}\left[\mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{\vartheta}{2}\varkappa + \frac{2-\vartheta}{2}\upsilon\right)\right) + \mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{2-\vartheta}{2}\varkappa + \frac{\vartheta}{2}\upsilon\right)\right)\right]. \end{aligned} \tag{3.9}$$

Multiplying both sides of (3.9) by $\vartheta^{\alpha-1}$ and integrating the obtaining inclusion with respect to ϑ over $[0, 1]$, we get

$$\begin{aligned} & \frac{1}{\alpha}\mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) \\ \supseteq\ & \frac{1}{2}\left[\int_0^1 \vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{\vartheta}{2}\varkappa + \frac{2-\vartheta}{2}\upsilon\right)\right) d\vartheta \right. \\ & \left. + \int_0^1 \vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{2-\vartheta}{2}\varkappa + \frac{\vartheta}{2}\upsilon\right)\right) d\vartheta\right] \\ =\ & \frac{2^{\alpha-1}}{(\upsilon - \varkappa)^{\alpha}}\left[\int_{\varpi_1+\varpi_2-\upsilon}^{\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}} (w - (\varpi_1 + \varpi_2 - \upsilon))^{\alpha-1}\mathcal{F}(w)\, dw \right. \\ & \left. + \int_{\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}}^{\varpi_1+\varpi_2-\varkappa} ((\varpi_1 + \varpi_2 - \varkappa) - w)^{\alpha-1}\mathcal{F}(w)\, dw\right] \\ =\ & \frac{2^{\alpha-1}\Gamma(\alpha)}{(\upsilon - \varkappa)^{\alpha}}\left[J^{\alpha}_{\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)-}\mathcal{F}(\varpi_1 + \varpi_2 - \upsilon) + J^{\alpha}_{\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)+}\mathcal{F}(\varpi_1 + \varpi_2 - \varkappa)\right] \end{aligned}$$

and first inclusion in (3.7) is proved. To prove the second inclusion in (3.7), from Jensen–Mercer inclusion, we have

$$\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{\vartheta}{2}\varkappa+\frac{2-\vartheta}{2}\upsilon\right)\right) \supseteq \mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\ominus_g\left[\frac{\vartheta}{2}\mathcal{F}(\varkappa)+\frac{2-\vartheta}{2}\mathcal{F}(\upsilon)\right] \tag{3.10}$$

$$\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{2-\vartheta}{2}\varkappa+\frac{\vartheta}{2}\upsilon\right)\right) \supseteq \mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\ominus_g\left[\frac{2-\vartheta}{2}\mathcal{F}(\varkappa)+\frac{\vartheta}{2}\mathcal{F}(\upsilon)\right]. \tag{3.11}$$

By adding (3.10) and (3.11), we obtain

$$\begin{aligned}&\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{\vartheta}{2}\varkappa+\frac{2-\vartheta}{2}\upsilon\right)\right)+\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{2-\vartheta}{2}\varkappa+\frac{\vartheta}{2}\upsilon\right)\right)\\ \supseteq\ & 2\left[\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\right]\ominus_g\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon).\end{aligned} \tag{3.12}$$

Multiplying both sides of inclusion (3.12) by $\vartheta^{\alpha-1}$ and integrating the resulting inclusion with respect to ϑ over $[0,1]$, we get

$$\begin{aligned}&\int_0^1\vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{\vartheta}{2}\varkappa+\frac{2-\vartheta}{2}\upsilon\right)\right)d\vartheta+\int_0^1\vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{2-\vartheta}{2}\varkappa+\frac{\vartheta}{2}\upsilon\right)\right)d\vartheta\\ \supseteq\ & \frac{2}{\alpha}\left[\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\right]\ominus_g\frac{1}{\alpha}\left[\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon)\right].\end{aligned}$$

This proof is completed. □

Remark 3.5 *Under the assumption of Theorem 3.4 with* $\underline{\mathcal{F}}(\vartheta)=\overline{\mathcal{F}}(\vartheta)$, *we get*

$$\begin{aligned}&\mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)\\ \leq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\varkappa)^{\alpha}}\left[J^{\alpha}_{\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)-}\mathcal{F}(\varpi_1+\varpi_2-\upsilon)+J^{\alpha}_{\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)+}\mathcal{F}(\varpi_1+\varpi_2-\varkappa)\right]\\ \leq\ & \mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)-\frac{\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon)}{2},\end{aligned}$$

which is given by Öğülmüş and Sarıkaya in [28].

Remark 3.6 *Under the assumption of Theorem 3.4 with* $\varkappa=\varpi_1$ *and* $\upsilon=\varpi_2$,

$$\begin{aligned}\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right) &\supseteq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varpi_2-\varpi_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varpi_1+\varpi_2}{2}\right)-}\mathcal{F}(\varpi_1)+J^{\alpha}_{\left(\frac{\varpi_1+\varpi_2}{2}\right)+}\mathcal{F}(\varpi_2)\right]\\ &\supseteq \frac{\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)}{2},\end{aligned}$$

which is given by Ali et al. in [3].

Theorem 3.7 *Let $\mathcal{F} : [\varpi_1, \varpi_2] \to I_c^+$ be an interval-valued convex function such that $\mathcal{F}(\vartheta) = [\underline{\mathcal{F}}(\vartheta), \overline{\mathcal{F}}(\vartheta)]$ and $\mathcal{F}(\varpi_2) \geq \mathcal{F}(\kappa_0)$ for all $\kappa_0 \in [\varpi_1, \varpi_2]$. Then, we have the following inclusions for generalized fractional integrals:*

$$\begin{aligned} & \mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) \\ \supseteq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon - \varkappa)^{\alpha}} \left[J^{\alpha}_{(\varpi_1+\varpi_2-\upsilon)+}\mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) + J^{\alpha}_{(\varpi_1+\varpi_2-\varkappa)-}\mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right)\right] \\ \supseteq\ & \mathcal{F}(\varpi_1) + \mathcal{F}(\varpi_2) \ominus_g \frac{\mathcal{F}(\varkappa) + \mathcal{F}(\upsilon)}{2}. \end{aligned} \tag{3.13}$$

Proof Since $\mathcal{F}$ is interval-valued convex function, so we have

$$\begin{aligned} \mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{u + v}{2}\right) &= \mathcal{F}\left(\frac{\varpi_1 + \varpi_2 - u + \varpi_1 + \varpi_2 - v}{2}\right) \\ &\supseteq \frac{1}{2}\left[\mathcal{F}(\varpi_1 + \varpi_2 - u) + \mathcal{F}(\varpi_1 + \varpi_2 - v)\right]. \end{aligned}$$

By setting $u = \frac{1-\vartheta}{2}\varkappa + \frac{1+\vartheta}{2}\upsilon$, $v = \frac{1+\vartheta}{2}\varkappa + \frac{1-\vartheta}{2}\upsilon$ for all $\varkappa, \upsilon \in [\varpi_1, \varpi_2]$ and $\vartheta \in [0,1]$, we obtain

$$\begin{aligned} & \mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) \\ \supseteq\ & \frac{1}{2}\left[\mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{1-\vartheta}{2}\varkappa + \frac{1+\vartheta}{2}\upsilon\right)\right) + \mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{1+\vartheta}{2}\varkappa + \frac{1-\vartheta}{2}\upsilon\right)\right)\right]. \end{aligned} \tag{3.14}$$

Multiplying both sides of (3.14) by $\vartheta^{\alpha-1}$ and integrating the obtaining inclusion with respect to ϑ over $[0,1]$, we get

$$\begin{aligned} & \frac{1}{\alpha}\mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) \\ \supseteq\ & \frac{1}{2}\left[\int_0^1 \vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{1-\vartheta}{2}\varkappa + \frac{1+\vartheta}{2}\upsilon\right)\right)d\vartheta\right. \\ & \left. + \int_0^1 \vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1 + \varpi_2 - \left(\frac{1+\vartheta}{2}\varkappa + \frac{1-\vartheta}{2}\upsilon\right)\right)d\vartheta\right] \\ =\ & \frac{2^{\alpha-1}}{(\upsilon - \varkappa)^{\alpha}}\left[\int_{\varpi_1+\varpi_2-\upsilon}^{\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}} \left(\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) - w\right)^{\alpha-1}\mathcal{F}(w)\,dw\right. \\ & \left. + \int_{\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}}^{\varpi_1+\varpi_2-\varkappa} \left(w - \left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right)\right)^{\alpha-1}\mathcal{F}(w)\,dw\right] \\ =\ & \frac{2^{\alpha-1}\Gamma(\alpha)}{(\upsilon - \varkappa)^{\alpha}}\left[J^{\alpha}_{(\varpi_1+\varpi_2-\upsilon)+}\mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right) + J^{\alpha}_{(\varpi_1+\varpi_2-\varkappa)-}\mathcal{F}\left(\varpi_1 + \varpi_2 - \frac{\varkappa + \upsilon}{2}\right)\right] \end{aligned}$$

and first inclusion in (3.13) is proved. To prove the second inclusion in (3.13), from Jensen–Mercer inclusion, we have

$$\begin{aligned}&\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{1-\vartheta}{2}\varkappa+\frac{1+\vartheta}{2}\upsilon\right)\right)\\ \supseteq\quad &\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\ominus_g\left[\frac{1-\vartheta}{2}\mathcal{F}(\varkappa)+\frac{1+\vartheta}{2}\mathcal{F}(\upsilon)\right]\end{aligned}\tag{3.15}$$

$$\begin{aligned}&\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{1+\vartheta}{2}\varkappa+\frac{1-\vartheta}{2}\upsilon\right)\right)\\ \supseteq\quad &\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\ominus_g\left[\frac{1+\vartheta}{2}\mathcal{F}(\varkappa)+\frac{1-\vartheta}{2}\mathcal{F}(\upsilon)\right].\end{aligned}\tag{3.16}$$

By adding (3.15) and (3.16), we obtain

$$\begin{aligned}&\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{1-\vartheta}{2}\varkappa+\frac{1+\vartheta}{2}\upsilon\right)\right)+\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{1+\vartheta}{2}\varkappa+\frac{1-\vartheta}{2}\upsilon\right)\right)\\ \supseteq\quad &2\left[\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\right]\ominus_g\left[\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon)\right].\end{aligned}\tag{3.17}$$

Multiplying both sides of inclusion (3.17) by $\vartheta^{\alpha-1}$ and integrating the resulting inclusion with respect to ϑ over $[0,1]$, we get

$$\begin{aligned}&\int_0^1\vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{1-\vartheta}{2}\varkappa+\frac{1+\vartheta}{2}\upsilon\right)\right)d\vartheta\\ &+\int_0^1\vartheta^{\alpha-1}\mathcal{F}\left(\varpi_1+\varpi_2-\left(\frac{1-\vartheta}{2}\varkappa+\frac{1+\vartheta}{2}\upsilon\right)\right)d\vartheta\\ \supseteq\quad &\frac{2}{\alpha}\left[\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)\right]\ominus_g\frac{1}{\alpha}\left[\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon)\right].\end{aligned}$$

This proof is completed. □

Corollary 3.8 *Under the assumption of Theorem 3.7 with* $\underline{\mathcal{F}}(\vartheta)=\overline{\mathcal{F}}(\vartheta)$,

$$\begin{aligned}&\mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)\\ \le\quad &\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\varkappa)^{\alpha}}\left[J^{\alpha}_{(\varpi_1+\varpi_2-\upsilon)+}\mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)+J^{\alpha}_{(\varpi_1+\varpi_2-\varkappa)-}\mathcal{F}\left(\varpi_1+\varpi_2-\frac{\varkappa+\upsilon}{2}\right)\right]\\ \le\quad &\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)-\frac{\mathcal{F}(\varkappa)+\mathcal{F}(\upsilon)}{2}.\end{aligned}$$

Remark 3.9 *Under the assumption of Theorem 3.7 with* $\varkappa=\varpi_1$ *and* $\upsilon=\varpi_2$,

$$\begin{aligned}\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right)\quad&\supseteq\quad\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varpi_2-\varpi_1)^{\alpha}}\left[J^{\alpha}_{\varpi_1+}\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\ J^{\alpha}_{\varpi_2-}\mathcal{F}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right]\\ &\supseteq\quad\frac{\mathcal{F}(\varpi_1)+\mathcal{F}(\varpi_2)}{2},\end{aligned}$$

which is given by Ali et al. in [3].

4. Conclusion

We used Riemann–Liouville fractional integrals to prove Hermite–Hadamard–Mercer inclusion for convex interval-valued functions. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for different kinds of convexities and fractional integrals.

Acknowledgement

The researchers of our article would like to thank the editors, anonymous referees, and journal staff for their comments and suggestions.

References

[1] Abdeljawad T, Ali MA, Mohammed PO, Kashuri A. On inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional integrals. AIMS Mathematics 2021; 6 (1): 712-725. doi: 10.3934/math.2021043

[2] Ali MM, Khan AR. Generalized integral Mercer's inequality and integral means. Journal of Inequalities and Special Functions 2019; 10 (1): 60-76.

[3] Ali MA, Budak H, Sarikaya MZ, Kashuri A. On the generalized Hermite-Hadamard inequalities for interval-valued functions. Preprint 2021; https://www.researchgate.net/publication/337473865

[4] Aubin JP, Cellina A. Differential Inclusions. Berlin, Heidelberg, New York, Germany: Springer, 1984.

[5] Aubin JP, Frankowska H. Set-valued analysis. Berlin, Germany: Springer Science, 2009.

[6] Breckner WW. Continuity of generalized convex and generalized concave set-valued functions. Revue d'analyse numérique et de théorie de l'approximation 1993; 22 (1): 39-51.

[7] Budak H, Tunç T, Sarikaya MZ. Fractional Hermite-Hadamard-type inequalities for interval-valued functions. Proceedings of the American Mathematical Society 2019; 148 (2): 705-718.

[8] Chalco-Cano Y, Flores-Franulic A, Roman-Flores H. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. Computational Applied Mathematics 2021; 31 (3): 457-472.

[9] Chalco-Cano Y, Lodwick WA, Condori-Equice W. Ostrowski type inequalities and applications in numerical integration for interval-valued functions. Soft Computing 2015; 19 (11): 3293-3300. doi: 10.1007/s00500-014-1483-6

[10] Chu HH, Rashid S, Hammouch H, Chu YM. New fractional estimates for Hermite-Hadamard-Mercer's type inequalities. Alexadria Engineering Journal 2020; 59 (5): 3079-3089. doi: 10.1016/j.aej.2020.06.040

[11] Costa TM. Jensen's inequality type integral for fuzzy interval-valued functions. Fuzzy Sets and Systems 2017; 327 (2017): 31-47.

[12] Costa TM, Roman-Flores H. Some integral inequalities for fuzzy-interval-valued functions. Information Sciences 2017; 2017 (420): 110-125.

[13] Dragomir SS, Pearce CEM. Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University, Berlin, Germany: Science Direct Working Paper, 2000.

[14] Flores-Franulic A, Chalco-Cano Y, Roman-Flores H. An Ostrowski type inequality for interval-valued functions. IFSA World Congress and NAFIPS Annual Meeting IEEE 2013; 2013 (35): 1459-1462.

[15] Kara H, Ali MA, Budak H, Hermite-Hadamard-type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals. Mathematical Methods in the Applied Sciences 2021; 44 (1): 104-123.

[16] Kian M, Moslehian MS. Refinements of the operator Jensen-Mercer inequality. The Electronic Journal of Linear Algebra 2013; 2013 (26): 742-753.

[17] Liu X, Ye G, Zhao D, Liu W. Fractional Hermite–Hadamard type inequalities for interval-valued functions. Journal of Inequalities and Applications 2019; 2019 (1): 1-11.

[18] Lupulescu V. Fractional calculus for interval-valued functions. Fuzzy Sets and Systems 2015; 2015 (265): 63-85.

[19] Markov S. On the algebraic properties of convex bodies and some applications. Journal of Convex Analysis 2000; 7 (1): 129-166.

[20] Mercer AMD. A Variant of Jensen Inequality, Journal of Inequalities in Pure and Applied Mathematics 2003; 4 (4): 73.

[21] Mitrinović DS, Pečarić JE, Fink AM. Classical and new inequalities in analysis. Kluwer, Dordrecht, Holland: Academic Publishers Group, 1993.

[22] Mitroi FC, Nikodem K, Wasowicz S. Hermite–Hadamard inequalities for convex set-valued functions. Demonstratio Mathematica 2013; 46 (4): 655-662.

[23] Moore RE. Interval analysis. Prentice-Hall, Hoboken, ABD: Englewood Cliffs, 1966.

[24] Moore RE, Kearfott RB, Cloud MJ. Introduction to interval analysis. Siam, Thailand: Society for Industrial and Applied Mathematics, 2009.

[25] Niezgoda M. A generalization of Mercer's result on convex functions. Nonlinear Analysis 2009; 71 (7-8): 2771-2779.

[26] Nikodem K. On midpoint convex set-valued functions. Aequationes Mathematicae 1987; 1987 (33): 46-56.

[27] Nikodem K, Sánchez JL, Sánchez L. Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps. Mathematica Aeterna 2014; 4 (8): 979-987.

[28] Ogulmus H, Sarikaya MZ. Hermite-Hadamard-Mercer type inequalities for fractional integrals. Filomat 2021; 35 (7): 2425-2436.

[29] Osuna-Gómez R, Jiménez-Gamero MD, Chalco-Cano Y, Rojas-Medar MA. Hadamard and Jensen inequalities for s-convex fuzzy processes. In Soft Methodology and Random Information Systems 2004; 2004 (1): 645-652.

[30] Pachpatte BG. On some inequalities for convex functions. Research Group in Mathematical Inequalities and Applications 2003; 6 (1): 1-9.

[31] Pečarić JE, Proschan F, Tong YL. Convex Functions, Partial Orderings and Statistical Applications. Boston, USA: Academic Press, 1992.

[32] Roman-Flores H, Chalco-Cano Y, Lodwick WA. Some integral inequalities for interval-valued functions. Computational and Applied Mathematics 2018; 37 (2): 1306-1318.

[33] Roman-Flores H, Chalco-Cano Y, Silva GN. A note on Gronwall type inequality for interval-valued functions. IFSA World Congress and NAFIPS Annual Meeting IEEE 2013; 2013 (35): 1455-1458.

[34] Sadowska E. Hadamard inequality and a refinement of Jensen inequality for set-valued functions. Results in Mathematics 1997; 32 (3): 332-337.

[35] Sial IB, Mei S, Ali MA, Budak H. A new variant of Jensen inclusion and Hermite-Hadamard type inclusions for interval-valued functions. Preprint, 2020.

[36] Sharma N, Singh SK, Mishra SK, Hamdi A. Hermite–Hadamard-type inequalities for interval-valued preinvex functions via Riemann–Liouville fractional integrals. Journal of Inequalities and Applications 2021; 2021 (1): 1-15.

[37] Shi F, Ye G, Zhao D, Liu W. Some fractional Hermite–Hadamard-type inequalities for interval-valued co-ordinated functions. Advances in Difference Equations 2021; 2021 (1): 1-17.

[38] Zhao D, Ali MA, Murtaza G, Zhang Z. On the Hermite-Hadamard inequalities for interval-valued co-ordinated convex functions. Advances in Difference Equations 2020; 2020 (570): 14.

[39] Zhao D, An T, Ye G, Liu W. Chebyshev type inequalities for interval-valued functions. Fuzzy Sets and Systems 2020; 2020 (396): 82-101.

[40] Zhao D, An T, Ye G, Liu W. New Jensen and Hermite-Hadamard type inequalities for $h-$ convex interval-valued functions. Journal of Inequalities and Applications 2018; 2018 (1): 1-14.

[41] Zhou H, Saleem MS, Nazeer W, Shah AF. Hermite-Hadamard type inequalities for interval-valued exponential type pre-invex functions via Riemann-Liouville fractional integrals. AIMS Mathematics 2021; 7 (2): 2602-2617.

[42] Zhao D, Ye G, Liu W, Torres DF. Some inequalities for interval-valued functions on time scales. Soft Computing 2019; 23 (15): 6005-6015.