Turkish Journal of Mathematics

Volume 46 | Number 6 Article 10

1-1-2022

Inverse nodal problem for Sturm-Liouville operator on a star graph with nonequal edges

SEVİM DURAK

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2178 – 2192
© TÜBİTAK
doi:10.55730/1300-0098.3262

Inverse nodal problem for Sturm–Liouville operator on a star graph with nonequal edges

Sevim DURAK*
Department of Mathematics, Faculty of Sciences, Cumhuriyet University, Sivas, Turkey

Received: 28.02.2022 • **Accepted/Published Online:** 21.04.2022 • **Final Version:** 04.07.2022

Abstract: In this study, Sturm–Liouville operator was investigated on a star graph with nonequal edges. First, the behaviors of sufficiently large eigenvalues were learned, then the solution of the inverse problem was given to determine the potential functions and parameters of the boundary condition on the star graph with the help of a dense set of nodal points and obtain a constructive solution to the inverse problems of this class.

Key words: Sturm–Liouville operator, nodal problem, star graph

1. Introduction

In this work we consider a star graph G with vertex set $V = \{\upsilon_0, \upsilon_1, ..., \upsilon_\nu\}$ and edge set $E = \{e_1, e_2, ..., e_\nu\}$, where $\upsilon_1, \upsilon_2, ..., \upsilon_\nu$ are the boundary vertices, υ_0 is the interior vertex and $e_j = [\upsilon_j, \upsilon_0]$ for $j = \overline{1, \nu}$, positive integer $\nu \geq 2$. We suppose that the length of edge $e_j \in E$ is equal to ℓ_j, where $\ell_j = \ell > 0,\ j = \overline{1, \nu - p}$, and $\ell_j = 2\ell, j = \overline{\nu - p + 1, \nu}$. We introduce a parameter x for each edge $e_j \in E$, $x \in [0, \ell_j]$. The following choice of orientation is convenient for us: $x = 0$ corresponds to the boundary vertices $\upsilon_1, \upsilon_2, ..., \upsilon_\nu$ and $x = \ell_j$ to the interior vertex υ_0.

Let $y = \{y_j(x)\}_{j=\overline{1,\nu}}$ be a vector function on G. Consider a second order differential expression

$$(L_j y_j)(x) := -y_j^{''}(x) + q_j(x) y_j(x), \quad x \in [0, \ell_j], j = \overline{1, \nu},$$

$q_j(x), j = \overline{1, \nu}$, are real valued functions from $L_2[0, \ell_j]$. The domain of expressions L_j

$$D(L_j) := \left\{ y_j \in W_2^1[0, \ell_j] : y_j^{'} \in AC[0, \ell_j], L_j y_j \in L_2[0, \ell_j] \right\}, j = \overline{1, \nu}.$$

We study the boundary value problem $L = L(q, p, h)$ for the Sturm–Liouville equations on a star graph G:

$$(L_j y_j)(x) = \lambda y_j(x), \quad x \in [0, \ell_j], j = \overline{1, \nu}, \lambda = s^2 \tag{1.1}$$

with the matching conditions

$$y_1(\ell_1) = y_j(\ell_j), j = \overline{2, \nu}, \sum_{j=1}^{\upsilon} y_j^{'}(\ell_j) = 0 \tag{1.2}$$

*Correspondence: sdurak@cumhuriyet.edu.tr
2000 *AMS Mathematics Subject Classification:* 34A55, 34B24, 34L05

in the interior vertex υ_0, and the boundary conditions

$$y_j^{'}(0) - h_j y_j(0) = 0, j = \overline{1,\nu} \tag{1.3}$$

in the boundary vertices $\upsilon_1, ..., \upsilon_\nu$, where h_j are real, $h = \{h_j\}_{j=\overline{1,\nu}}$, and $q = \{q_j(x)\}_{j=\overline{1,\nu}}$ are called potential functions on a star graph G.

The solution of many problems in different fields of mathematics, mechanics, physics, geophysics, electronics, nanotechnology, natural sciences and engineering is reduced to the problems of studying differential operators on graphs (trees) (see [2, 8, 14–17, 22–24, 27] references there in). Spectral inverse problems for differential operators given on graphs began to be studied more rapidly in the twenty-first century.

In 1988, by bringing a different perspective to this problem, it was reduced to the solution of inverse nodal problems for Sturm–Liouville operators by Hald and McLaughlin [9–12, 21]. In recent years, many studies have been conducted on nodal inverse problems, which is one of the important topics of inverse problems theory, and some of these studies are studies on graphs. The articles [2, 4, 5, 12, 18, 19, 25, 26, 28, 34] can be shown as the basis for such studies.

Inverse nodal problems on a graphs for the Sturm–Liouville operators and Dirac operators and for the diffusion operators are studied in sufficient detail in [5, 29]. The works [6, 33, 34] deal with inverse nodal problems for the Sturm–Liouville operators on star-shaped graphs with equal edges. The work [30] is related to inverse nodal problems on a equal edge graph with loops. The work [29] is conserned with inverse nodal problems for diffusion operators on a star-shaped graph with equal edges. In [13] authors consider inverse nodal problems for Dirac-type integro-differential operators on a star graph with equal edges. If the lengths of edges of graphs are nonequal, the problems will become more difficult (see [3, 20]). In particular, the asymptotic expressions of eigenvalues are difficult to find. For inverse nodal problems on graphs in [7] proved that the specification of the spectrum and the set of all nodal points uniquely determine the potential.

In the present paper unlike other publications we consider inverse nodal problems on a special kind of star graph G with nonequal edges and want to reconstruct the potential only by the dense subset of nodal points.

This paper is organized as follows: in Section 2, we find asymptotical formulas of eigenvalues of the problem L, Section 3 is concerned with the asymptotical formulas of the nodal points and an inverse nodal problem.

2. Properties of the spectrum

Let $C_j(x,s), S_j(x,s), j = \overline{1,\nu}$, be solutions of equations (1.1) under the initial conditions $C_j(0,s) = S_j^{'}(0,s) = 1, C_j^{'}(0,s) = S_j(0,s) = 0$, and denote by $\varphi_j(x,s), j = \overline{1,\nu}$, the solutions of equations (1.1) satisfying initial conditions $\varphi_j(0,s) = 1, \varphi_j^{'}(0,s) = h_j$. For each fixed $x \in [0, \ell_j]$, the functions

$$C_j(x,s), S_j(x,s), C_j^{'}(x,s), S_j^{'}(x,s), \varphi_j(x,s)$$

and $\varphi_j^{'}(x,s), j = \overline{1,\nu}$, are entire in s and

$$\varphi_j(x,s) = C_j(x,s) + h_j S_j(x,s).$$

From [31] one gets the following asymptotical formulas as $|s| \to \infty$,

$$\varphi_j(x,s) = \cos sx + A_j(x)\frac{\sin sx}{s} + O\left(\frac{e^{|\tau|x}}{s}\right), \tag{2.1}$$

$$\varphi_j^{'}(x,s) = -s\sin sx + A_j(x)\cos sx + O\left(e^{|\tau|x}\right), \tag{2.2}$$

where $\tau = \operatorname{Im} s,$ and

$$A_j(x) = h_j + \frac{1}{2}\int_0^x q_j(t)\,dt \tag{2.3}$$

for $j = \overline{1,\nu}$.

Then the solutions of equations (1.1) satisfying the boundary conditions (1.3) are

$$y_j(x,s) = H_j(s)\,\varphi_j(x,s), \tag{2.4}$$

where $H_j(s)$ are functions independent of x. Substituting (2.4) into the martching conditions (1.2), we obtain the characteristic function of the problem L.

$$\begin{aligned}\Delta(s) &= \sum_{j=1}^{\nu}\varphi_j^{'}(\ell_j,s)\prod_{j\neq k\in\{1,\ldots,\nu\}}\varphi_k(\ell_j,s) = \\ &= \left\{\sum_{j=1}^{\nu-p}\varphi_j^{'}(\ell,s)\times\prod_{j\neq k\in\{1,\ldots,\nu-p\}}\varphi_k(\ell,s)\right\}\times\prod_{k=\nu-p+1}^{\nu}\varphi_k(2\ell,s) + \\ &+ \left\{\sum_{j=\nu-p+1}^{\upsilon}\varphi_j^{'}(2\ell,s)\times\prod_{k=1}^{p}\varphi_k(\ell,s)\right\}\times\prod_{j\neq k\in\{p+1,\ldots\nu\}}\varphi_k(2\ell,s).\end{aligned} \tag{2.5}$$

As defined in [33] if λ_0 is a zero of $\Delta(\lambda)$ $\left(\lambda = s^2\right)$ then the function $y(x,\lambda_0) = y\left(x,s_0^2\right)$ of the form $y(x,\lambda) = \{y_i(x,\lambda)\}_{i=\overline{1,\nu}}$, $y_i(x,\lambda) = A_i(\lambda)\,\varphi_i(x,\lambda)$ is an eigenfunction and $\lambda = s_0^2$ is an eigenvalue of problem (1.1)–(1.3). Conversely, if λ_0 is an eigenvalue then the corresponding eigenfunction is of the form $y(x,\lambda) = \{y_i(x,\lambda)\}_{i=\overline{1,\nu}}$, $y_i(x,\lambda) = A_i(\lambda)\,\varphi_i(x,\lambda)$ with $\lambda = \lambda_0$. Since $y(x,\lambda_0) \neq 0$, the algebraic system has a nontrivial solution: consequantly, $\Delta(\lambda_0) = 0$.

Let us introduce the auxiliary function

$$\begin{aligned}\Delta_0(s) &= \sum_{j=1}^{\nu}(-s\sin s\ell_j)\prod_{j\neq k\in\{1,\ldots,\nu\}}\cos s\ell_k = \\ &= (-s\sin s\ell)\,(\cos s\ell)^{\nu-p-1}(\cos 2s\ell)^{p-1}\left\{2\upsilon\cos^2 s\ell - (\nu-p)\right\}.\end{aligned} \tag{2.6}$$

Using (2.6) and Rouche method (see [31]), we deduce that the function of $\Delta_0(s)$ has countably many eigenvalues

$$\left\{t_n^{\theta}\right\}_{n\in\mathbb{Z}} = \left(\left\{s_n^{(01)}\right\}_{n=-\infty}^{n=-0} \bigcup \left\{s_n^{(01)}\right\}_{n=0}^{+\infty}\right) \left(\bigcup_{j=2}^{\nu-p}\left\{s_n^{(0j)}\right\}_{n\in\mathbb{Z}}\right) \left(\bigcup_{j=\nu-p+1}^{\nu+1}\left\{s_n^{(0j)}\right\}_{n\in\mathbb{Z}}\right),$$

we arrange the zeros in the following way:

$$\begin{aligned}
&s_{-0}^{(01)} = s_0^{(01)} = 0, s_n^{(01)} = \frac{n\pi}{\ell}, n \in \mathbb{Z}\setminus\{0\};\\
&s_n^{(02)} = \frac{n\pi+\theta}{\ell}, s_n^{(03)} = \frac{n\pi-\theta}{\ell}, n\in\mathbb{Z};\\
&s_n^{(0j)} = \frac{\left(n+\frac{1}{2}\right)\pi}{\ell}, j=\overline{1,\nu-p}, n\in\mathbb{Z};\\
&s_n^{(0j)} = \frac{\left(n+\frac{1}{2}\right)\pi}{2\ell}, j=\overline{\nu-p+1,\nu+1}, n\in\mathbb{Z},
\end{aligned}$$

where $\theta = \arccos\sqrt{\dfrac{\nu-p}{2\nu}}$. The set of zeros of $\Delta(s)$ denote by

$$\{t_n\}_{n\in\mathbb{Z}} = \left(\left\{s_n^{(1)}\right\}_{n=-\infty}^{-0} \bigcup \left\{s_n^{(1)}\right\}_{n=0}^{+\infty}\right) \left(\bigcup_{j=2}^{\nu-p}\left\{s_n^{(j)}\right\}_{n\in\mathbb{Z}}\right) \left(\bigcup_{j=\nu-p+1}^{\nu+1}\left\{s_n^{(j)}\right\}_{n\in\mathbb{Z}}\right),$$

we arrange the zeros in the following usual way: (1) $s_{-n}^{(j)} = -s_n^{(j)}$ for all not pure real $s_n^{(j)}$, (2) $\operatorname{Re} s_{n+1}^{(j)} \geq \operatorname{Re} s_n^{(j)}$. Using Lemma 2.1 in [29] we can obtain the following proposition.

Lemma 2.1 *The zeros of $\Delta(s)$ can be enumerated as follows:*

$$t_n = t_n^{(0)} + o(1). \tag{2.7}$$

Moreover, using Lemma 2.1 we can obtain the following result.

Lemma 2.2 *The following asymptotic behavior for the set $\{t_n\}$ of zeros of the function $\Delta(s)$ is true:*

(a)

$$s_n^{(1)} = \frac{n\pi}{\ell} + \frac{\sum\limits_{j=1}^{\nu-p}\eta_j + \sum\limits_{j=\nu-p+1}^{\nu}\eta_j}{(\nu+p)\pi n} + O\left(\frac{1}{n^2}\right), \tag{2.8}$$

(b)

$$s_n^{(2)} = \frac{n\pi+\theta}{\ell} + \frac{1}{n\pi+\theta}\left\{\frac{p}{\nu^2-p^2}\left(\sum_{j=1}^{\nu-p}\eta_j\right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2-p(p-1)}{2\nu(\nu+p)}\right]\left(\sum_{j=\nu-p+1}^{\nu}\eta_j\right)\right\} + O\left(\frac{1}{n^{\overline{2}}}\right), \tag{2.9}$$

(c)

$$s_n^{(2)} = \frac{n\pi-\theta}{\ell} + \frac{1}{n\pi-\theta}\left\{\frac{p}{\nu^2-p^2}\left(\sum_{j=1}^{\nu-p}\eta_j\right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2-p(p-1)}{2\nu(\nu+p)}\right]\left(\sum_{j=\nu-p+1}^{\nu}\eta_j\right)\right\} + O\left(\frac{1}{n^2}\right), \tag{2.10}$$

(d)

$$s_n^{(j)} = \frac{(n+\frac{1}{2})\pi}{\ell} + \frac{\tau_{j-3}^{(1)}}{(n+\frac{1}{2})\pi} + O\left(\frac{1}{n^2}\right), j=\overline{4,\nu-p+1}, \tag{2.11}$$

(e)

$$s_n^{(j)} = \frac{(n+\frac{1}{2})\pi}{2\ell} + \frac{2\tau_j^{(2)}}{(n+\frac{1}{2})\pi} + O\left(\frac{1}{n^2}\right), j=\overline{\nu-p+2,\nu+1}, \tag{2.12}$$

where $\tau_{j-3}^{(1)}$ and $\tau_j^{(2)}$ are the solutions of the equations

$$P_1(x) := \sum_{i=1}^{\nu-p}\prod_{i\neq k\in\{1,\dots,\nu-p\}}(x-\eta_k), P_2(x) := \sum_{i=\nu-p+1}^{\nu}\prod_{i\neq k\in\{p+1,\dots,\nu\}}(x+\eta_k)$$

respectively, and $\eta_j = h_j + \frac{1}{2}\int\limits_0^{\ell_j} q_j(t)dt$, i.e. $\eta_j = A_j(\ell_j)$ for $j=\overline{1,\nu}$.

Proof (a) Using Rouche's theorem, for sufficiently large integer $|n|$, $\Delta(s)$ has exactly one zero in a suitable neighborhood of $s_n^{(0)} = \dfrac{n\pi}{\ell}$, and denote

$$s_n^{(1)} = \frac{n\pi}{\ell} + \beta_n^{(1)}, \tag{2.13}$$

where $\beta_n^{(1)} = o(1)$ as $|n|\to+\infty$. Substituting (2.13) into $\Delta(s)=0$, then from (2.5), we have

$$\begin{aligned}\Delta\left(s_n^{(1)}\right) = &\sum_{j=1}^{\nu-p}\left[-\frac{n\pi}{\ell}\sin\beta_n^{(1)}\ell + \eta_j\cos\beta_n^{(1)}\ell + o(1)\right]\times\prod_{j\neq k\in\{1,\dots,\nu-p\}}\left[\cos\beta_n^{(1)}\ell + o(\frac{1}{n})\right]\times\\ &\times\prod_{k=\nu-p+1}^{\nu}\left[\cos 2\beta_n^{(1)}\ell + o(\frac{1}{n})\right] + \sum_{j=\upsilon-p+1}^{\upsilon}\left[-\frac{n\pi}{\ell}\sin 2\beta_n^{(1)}\ell + \eta_j\cos 2\beta_n^{(1)}\ell + o(1)\right]\times\\ &\times\prod_{k=1}^{p}\left[\cos\beta_n^{(1)}\ell + o(\frac{1}{n})\right]\times\prod_{j\neq k\in\{p+1,\dots,\nu\}}\left[\cos 2\beta_n^{(1)}\ell + o(\frac{1}{n})\right] = 0.\end{aligned} \tag{2.14}$$

From (2.14), we have $\sin\beta_n^{(1)}\ell = O\left(\dfrac{1}{n}\right)$. Using the inversion formula, we get

$$\beta_n^{(1)} = \frac{C^{(1)}}{n\ell} + O\left(\frac{1}{n^2}\right), \tag{2.15}$$

where $C^{(1)}$ is a constant. Substituting (2.15) into (2.14) and let $|n| \to +\infty$, we obtain

$$C^{(1)} = \frac{\ell \left\{ \sum\limits_{j=1}^{\nu-p} \eta_j + \sum\limits_{j=\nu-p+1}^{\nu} \eta_j \right\}}{(\nu+p)\pi}. \tag{2.16}$$

In this case, if we make use of (2.13), (2.15) and (2.16) we get (2.8).

(b) As in case (a) using Rouche's theorem, for sufficiently large integer $|n|$, $\Delta(s)$ has only one zero in a suitable neighborhood of $s_n^{(02)} = \dfrac{n\pi+\theta}{\ell}$, and denote

$$s_n^{(2)} = \frac{n\pi+\theta}{\ell} + \beta_n^{(2)}, \tag{2.17}$$

where $\beta_n^{(2)} = o(1)$ as $|n| \to +\infty$. Substituting (2.17) into $\Delta(s) = 0$, then from (2.5), we have

$$\begin{aligned} \Delta\left(s_n^{(2)}\right) = & \sum_{j=1}^{\nu-p} \left[\frac{n\pi+\theta}{\ell} A_1^{(n)} - \eta_j A_2^{(n)} + o(1) \right] \times \prod_{j\neq k \in \{1,\ldots,\nu-p\}} \left[A_2^{(n)} + \frac{\ell\eta_k}{n\pi+\theta} A_1^{(n)} + o(\frac{1}{n}) \right] \times \\ & \times \prod_{k=\nu-p+1}^{\nu} \left[A_3^{(n)} - \frac{\ell\eta_k}{n\pi+\theta} A_4^{(n)} + o(\frac{1}{n}) \right] + \sum_{j=\nu-p+1}^{\nu} \left[-\frac{n\pi+\theta}{\ell} A_4^{(n)} + \eta_j A_3^{(n)} + o(1) \right] \times \\ & \times \prod_{k=1}^{p} \left[A_2^{(n)} + \frac{\ell\eta_k}{n\pi+\theta} A_1^{(n)} + o(\frac{1}{n}) \right] \times \prod_{j\neq k \in \{p+1,\ldots,\nu\}} \left[A_3^{(n)} - \frac{2\ell\eta_k}{n\pi+\theta} A_4^{(n)} + o(\frac{1}{n}) \right] \\ & = 0, \end{aligned} \tag{2.18}$$

where

$$A_1^{(n)} = \sqrt{\frac{\nu+p}{2\nu}} \cos\beta_n^{(2)}\ell + \sqrt{\frac{\nu-p}{2\nu}} \sin\beta_n^{(2)}\ell,$$

$$A_2^{(n)} = \sqrt{\frac{\nu-p}{2\nu}} \cos\beta_n^{(2)}\ell - \sqrt{\frac{\nu+p}{2\nu}} \sin\beta_n^{(2)}\ell,$$

$$A_3^{(n)} = \frac{p}{\nu} \cos 2\beta_n^{(2)}\ell + \frac{\sqrt{\nu^2-p^2}}{\nu} \sin 2\beta_n^{(2)}\ell,$$

$$A_4^{(n)} = \frac{\sqrt{\nu^2-p^2}}{\nu} \cos 2\beta_n^{(1)}\ell - \frac{p}{\nu} \sin 2\beta_n^{(2)}\ell,$$

or

$$\frac{n\pi+\theta}{\ell}A_2^{(n)}A_3^{(n)}\left[(\nu-p)A_1^{(n)}A_3^{(n)}-pA_2^{(n)}A_4^{(n)}\right]+ \tag{2.19}$$
$$+\left(\sum_{j=1}^{\nu-p}\eta_j\right)A_3^{(n)}\left[(\nu-p-1)(A_1^{(n)})^2A_3^{(n)}-(A_2^{(n)})^2A_3^{(n)}-pA_1^{(n)}A_2^{(n)}A_4^{(n)}\right]-$$
$$-\left(\sum_{j=\nu-p+1}^{\nu}\eta_j\right)A_2^{(n)}\left[(\nu-p)A_1^{(n)}A_3^{(n)}A_4^{(n)}+(p-1)A_2^{(n)}(A_4^{(n)})^2+A_2^{(n)}(A_3^{(n)})^2\right]+o(1)$$
$$=0.$$

If we make use of the identity

$$(\nu-p)A_1^{(n)}A_3^{(n)}-pA_2^{(n)}A_4^{(n)}=\sin\beta_n^{(2)}\ell\left[2\nu\sqrt{\frac{\nu-p}{2\nu}}\cos^2\beta_n^{(2)}\ell+\right.$$
$$\left.+2p\sqrt{\frac{\nu-p}{2\nu}}\cos 2\beta_n^{(2)}\ell+(\nu-2p)\sqrt{\frac{\nu+p}{2\nu}}\sin 2\beta_n^{(2)}\ell\right]$$

Equation (2.19) implies

$$\frac{n\pi+\theta}{\ell}\left(\sin\beta_n^{(2)}\ell\right)A_2^{(n)}A_3^{(n)}A_5^{(n)}+ \tag{2.20}$$
$$+\left(\sum_{j=1}^{\nu-p}\eta_j\right)A_3^{(n)}\left[(\nu-p-1)(A_1^{(n)})^2A_3^{(n)}-(A_2^{(n)})^2A_3^{(n)}-pA_1^{(n)}A_2^{(n)}A_4^{(n)}\right]-$$
$$-\left(\sum_{j=\nu-p+1}^{\nu}\eta_j\right)A_2^{(n)}\left[(\nu-p)A_1^{(n)}A_3^{(n)}A_4^{(n)}+(p-1)A_2^{(n)}(A_4^{(n)})^2+A_2^{(n)}(A_3^{(n)})^2\right]+o(1)$$
$$=0$$

where

$$A_5^{(n)}=\left[2\nu\sqrt{\frac{\nu-p}{2\nu}}\cos^2\beta_n^{(2)}\ell+2p\sqrt{\frac{\nu-p}{2\nu}}\cos 2\beta_n^{(2)}\ell+(\nu-2p)\sqrt{\frac{\nu+p}{2\nu}}\sin 2\beta_n^{(2)}\ell\right].$$

Using the limits of expressions $A_i^{(n)}, i=\overline{1,5},$ when $|n|\to+\infty,$ we have

$$\sin\beta_n^{(2)}\ell=O\left(\frac{1}{n\pi+\theta}\right)$$

from (2.20). Using the inversion formula, we get

$$\beta_n^{(2)}=\frac{C^{(2)}}{(n\pi+\theta)\ell}+O\left(\frac{1}{n^2}\right), \tag{2.21}$$

where $C^{(2)}$ is a constant. Substituting (2.21) into (2.20) and let $|n| \to +\infty$, we obtain

$$C^{(2)} = \frac{p\ell}{\nu^2 - p^2} \left(\sum_{j=1}^{\nu-p} \eta_j \right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2 - p(p-1)}{2\nu(\nu+p)} \right] \ell \left(\sum_{j=\nu-p+1}^{\nu} \eta_j \right). \tag{2.22}$$

In this case if we make use of (2.17), (2.21) and (2.22) we get (2.9). Similarly, in the case (c) the formula (2.10) can be proved.

(d) Using Rouche's theorem, for sufficiently large integer $|n|$, there lie exactly $\nu - p - 1$ zeros of $\Delta(s)$ in a suitable neighborhood of

$$s_n^{(0j)} = \frac{(n+\frac{1}{2})\pi}{\ell},$$

and denote

$$s_n^{(j)} = \frac{(n+\frac{1}{2})\pi}{\ell} + \beta_n^{(j)}, \quad j = \overline{4, \nu-p+1}, \tag{2.23}$$

where $\beta_n^{(j)} = o(1)$ as $|n| \to +\infty$. Substituting (2.23) into $\Delta(s) = 0$, then from (2.5), we have

$$\begin{aligned} \Delta\left(s_n^{(j)}\right) = & \sum_{i=1}^{\nu-p} \left[\left(\frac{(n+\frac{1}{2})\pi}{\ell} + \beta_n^{(j)} \right) \cos\beta_n^{(j)}\ell + \eta_i \sin\beta_n^{(j)}\ell + o(1) \right] \times \\ & \times \prod_{i\neq k\in\{1,\dots,\nu-p\}} \left[\sin\beta_n^{(j)}\ell - \eta_k \frac{\cos\beta_n^{(j)}\ell}{\frac{(n+\frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o(\frac{1}{n}) \right] \times \\ & \times \prod_{k=\nu-p+1}^{\nu} \left[-\cos 2\beta_n^{(j)}\ell - \eta_k \frac{\sin 2\beta_n^{(j)}\ell}{\frac{(n+\frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o(\frac{1}{n}) \right] + \\ & + \sum_{i=\nu-p+1}^{\nu} \left[\left(\frac{(n+\frac{1}{2})\pi}{\ell} + \beta_n^{(j)} \right) \sin 2\beta_n^{(j)}\ell - \eta_i \cos 2\beta_n^{(j)}\ell + o(1) \right] \times \\ & \times \prod_{k=1}^{p} \left[-\sin\beta_n^{(j)}\ell - \eta_k \frac{\cos\beta_n^{(j)}\ell}{\frac{(n+\frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o(\frac{1}{n}) \right] \times \\ & \times \prod_{i\neq k\in\{p+1,\dots,\nu\}} \left[-\cos 2\beta_n^{(j)}\ell - \eta_k \frac{\sin 2\beta_n^{(j)}\ell}{\frac{(n+\frac{1}{2})\pi}{\ell} + \beta_n^{(j)}} + o(\frac{1}{n}) \right] \\ & = 0. \end{aligned} \tag{2.24}$$

Acting as in case (a) from (2.24) we find

$$\sin\beta_n^{(j)}\ell = O\left(\frac{1}{(n+\frac{1}{2})\pi} \right)$$

as $|n| \to +\infty$. Using the inversion formula, we get

$$\beta_n^{(j)} = \frac{C^{(j)}}{(n+\frac{1}{2})\pi\ell} + O\left(\frac{1}{n^2}\right), \tag{2.25}$$

where $C^{(j)}, j = \overline{4, \nu-p+1}$ are all constants. Next, substituting (2.25) into (2.24) we have

$$\sum_{i=1}^{\nu-p}\left[1+o\left(\frac{1}{n}\right)\right] \times \prod_{i\neq k\in\{1,\ldots,\nu-p\}}\left[\frac{C^{(j)}}{(n+\frac{1}{2})\pi} - \frac{\ell\eta_k}{(n+\frac{1}{2})\pi} + o(\frac{1}{n})\right] \times \prod_{k=\nu-p+1}^{\nu}\left[1+o\left(\frac{1}{n}\right)\right] +$$
$$+\sum_{i=\nu-p+1}^{\nu}\left[\frac{2C^{(j)}}{(n+\frac{1}{2})\pi} - \frac{\ell\eta_i}{(n+\frac{1}{2})\pi} + o\left(\frac{1}{n}\right)\right] \times \prod_{k=1}^{p}\left[\frac{C^{(j)}}{(n+\frac{1}{2})\pi} - \frac{\ell\eta_k}{(n+\frac{1}{2})\pi} + o(\frac{1}{n})\right] \times$$
$$\times \prod_{i\neq k\in\{p+1,\ldots,\nu\}}\left[1+o\left(\frac{1}{n}\right)\right] = 0.$$

Letting $|n| \to +\infty$ yields

$$\sum_{i=1}^{\nu-p}\prod_{i\neq k\in\{1,\ldots,\nu-p\}}\left(C^{(j)} - \ell\eta_k\right) = 0, \quad j = \overline{4, \nu-p+1}. \tag{2.26}$$

Denote

$$\tau_{j-3} = \frac{C^{(j)}}{\ell}, \quad j = \overline{4, \nu-p+1}, \tag{2.27}$$

$$P_1(x) = \sum_{i=1}^{\nu-p}\prod_{i\neq k\in\{1,\ldots,\nu-p\}}(x-\eta_k)\,dx. \tag{2.28}$$

In view of (2.23), (2.25), (2.26) and (2.28) we arrive at (2.11). Similarly, in the case (e) the formula (2.12) can be proved. □

3. Inverse nodal problem

Let

$$y(x,s) = \{y_j(x,s)\}_{j=\overline{1,\nu}}, y_j(x,s) = D_j\varphi_j(x,s), \tag{3.1}$$

where $D_j, j = \overline{1,\nu}$ are constants depending on s, then the function $y(x,s)$ satisfies (1.1) and (1.3). In this we denote $s_n = s_n^{(2)}$. It follows from (2.9) that

$$s_n = \frac{n\pi+\theta}{\ell} + \frac{M(\nu,p)}{n\pi+\theta} + O\left(\frac{1}{n^2}\right), \tag{3.2}$$

where

$$M(\nu,p) = \frac{p}{\nu^2-p^2}\left(\sum_{j=1}^{\nu-p}\eta_j\right) + \left[\frac{(p-1)(\nu-p)}{2p\nu} + \frac{\nu^2-p(p-1)}{2\nu(\nu+p)}\right]\left(\sum_{j=\nu-p+1}^{\nu}\eta_j\right). \tag{3.3}$$

Using asymptotics formulas (2.1) and (3.2), we obtain as $|n| \to +\infty$, uniformly in $x \in [0, \ell_j]$,

$$\varphi_j(x, s_n) = \cos s_n x + A_j(x) \frac{\sin s_n x}{s_n} + o\left(\frac{1}{n}\right), j = \overline{1, \nu}, \tag{3.4}$$

where

$$A_j(x) = 2h_j - M(\nu, p)x + \int_0^x q_j(t)\, dt. \tag{3.5}$$

Fix $j = \overline{1, \nu}$ and assume that x_{nj}^k are the zeros (i.e. nodal points) of the functions $\varphi_j(x, s_n)$. Taking (3.4) into account, from equation $\varphi_j(x, s_n) = 0$, we obtain the following asymptotic formula, uniform in j, for the zeros i.e. nodal points as $|n| \to +\infty$

$$x_{nj}^i = \frac{\left(i - \frac{1}{2}\right)\pi}{s_n} + \frac{A_j\left(x_{nj}^i\right)}{s_n^2} + o\left(\frac{1}{n^2}\right), i \in \mathbb{Z}. \tag{3.6}$$

From (3.2) we have

$$\frac{1}{s_n} = \frac{\ell}{n\pi + \theta} - \frac{\ell^2 M(\nu, p)}{(n\pi + \theta)^3} + o\left(\frac{1}{n^3}\right). \tag{3.7}$$

Putting (3.7) into (3.6), we obtain the following asymptotical formulas for nodes (i.e. nodal points) as $|n| \to +\infty$ uniformly in $i \in \mathbb{Z}$:

$$x_{nj}^i = \frac{\left(i - \frac{1}{2}\right)\pi\ell}{n\pi + \theta} + \frac{\ell^2 A_j\left(x_{nj}^i\right)}{(n\pi + \theta)^2} - \frac{\left(i - \frac{1}{2}\right)\pi\ell^2 M(\nu, p)}{(n\pi + \theta)^3} + o\left(\frac{1}{n^3}\right). \tag{3.8}$$

Fix $j = \overline{1, \nu - p}$. It is easy to see from (3.8) that there exists $N_1 \in \mathbb{N}$ such that for all $|n| > N_1$ the function $\varphi_j(x, s_n)$ has exactly $|n|$ (simple) zeros in the interval $(0, \ell)$, namely,

$$0 < x_{nj}^1 < x_{nj}^2 < ... < x_{nj}^n < \ell \text{ for } n > 0,$$

$$0 < x_{nj}^0 < x_{nj}^{-1} < ... < x_{nj}^{n+1} < \ell \text{ for } n < 0.$$

Analogously, for the fix $j = \overline{\nu - p + 1, \nu}$ there exists $N_2 \in \mathbb{N}$ such that for all $|n| > N_2$ the function $\varphi_j(x, s_n)$ has exactly $2|n|$ (simple) nodes in the interval $(0, 2\ell)$, i.e.

$$0 < x_{nj}^1 < x_{nj}^2 < ... < x_{nj}^{2n} < 2\ell \text{ for } n > 0,$$

$$0 < x_{nj}^0 < x_{nj}^{-1} < ... < x_{nj}^{2n+1} < 2\ell \text{ for } n < 0.$$

The sets

$$X_j = \left\{x_{nj}^i : n > N_1, i = \overline{1, n}\right\} \bigcup \left\{x_{nj}^i : n < -N_1, i = \overline{0, n+1}\right\}, j = \overline{1, \nu - p}$$

and

$$X_j = \left\{x_{nj}^i : n > N_2, i = \overline{1,2n}\right\} \bigcup \left\{x_{nj}^i : n < -N_2, i = \overline{0,2n+1}\right\}, j = \overline{\nu - p + 1, \upsilon}$$

are called nodes sets on the $e_j \left(j = \overline{1, \nu - p}\right)$ and $e_j \left(j = \overline{\nu - p + 1, \nu}\right)$, respectively. Clearly, the set X_j is dense in $(0, \ell_j)$ for $j = \overline{1, \nu}$.

Theorem 3.1 *Fix $j = \overline{1, \nu}$ and $x \in [0, \ell_j]$ suppose that $\left\{x_{nj}^i\right\} \subset X_j$ are chosen such that $\lim\limits_{|n| \to +\infty} x_{nj}^i = x$. Then there exists a finite limit*

$$g_j(x) := \lim_{|n| \to +\infty} \left(x_{nj}^i - \frac{\left(i - \frac{1}{2}\right) \pi \ell}{n\pi + \theta} \right) \frac{(n\pi + \theta)^2}{\ell^2}, \tag{3.9}$$

and

$$g_j(x) = A_j(x) - \pi M(\nu, p) x, \tag{3.10}$$

where $A_j(x)$ and $M(\nu, p)$ are defined by (2.3) and (3.3), respectively.

Proof If we use the asymptotical formula (3.8), we get that

$$\left(x_{nj}^i - \frac{\left(i - \frac{1}{2}\right) \pi \ell}{n\pi + \theta} \right) \frac{(n\pi + \theta)^2}{\ell^2} = A_j\left(x_{nj}^i\right) - \frac{\left(i - \frac{1}{2}\right) \pi M(\nu, p)}{n\pi + \theta} + o(1). \tag{3.11}$$

Since,

$$\lim_{|n| \to +\infty} x_{nj}^i = x, \lim_{|n| \to +\infty} \frac{\left(i - \frac{1}{2}\right) \pi \ell}{n\pi + \theta} = x$$

and

$$\lim_{|n| \to +\infty} A_j\left(x_{nj}^i\right) = A_j(x),$$

from this and (3.11), we conclude that as $|n| \to +\infty$ the limit of left hand side of (3.11) exist, and (3.10) holds. Theorem 3.1 is proved. □

Let us now state a uniqueness theorem and present a constractive prosedure for solving the inverse nodal problem.

Theorem 3.2 *Let $X_j^0 \subset X_j$ be a subset of nodes which is dense $(0, \ell_j)$ for $j = \overline{1, \nu}$. Then, the specification of $\bigcup\limits_{j=1}^{\nu} X_j^0$ uniquely determines the potential $q_j(x) - \langle q \rangle$ a.e. on $(0, \ell_j)$ and the coefficients $h_j, j = \overline{1, \nu}$ of the boundary conditions. The potentials $q_j(x) - \langle q \rangle$ and the numbers h_j can be constructed via the following algorithm:*

1. *For $j = \overline{1, \nu}$, and each $x \in [0, \ell_j]$, we choose a sequence $\left\{x_{nj}^i\right\} \subset X_j^0$ such that $\lim\limits_{|n| \to +\infty} x_{nj}^i = x$:*

2. From (3.9), we find the function $g_j(x)$ and calculate values for $g_j(x)$ at $x=0$, i.e.

$$h_j = g_j(0), j = \overline{1,\upsilon}. \tag{3.12}$$

3. The functions $q_j(x)$ can be determined as

$$q_j(x) - \langle q \rangle = 2g_j'(x) + \frac{\pi p}{\nu^2 - p^2} \sum_{j=1}^{\nu-p} h_j + \tag{3.13}$$

$$+ \frac{\pi}{2} \left[\frac{(p-1)(\nu-p)}{p\nu} + \frac{\nu^2 - p(p-1)}{\nu(\nu+p)} \right] \sum_{j=\nu-p+1}^{\nu} h_j,$$

where

$$\langle q \rangle := \frac{\pi p}{\nu^2 - p^2} \sum_{j=1}^{\nu-p} \int_0^{\ell} q_j(t)dt +$$

$$+ \frac{\pi}{2} \left[\frac{(p-1)(\nu-p)}{p\nu} + \frac{\nu^2 - p(p-1)}{\nu(\nu+p)} \right] \sum_{j=\nu-p+1}^{\nu} \int_0^{2\ell} q_j(t)dt.$$

Proof Formulas (3.12) and (3.13) can be derived from (3.10) step by step. We obtain the following reconstruction prosedure:

i) Taking value for $g_j(x)$ at $x=0$, then it yields $h_j = g_j(0)$, $j = \overline{1,\nu}$.

ii) After h_j are reconstructed, on takes derivatives of the function $g_j(x)$ we have (3.13). □

4. Example

Let graph G with vertex set $V = \{\upsilon_0, \upsilon_1, \upsilon_2\}$ and edge set $E = \{e_1, e_2\}$, where υ_1, υ_2 are the boundary vertices, υ_0 is the interior vertex and $e_j = [\upsilon_j, \upsilon_0]$ for $j = \overline{1,2}$. We suppose that the length of edge $e_1, e_2 \in E$ is equal to ℓ_1 and ℓ_2 $(\ell_1 \neq \ell_2)$ respectively. In this case,

$$\Delta_0(s) = -s \sin s(\ell_1 + \ell_2).$$

From here,

$$s_n^0 = \frac{\pi n}{\ell_1 + \ell_2}, n \in \mathbb{Z}.$$

Next, it follows from Equation (2.5) that,

$$\Delta(s) = -s \sin s(\ell_1 + \ell_2) + (A_1 + A_2) \cos s(\ell_1 + \ell_2) + o(1).$$

From $\Delta(s) = 0$, we have

$$s_n = \frac{\pi n}{\ell_1 + \ell_2} + \frac{(A_1 + A_2)(\ell_1 + \ell_2)}{n\pi} + O\left(\frac{1}{n^2}\right).$$

Fix $j=\overline{1,2}$ and assume that x_{nj}^{k} are the zeros of the functions $\varphi_j\left(x,s_n\right)$. Taking (3.4) in to account, from equation $\varphi_j\left(x,s_n\right)=0$, we obtain the following asymptotic formula, uniform in j, for the zeros as $|n|\to+\infty$

$$x_{nj}^{k}=\frac{\left(k-\frac{1}{2}\right)\left(\ell_1+\ell_2\right)}{n}+\frac{A_j\left(x_{nj}^{k}\right)\left(\ell_1+\ell_2\right)^2}{n^2\pi^2}-$$
$$-\frac{\left(k-\frac{1}{2}\right)\left(\ell_1+\ell_2\right)^2\left(A_1+A_2\right)}{n^3\pi^2}+o\left(\frac{1}{n^2}\right).$$

Denote

$$g_j\left(x\right):=\lim_{|n|\to+\infty}\left(x_{nj}^{k}-\frac{\left(k-\frac{1}{2}\right)\left(\ell_1+\ell_2\right)}{n}\right)\frac{n^2\pi^2}{\left(\ell_1+\ell_2\right)^2}.$$

then

$$g_j\left(x\right)=A_j\left(x\right)-\frac{\left(A_1+A_2\right)x}{\left(\ell_1+\ell_2\right)}. \tag{4.1}$$

1. For $j=1,2,$ and each $x\in\left[0,\ell_j\right]$, we choose a sequence $\left\{x_{nj}^{k}\right\}\subset X_j^0$ such that

$$\lim_{|n|\to+\infty}x_{nj}^{k}=x;$$

2. From (4.1), we find the function $g_j\left(x\right)$ and calculata values for $g_j\left(x\right)$ at $x=0,$ i.e.

$$h_j=g_j\left(0\right),\ j=1,2.$$

3. The functions $q_j\left(x\right)$ can be determined as

$$q_j\left(x\right)-\langle q\rangle=2g_j^{'}\left(x\right)+\frac{2\left(h_1+h_2\right)}{\left(\ell_1+\ell_2\right)},\ j=1,2,$$

where

$$\langle q\rangle=\frac{1}{\left(\ell_1+\ell_2\right)}\left[\int_0^{\ell_1}q_1\left(t\right)dt+\int_0^{\ell_2}q_2\left(t\right)dt\right].$$

References

[1] Amirov R, Ergun A, Durak S. Half inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse. Numerical Methods for Partial Differential Equations 2021; 37 (1): 915-924.

[2] Binding PA, Watson BA. An inverse nodal problem for two- parameter Sturm-Liouville systems. Inverse Problems 2009; 25: 1-1.

[3] Boman J, Kurasov P, Suhr R. Schrödinger operators on graphs and geometry II. Spectral estimates for L1-potentials and an Ambartsumian theorem. Integral Equations and Operator Theory 2018; 90 (40).

[4] Browne PJ, Sleeman BD. Inverse nodal problem for Sturm-Liouville equation with eigenparameter dependent boundary conditions. Inverse Problems 1996; 12: 377-381.

[5] Buterin SA, Shieh CT. Inverse nodal problem for differential penciles. Applied. Mathematics Letters 2009; 22: 1240-1247.

[6] Cheng YH. Reconstruction of the Sturm-Liouville operator on a p-star graphs with nodal data. Rocky Mountain Journal Mathematics 2012; 42: 1431-1446.

[7] Currie S, Watson BA. Inverse nodal problems for Sturm-Liouville equations on graphs. Inverse Problems 2007; 23: 2029-2040.

[8] Dekoninck B, Nicaise S. The eigenvalue problem for networks of beams. Linear Algebra Applications 2000;.314 (3): 165-189.

[9] Ergun A, Amirov R. Half inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse. Numerical Methods for Partial Differential Equations 2022; 38: 577–590.

[10] Ergun A. A half inverse problem for the singular diffusion operator with jump condition. Miskolch Mathematical Notes 2020; 21 (2): 805-821.

[11] Ergun A. Inverse problem for singular diffusion operatör. Miskolch Mathematical Notes 2021; 22 (1): 173-192.

[12] Hald O H, McLaughlin JR. Solutions of inverse nodal problems. Inverse Problems 1989; 5: 307-347.

[13] Hu YT, Bondarenko NP, Shieh CT, Yang CF. Traces and inverse nodal problems for Dirac-type integro-differential operators on a graph. Applied Mathematics and Computation 2019; 363: 1-10.

[14] Kottos T, Smilansky U. Quantum chaos on graphs. Phys. Review. Letters 1997; 79 (24): 4794-4797.

[15] Kuchment P. Graph models for waves in thin structures. Waves in Random Media 2002; 12 (4): 1-24.

[16] Kuchment P. Quantum Graphs. Some basic structures. Waves in Random Media 2004; 14: 107-128.

[17] Lagnese JE, Leugering G, Schmidt EJPG. Modelling, Analysis and Control of Dynamic Elastic Multi-Link Structures. New York, USA:Systems Control Found Applications, 1994.

[18] Law CK, Yang CF. Reconstructing the potential function and its derivatives using nodal data. Inverse Problems 1998; 14: 299-312.

[19] Law CK, Shen CL, Wang WC. The inverse nodal problem and the Ambarzumyan problem for the p-laplacian. Proceedings of the Royal Society of Edinburgh A 2009; 139: 1261-1273.

[20] Lin CR. Ambarzumyan Problem on Trees. Kaohsiung, Taiwan: Unpublished Master Thesis, National Sun Yat-sen University, 2008.

[21] McLaughlin JR. Inverse spectral theory using nodal points as data-a uniqueness result. Journal of Differential Equations 1988; 73: 354-362.

[22] Montroll EW. Quantum theory on a network. I. A solvable model whose wavefunctions are elementary functions. Journal of Mathematical Physics 1970; 11 (2): 635-648.

[23] Nicaise S. Some results on spectral theory over networks, applied to nerve impulse transmission. Lecture Notes in Mathematics 1985; 532-541.

[24] Sandeman DC, Tautz J, Lindauer M. Transmission of vibration across honeycombs and its detection by bee leg receptors. Journal of Experimental. Biology 1996; 199: 2585-2594.

[25] Shen CL, Shieh CT. An inverse nodal problem for vectorial Sturm-Liouville equation. Inverse Problems 2000; 16: 349-356.

[26] Shieh CT, Yurko VA. Inverse nodal and inverse spectral problems for discontinuous boundary value problems. Journal of Mathematical Analysis and Applications 2008; 347: 266-272.

[27] Sobolev AV, Solomyak M. Schrödinger operator on homogeneous metric trees: spectrum in gaps. Reviews in Mathematical Physics 2002; 14 (5): 421-467.

[28] Yang XF. A solution of the inverse nodal problem. Inverse Problems 1997; 13: 203-213.

[29] Yang CF, Yang XP. Inverse nodal problems for differential pencils on a star-shaped graph. Inverse problems 2010; 26.

[30] Yang CF, Wang F. Inverse problems on a graph with loops. Journal of Inverse and III-posed Problems 2017; 25: 373-380.

[31] Yurko VA. Method of spectral mapping in the Inverse Problem Theory, Journal of Inverse and III-posed Problems 2002; 303.

[32] Yurko VA. Inverse spectral problems for Sturm-Liouville operators on graphs. Inverse Problems 2005; 21 (3): 1075-1086.

[33] Yurko VA. Inverse nodal problems for Sturm-Liouville operators on star-type graphs. Journal of Inverse and III-posed Problems 2008; 16 (7): 715-722.

[34] Yurko VA. Inverse nodal problems for Sturm-Liouville operators on a star-type graph. Siberian Mathematical Journal 2009; 50 (2): 373-378.