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Birotational hypersurface and the second Laplace–Beltrami operator in the four dimensional Euclidean space \mathbb{E}^4

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Abstract: We consider the birotational hypersurface $\mathbf{x}(u, v, w)$ with the second Laplace–Beltrami operator in the four dimensional Euclidean space \mathbb{E}^4 . We give the i -th curvatures of \mathbf{x} . In addition, we compute the second Laplace–Beltrami operator of the birotational hypersurface satisfying $\Delta^{II} \mathbf{x} = \mathcal{A} \mathbf{x}$ for some 4×4 matrix \mathcal{A} .

Key words: Euclidean spaces, four space, birotational hypersurface, Gauss map, i -th curvature, second Laplace–Beltrami operator

1. Introduction

With Chen [13–16], the researches of the submanifolds of the finite type whose immersion into the \mathbb{E}^m (or \mathbb{E}_ν^m) by using a finite number of eigenfunctions of their Laplacian have been examined for almost 50 years.

Takahashi [46] introduced that a connected Euclidean submanifold is of 1-type, iff it is either minimal in \mathbb{E}^m or minimal in some hypersphere of \mathbb{E}^m . Submanifolds of the finite type closest in simplicity to the minimal ones are the 2-type spherical submanifolds (where spherical means into a sphere). Some results of the 2-type spherical closed submanifolds were studied by [9, 10, 14]. Garay [28] worked an extension of the Takahashi’s theorem in \mathbb{E}^m . Cheng and Yau gave the hypersurfaces with constant scalar curvature; Chen and Piccinni [17] considered the submanifolds with the finite type Gauss map in \mathbb{E}^m . Dursun [23] focused on the hypersurfaces with pointwise 1-type Gauss map in \mathbb{E}^{n+1} .

In \mathbb{E}^3 ; Takahashi [46] gave that the minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez et al. [25] classified that the surfaces satisfying $\Delta H = AH$, $A \in Mat(3, 3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [20] found the minimal helicoid in terms of the pointwise 1-type Gauss map of the first kind; Garay [27] worked on the certain class of the finite type surfaces of revolution; Dillen et al. [21] focused that the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3, 3)$, $B \in Mat(3, 1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [45] obtained the surfaces of revolution satisfying $\Delta^{III} x = Ax$; Senoussi and Bekkar [44] introduced the helicoidal surfaces M^2 which are of the finite type with respect to the fundamental forms I, II and III , i.e.

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their position vector field $r(u, v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3, 3)$; Kim et al. [37] gave the Cheng–Yau’s operator and the Gauss map of the surfaces of revolution.

In \mathbb{E}^4 ; Moore [41, 42] gave the general rotational surfaces; Hasanis and Vlachos [34] studied the hypersurfaces with the harmonic mean curvature vector field; Cheng and Wan [18] considered the complete hypersurfaces with CMC; Kim and Turgay [38] studied the surfaces with the L_1 -pointwise 1-type Gauss map; Arslan et al. [3] introduced Vranceanu surface with pointwise 1-type Gauss map; Arslan et al. [4] introduced generalized rotational surfaces; Arslan et al. [5] obtained the tensor product surfaces with pointwise 1-type Gauss map; Kahraman Aksoyak and Yaylı [35] considered the rotational surfaces with the pointwise 1-type Gauss map; Güler et al. [32] worked the helicoidal hypersurfaces; Güler et al. [31] introduced the Gauss map and the third Laplace–Beltrami operator of the rotational hypersurface; Güler and Turgay [33] obtained the Cheng–Yau’s operator and the Gauss map of the rotational hypersurfaces; Güler [30] worked the rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4, 4)$. He [29] also examined the fundamental form IV and the curvature formulas of the hypersphere; Arslan et al. [7] introduced the rotational λ -hypersurfaces in the Euclidean spaces.

In Minkowski 4-space \mathbb{E}_1^4 ; Ganchev and Milousheva [26] indicated analogue of surfaces of [41, 42]; Arvanitoyeorgos et al. [8] studied that if the mean curvature vector field of M_1^3 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M_1^3 has CMC; Arslan and Milousheva [6] considered the meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map; Turgay [47] introduced some classifications of the Lorentzian surfaces with the finite type Gauss map; Dursun and Turgay [24] gave the space-like surfaces in with the pointwise 1-type Gauss map. Kahraman Aksoyak and Yaylı [36] worked the general rotational surfaces with pointwise 1-type Gauss map in \mathbb{E}_2^4 ; Bektaş et al. [11] considered surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in \mathbb{E}_2^5 . They [12] also gave the pseudo-spherical submanifolds with 1-type pseudo-spherical Gauss map.

We consider the birotational hypersurface with the second Laplace–Beltrami operator in the four dimensional Euclidean space \mathbb{E}^4 . In Section 2, we indicate the fundamental notions of the four dimensional Euclidean geometry. We obtain the curvature formulas of a hypersurface in \mathbb{E}^4 in Section 3. In Section 4, we give the birotational hypersurface. Additionally, we examine the birotational hypersurface satisfying $\Delta^I \mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix \mathcal{A} in \mathbb{E}^4 in Section 5. Finally, we give some results in Section 6.

2. Preliminaries

In this section, giving some of basic facts and definitions, we describe notations used whole paper. Let \mathbb{E}^m denote the Euclidean m -space with the canonical Euclidean metric tensor given by $\tilde{g} = \langle \cdot, \cdot \rangle = \sum_{i=1}^m dx_i^2$, where (x_1, x_2, \dots, x_m) is a rectangular coordinate system in \mathbb{E}^m . Consider an m -dimensional Riemannian submanifold of the space \mathbb{E}^m . We denote the Levi–Civita connections of \mathbb{E}^m and M by $\tilde{\nabla}$ and ∇ , respectively. We shall use the letters X, Y, Z, W (resp., ξ, η) to denote the vectors fields tangent (resp., normal) to M . The Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.1)$$

$$\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \quad (2.2)$$

where h , D and A are the second fundamental form, the normal connection and the shape operator of M , respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X, Y)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.4}$$

where R , R^D are the curvature tensors associated with connections ∇ and D , respectively, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space \mathbb{E}^{n+1} , \mathbf{S} its shape operator (i.e. Weingarten map) and x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ of consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \dots, n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \dots, \theta_n\}$. Then the first structural equation of Cartan is given by

$$d\theta_i = \sum_{j=1}^n \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \dots, n, \tag{2.5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of M and \mathbb{E}^{n+1} by ∇ and $\tilde{\nabla}$, respectively. Then, from the Codazzi equation (2.3), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{2.6}$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{2.7}$$

for distinct $i, j, l = 1, 2, \dots, n$.

We put $s_j = \sigma_j(k_1, k_2, \dots, k_n)$, where σ_j is the j -th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq n} a_{i_1} a_{i_2} \dots a_{i_j}.$$

We use the following notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, k_{i+2}, \dots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \dots = 0$. We call the function s_k as the k -th mean curvature of M . We would like to note that functions $H = \frac{1}{n}s_1$ and $K = s_n$ are called the mean curvature and the Gauss-Kronecker curvature of M , respectively. In particular, M is said to be the j -minimal if $s_j \equiv 0$ on M .

In \mathbb{E}^{n+1} , to find the i -th curvature formulas \mathfrak{C}_i (Curvature formulas sometimes are represented as the mean curvature H_i , and sometimes as the Gaussian curvature K_i by different writers, such as [1] and [39]. We will call it just the i -th curvature \mathfrak{C}_i in this paper.), where $i = 0, \dots, n$, firstly, we use the characteristic polynomial of \mathbf{S} :

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^n (-1)^k s_k \lambda^{n-k}, \tag{2.8}$$

where $i = 0, \dots, n$, \mathcal{I}_n denotes the identity matrix of order n . Then, we get curvature formulas $\binom{n}{i} \mathfrak{C}_i = s_i$. That is, $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1} \mathfrak{C}_1 = s_1, \dots, \binom{n}{n} \mathfrak{C}_n = s_n = K$.

For a Euclidean submanifold $x: M \rightarrow \mathbb{E}^m$, the immersion (M, x) is called finite type, if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x) , i.e. $x = x_0 + \sum_{i=1}^k x_i$, where x_0 is a constant map, x_1, \dots, x_k nonconstant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, k$. If λ_i are different, M is called k -type. See [14] for details.

Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an isometric immersion from $M^3 \subset \mathbb{E}^3$ to \mathbb{E}^4 . The triple vector product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ of \mathbb{E}^4 is defined by:

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

For a hypersurface \mathbf{x} in 4-space, we see $(g_{ij})_{3 \times 3}$, $(h_{ij})_{3 \times 3}$, where (g_{ij}) and (h_{ij}) are the first, and the second fundamental form matrices, respectively, and $g_{11} = \mathbf{x}_u \cdot \mathbf{x}_u$, $g_{12} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{22} = \mathbf{x}_v \cdot \mathbf{x}_v$, $g_{13} = \mathbf{x}_u \cdot \mathbf{x}_w$, $g_{23} = \mathbf{x}_v \cdot \mathbf{x}_w$, $g_{33} = \mathbf{x}_w \cdot \mathbf{x}_w$, $h_{11} = \mathbf{x}_{uu} \cdot e$, $h_{12} = \mathbf{x}_{uv} \cdot e$, $h_{22} = \mathbf{x}_{vv} \cdot e$, $h_{13} = \mathbf{x}_{uw} \cdot e$, $h_{23} = \mathbf{x}_{vw} \cdot e$, $h_{33} = \mathbf{x}_{ww} \cdot e$. Here,

$$e = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|} \tag{2.9}$$

is the unit normal (i.e. the Gauss map) of the hypersurface \mathbf{x} .

The product matrices $(g_{ij})^{-1} \cdot (h_{ij})$ gives the matrix of the shape operator \mathbf{S} of the hypersurface \mathbf{x} in 4-space. See [31–33] for details.

3. i -th curvatures

In \mathbb{E}^4 , to compute the i -th mean curvature formula \mathfrak{C}_i , where $i = 0, 1, 2, 3$, we use the characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$:

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda I_3) = 0.$$

Then, obtain $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1} \mathfrak{C}_1 = \binom{3}{1} H = -\frac{b}{a}$, $\binom{3}{2} \mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3} \mathfrak{C}_3 = K = -\frac{d}{a}$.

Therefore, we find i -th curvature formulas depends on the coefficients of the first and second fundamental forms in 4-space.

Theorem 3.1 Any hypersurface \mathbf{x} in \mathbb{E}^4 has the following curvature formulas, $\mathfrak{C}_0 = 1$ (by definition),

$$\mathfrak{C}_1 = \frac{\left\{ \begin{array}{l} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})g_{33} + (g_{11}g_{22} - g_{12}^2)h_{33} - g_{23}^2h_{11} - g_{13}^2h_{22} \\ -2(g_{13}h_{13}g_{22} - g_{23}h_{13}g_{12} - g_{13}h_{23}g_{12} + g_{11}g_{23}h_{23} - g_{13}g_{23}h_{12}) \end{array} \right\}}{3[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{\left\{ \begin{array}{l} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})h_{33} + (h_{11}h_{22} - g_{12}^2)g_{33} - g_{11}h_{23}^2 - g_{22}h_{13}^2 \\ -2(g_{13}h_{13}h_{22} - g_{23}h_{13}h_{12} - g_{13}h_{23}h_{12} + g_{23}h_{23}h_{11} - h_{13}h_{23}g_{12}) \end{array} \right\}}{3[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{(h_{11}h_{22} - h_{12}^2)h_{33} - h_{11}h_{23}^2 + 2h_{12}h_{13}h_{23} - h_{22}h_{13}^2}{(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2}. \tag{3.3}$$

See [29] for details.

4. Birotational hypersurface

In this section, we define the rotational hypersurface, then find its differential geometric properties in Euclidean 4-space \mathbb{E}^4 . We would like to note that the definition of the rotational hypersurfaces in Riemannian space forms were defined in [22]. A rotational hypersurface $M \subset \mathbb{E}^{n+1}$ generated by a curve γ around an axis γ that does not meet γ is obtained by taking the orbit of γ under those orthogonal transformations of \mathbb{E}^{n+1} that leaves \mathbf{r} pointwise fixed (See [22, Remark 2.3]).

We use curve γ as $(\mathbf{f}(u), 0, \mathbf{g}(u), 0)$ with the following rotation matrix

$$\begin{pmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & \cos w & -\sin w \\ 0 & 0 & \sin w & \cos w \end{pmatrix},$$

and give the following definition:

Definition 4.1 A birotational hypersurface in \mathbb{E}^4 is defined by

$$\mathbf{x}(u, v, w) = (\mathbf{f}(u) \cos v, \mathbf{f}(u) \sin v, \mathbf{g}(u) \cos w, \mathbf{g}(u) \sin w), \tag{4.1}$$

where \mathbf{f}, \mathbf{g} are differentiable functions, and $0 \leq v, w \leq 2\pi$.

Remark 4.2 While $\mathbf{f}(u) = \mathbf{g}(u) = 1$ in (4.1), we obtain the Clifford torus in \mathbb{E}^4 . See [2, 48] for details. Moreover, when $v = w$ in (4.1), we get the tensor product surface in \mathbb{E}^4 . See [5, 43] for details.

Considering the following first order derivative of (4.1) with respect to u, v, w , respectively,

$$\mathbf{x}_u = \begin{pmatrix} \mathbf{f}' \cos v \\ \mathbf{f}' \sin v \\ \mathbf{g}' \cos w \\ \mathbf{g}' \sin w \end{pmatrix}, \mathbf{x}_v = \begin{pmatrix} -\mathbf{f} \sin v \\ \mathbf{f} \cos v \\ 0 \\ 0 \end{pmatrix}, \mathbf{x}_w = \begin{pmatrix} 0 \\ 0 \\ -\mathbf{g} \sin w \\ \mathbf{g} \cos w \end{pmatrix},$$

we find the following first quantities of (4.1):

$$(g_{ij}) = \text{diag}(\mathbf{f}'^2 + \mathbf{g}'^2, \mathbf{f}^2, \mathbf{g}^2), \tag{4.2}$$

where \mathbf{f}' and \mathbf{g}' denote the first order derivative of \mathbf{f} and \mathbf{g} respect to u , respectively. Here,

$$g = \det (g_{ij}) = \mathbf{f}'^2 \mathbf{g}'^2 (\mathbf{f}'^2 + \mathbf{g}'^2).$$

Using (2.9), we get the following Gauss map of the birotational hypersurface (4.1):

$$e = \frac{1}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}} (-\mathbf{g}' \cos v, -\mathbf{g}' \sin v, \mathbf{f}' \cos w, \mathbf{f}' \sin w). \tag{4.3}$$

With the help of the second differentials of (4.1) with respect to u, v, w , and the Gauss map (4.3) of the birotational hypersurface (4.1), we have the following second quantities:

$$(h_{ij}) = \text{diag} \left(\frac{\mathbf{f}' \mathbf{g}'' - \mathbf{g}' \mathbf{f}''}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}}, \frac{\mathbf{f} \mathbf{g}'}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}}, -\frac{\mathbf{g} \mathbf{f}'}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}} \right), \tag{4.4}$$

where \mathbf{f}'' and \mathbf{g}'' denote the second order derivative of \mathbf{f} and \mathbf{g} respect to u , respectively. So, we get

$$h = \det (h_{ij}) = -\frac{\mathbf{f} \mathbf{g} \mathbf{f}' \mathbf{g}' (\mathbf{f}' \mathbf{g}'' - \mathbf{f}'' \mathbf{g}')}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{3/2}}.$$

We calculate the shape operator matrix of the birotational hypersurface (4.1), by using (4.2) and (4.4), then, obtain the following

$$\mathbf{S} = \text{diag} \left(\frac{\mathbf{f}' \mathbf{g}'' - \mathbf{g}' \mathbf{f}''}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{3/2}}, \frac{\mathbf{g}'}{\mathbf{f} (\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}}, -\frac{\mathbf{f}'}{\mathbf{g} (\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}} \right).$$

Finally, by using (3.1), (3.2), and (3.3), with (4.2), (4.4), respectively, we find the following curvatures of the birotational hypersurface (4.1):

Corollary 4.3 *Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). The \mathbf{x} has the following (mean) 1-curvature*

$$\mathfrak{C}_1 = \frac{(\mathbf{f}' \mathbf{g}'' - \mathbf{g}' \mathbf{f}'') \mathbf{f} \mathbf{g} - (\mathbf{f}'^2 + \mathbf{g}'^2) (\mathbf{f} \mathbf{f}' - \mathbf{g} \mathbf{g}')}{3 \mathbf{f} \mathbf{g} (\mathbf{f}'^2 + \mathbf{g}'^2)^{3/2}}.$$

Corollary 4.4 *Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). The \mathbf{x} has the following 2-curvature*

$$\mathfrak{C}_2 = \frac{(\mathbf{f} \mathbf{f}' - \mathbf{g} \mathbf{g}') (\mathbf{g}' \mathbf{f}'' - \mathbf{f}' \mathbf{g}'') - (\mathbf{f}'^2 + \mathbf{g}'^2) \mathbf{f}' \mathbf{g}'}{3 \mathbf{f} \mathbf{g} (\mathbf{f}'^2 + \mathbf{g}'^2)^2}.$$

Corollary 4.5 *Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). The \mathbf{x} has the following (Gaussian) 3-curvature*

$$\mathfrak{C}_3 = -\frac{\mathbf{f}' \mathbf{g}' (\mathbf{f}' \mathbf{g}'' - \mathbf{g}' \mathbf{f}'')}{\mathbf{f} \mathbf{g} (\mathbf{f}'^2 + \mathbf{g}'^2)^{5/2}}.$$

Example 4.6 Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of \mathbf{x} is parametrized by the arc length, and $\mathbf{f}(u) = \cos u$, $\mathbf{g}(u) = \sin u$, the birotational hypersurface has the following curvatures

$$\begin{aligned} \mathfrak{C}_1 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive CMC,} \\ \mathfrak{C}_2 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive constant 2-curvature,} \\ \mathfrak{C}_3 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive CGC.} \end{aligned}$$

Example 4.7 Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of \mathbf{x} is parametrized by $\mathbf{f}(u) = \mathbf{g}(u) = \frac{u}{\sqrt{2}}$, the birotational hypersurface has the following curvatures

$$\begin{aligned} \mathfrak{C}_1 &= 0, \text{ i.e. } \mathbf{x} \text{ is 1-minimal,} \\ \mathfrak{C}_2 &= -\frac{1}{3u^2}, \text{ i.e. } \mathbf{x} \text{ has negative 2-curvature,} \\ \mathfrak{C}_3 &= 0, \text{ i.e. } \mathbf{x} \text{ is 3-minimal.} \end{aligned}$$

5. Birotational hypersurface satisfying $\Delta^{II} \mathbf{x} = A\mathbf{x}$

In this section, we give the second Laplace–Beltrami operator of a smooth function. Therefore, we calculate the second Laplace–Beltrami operator of the birotational hypersurface.

The inverse of the matrix II , i.e.

$$(h_{ij}) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

is given by

$$\frac{1}{h} \begin{pmatrix} h_{22}h_{33} - h_{23}h_{32} & -(h_{12}h_{33} - h_{13}h_{32}) & h_{12}h_{23} - h_{13}h_{22} \\ -(h_{21}h_{33} - h_{31}h_{23}) & h_{11}h_{33} - h_{13}h_{31} & -(h_{11}h_{23} - h_{21}h_{13}) \\ h_{21}h_{32} - h_{22}h_{31} & -(h_{11}h_{32} - h_{12}h_{31}) & h_{11}h_{22} - h_{12}h_{21} \end{pmatrix},$$

where

$$\begin{aligned} h &= \det(h_{ij}) \\ &= h_{11}h_{22}h_{33} - h_{11}h_{23}h_{32} + h_{12}h_{31}h_{23} - h_{12}h_{21}h_{33} + h_{21}h_{13}h_{32} - h_{13}h_{22}h_{31}. \end{aligned}$$

Definition 5.1 The second Laplace–Beltrami operator of a smooth function $\varphi = \varphi(x^1, x^2, x^3) |_{\mathbf{D}}$ ($\mathbf{D} \subset \mathbb{R}^3$) of class C^3 with respect to the second fundamental form of a hypersurface \mathbf{x} is the operator Δ^{II} defined by

$$\Delta^{II} \varphi = \frac{1}{h^{1/2}} \sum_{i,j=1}^3 \frac{\partial}{\partial x^i} \left(h^{1/2} h^{ij} \frac{\partial \varphi}{\partial x^j} \right), \tag{5.1}$$

where $(h^{ij}) = (h_{kl})^{-1}$ and $h = \det(h_{ij})$.

We can write (5.1), clearly, as follows:

$$\Delta^{II} \varphi = \frac{1}{|h|^{1/2}} \left\{ \begin{aligned} &\frac{\partial}{\partial x^1} \left(|h|^{1/2} h^{11} \frac{\partial \varphi}{\partial x^1} \right) - \frac{\partial}{\partial x^1} \left(|h|^{1/2} h^{12} \frac{\partial \varphi}{\partial x^2} \right) + \frac{\partial}{\partial x^1} \left(|h|^{1/2} h^{13} \frac{\partial \varphi}{\partial x^3} \right) \\ &- \frac{\partial}{\partial x^2} \left(|h|^{1/2} h^{21} \frac{\partial \varphi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(|h|^{1/2} h^{22} \frac{\partial \varphi}{\partial x^2} \right) - \frac{\partial}{\partial x^2} \left(|h|^{1/2} h^{23} \frac{\partial \varphi}{\partial x^3} \right) \\ &+ \frac{\partial}{\partial x^3} \left(|h|^{1/2} h^{31} \frac{\partial \varphi}{\partial x^1} \right) - \frac{\partial}{\partial x^3} \left(|h|^{1/2} h^{32} \frac{\partial \varphi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left(|h|^{1/2} h^{33} \frac{\partial \varphi}{\partial x^3} \right) \end{aligned} \right\}.$$

For any rotational hypersurface $h_{ij} = 0$, when $i \neq j$. Hence, we can rewrite the second Laplace–Beltrami operator:

$$\Delta^{II}\varphi = \frac{1}{|h|^{1/2}} \left\{ \frac{\partial}{\partial x^1} \left(|h|^{1/2} h^{11} \frac{\partial \varphi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(|h|^{1/2} h^{22} \frac{\partial \varphi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left(|h|^{1/2} h^{33} \frac{\partial \varphi}{\partial x^3} \right) \right\}.$$

Therefore, more clear form of the second Laplace–Beltrami operator of any rotational hypersurface $\mathbf{x}(u, v, w)$ is given by

$$\Delta^{II}\mathbf{x} = \frac{1}{|h|^{1/2}} \left\{ \frac{\partial}{\partial u} \left(\frac{h_{22}h_{33}}{|h|^{1/2}} \mathbf{x}_u \right) + \frac{\partial}{\partial v} \left(\frac{h_{11}h_{33}}{|h|^{1/2}} \mathbf{x}_v \right) + \frac{\partial}{\partial w} \left(\frac{h_{11}h_{22}}{|h|^{1/2}} \mathbf{x}_w \right) \right\}. \tag{5.2}$$

Differentiating $\frac{h_{22}h_{33}}{|h|^{1/2}} \mathbf{x}_u$, $\frac{h_{11}h_{33}}{|h|^{1/2}} \mathbf{x}_v$, $\frac{h_{11}h_{22}}{|h|^{1/2}} \mathbf{x}_w$, with respect to u, v, w , respectively, and substituting them into (5.2), we get the following.

Theorem 5.2 *The second Laplace–Beltrami operator of the birotational hypersurface (4.1) is given by*

$$\Delta^{II}\mathbf{x} = \begin{pmatrix} \Delta^{II}\mathbf{x}_1 \\ \Delta^{II}\mathbf{x}_2 \\ \Delta^{II}\mathbf{x}_3 \\ \Delta^{II}\mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{f}(u) \cos v \\ \mathbf{f}(u) \sin v \\ \mathbf{g}(u) \cos w \\ \mathbf{g}(u) \sin w \end{pmatrix},$$

where

$$\mathbf{f}(u) = \frac{\left\{ \begin{array}{l} -\mathbf{f}\mathbf{g}'\mathbf{g}'^2 (\mathbf{f}'^2 + \mathbf{g}'^2) \mathbf{f}''' + \mathbf{f}\mathbf{g}\mathbf{f}'^2\mathbf{g}' (\mathbf{f}'^2 + \mathbf{g}'^2) \mathbf{g}''' \\ -\mathbf{f}\mathbf{g}\mathbf{g}' (4\mathbf{f}'^2 + 5\mathbf{g}'^2) \mathbf{f}'' \\ + \left(\begin{array}{l} \mathbf{f}\mathbf{g}\mathbf{f}' (\mathbf{f}'^2 + 2\mathbf{g}'^2) \mathbf{g}'' \\ -\mathbf{f}'\mathbf{g}' (\mathbf{f}'^2 + \mathbf{g}'^2) (\mathbf{f}\mathbf{g}' + \mathbf{g}\mathbf{f}') \end{array} \right) \end{array} \right\} (\mathbf{f}'\mathbf{g}'' - \mathbf{f}''\mathbf{g}')}{\mathbf{f}\mathbf{g}\mathbf{g}' (\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2} (\mathbf{f}'\mathbf{g}'' - \mathbf{f}''\mathbf{g}')^2},$$

$$\mathbf{g}(u) = \frac{\left\{ \begin{array}{l} -\mathbf{f}\mathbf{g}'\mathbf{g}'^2 (\mathbf{f}'^2 + \mathbf{g}'^2) \mathbf{f}''' + \mathbf{f}\mathbf{g}\mathbf{f}'^2\mathbf{g}' (\mathbf{f}'^2 + \mathbf{g}'^2) \mathbf{g}''' \\ + \mathbf{f}\mathbf{g}\mathbf{g}' (2\mathbf{f}'^2 + \mathbf{g}'^2) \mathbf{f}'' \\ - \left(\begin{array}{l} \mathbf{f}\mathbf{g}\mathbf{f}' (5\mathbf{f}'^2 + 4\mathbf{g}'^2) \mathbf{g}'' \\ +\mathbf{f}'\mathbf{g}' (\mathbf{f}'^2 + \mathbf{g}'^2) (\mathbf{f}\mathbf{g}' + \mathbf{g}\mathbf{f}') \end{array} \right) \end{array} \right\} (\mathbf{f}'\mathbf{g}'' - \mathbf{f}''\mathbf{g}')}{\mathbf{f}\mathbf{g}\mathbf{f}' (\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2} (\mathbf{f}'\mathbf{g}'' - \mathbf{f}''\mathbf{g}')^2},$$

and \mathbf{f}''' and \mathbf{g}''' denote the third order derivative of \mathbf{f} and \mathbf{g} respect to u , respectively.

6. Conclusion

Considering the findings in the previous section, we obtain the following results:

Corollary 6.1 *Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). The birotational hypersurface \mathbf{x} satisfies $\Delta^{II}\mathbf{x} = \mathcal{A}\mathbf{x}$, where*

$$\mathcal{A} = \text{diag} \left(\frac{\mathbf{f}}{\mathbf{f}}\mathcal{I}_2, \frac{\mathbf{g}}{\mathbf{g}}\mathcal{I}_2 \right),$$

and $\mathcal{A} \in \text{Mat}(4, 4)$, \mathcal{I}_2 is the identity matrix.

Corollary 6.2 Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of birotational hypersurface \mathbf{x} is parametrized by the arc length, the \mathbf{x} holds $\Delta^{II}\mathbf{x} = \mathcal{B}\mathbf{x}$, where

$$\mathcal{B} = \text{diag}(\mathfrak{p}\mathcal{I}_2, \mathfrak{q}\mathcal{I}_2),$$

and

$$\begin{aligned} \mathfrak{p}(u) &= \mathbf{f}'(\mathbf{f}'\mathbf{g}''' - \mathbf{g}'\mathbf{f}''') - (4\mathbf{f}'^2 + 5\mathbf{g}'^2)\mathbf{f}'' + \frac{\mathbf{f}'(\mathbf{f}'^2 + 2\mathbf{g}'^2)}{\mathbf{g}'}\mathbf{g}'' - \frac{\mathbf{f}'(\mathbf{f}\mathbf{g}' + \mathbf{g}\mathbf{f}')}{\mathbf{f}\mathbf{g}}, \\ \mathfrak{q}(u) &= \mathbf{g}'(\mathbf{f}'\mathbf{g}''' - \mathbf{g}'\mathbf{f}''') + \frac{\mathbf{g}'(2\mathbf{f}'^2 + \mathbf{g}'^2)}{\mathbf{f}'}\mathbf{f}'' - (5\mathbf{f}'^2 + 4\mathbf{g}'^2)\mathbf{g}'' - \frac{\mathbf{g}'(\mathbf{f}\mathbf{g}' + \mathbf{g}\mathbf{f}')}{\mathbf{f}\mathbf{g}}, \end{aligned}$$

and $\mathcal{B} \in \text{Mat}(4, 4)$, \mathcal{I}_2 is the identity matrix.

Corollary 6.3 Let $\mathbf{x} : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of \mathbf{x} is parametrized $\mathbf{f}(u) = \cos u$, $\mathbf{g}(u) = \sin u$, the birotational hypersurface \mathbf{x} supplies $\Delta^{II}\mathbf{x} = \mathcal{C}\mathbf{x}$, where

$$\mathcal{C} = 3 \text{diag}(\cos u \mathcal{I}_2, \sin u \mathcal{I}_2),$$

and $\mathcal{C} \in \text{Mat}(4, 4)$, \mathcal{I}_2 is the identity matrix.

Example 6.4 Considering the hypersphere $S^3(r) = \{\xi \in \mathbb{E}^4 \mid \langle \xi, \xi \rangle = r^2\}$ for radius $r > 0$:

$$\xi(u, v, w) = (r \cos u \cos v, r \cos u \sin v, r \sin u \cos w, r \sin u \sin w), \tag{6.1}$$

we have the shape operator $\mathbf{S} = \frac{1}{r}\mathcal{I}_3$, and find the following curvatures of it:

$$\mathfrak{C}_0 = 1, \mathfrak{C}_1 = H = \frac{1}{r}, \mathfrak{C}_2 = \frac{1}{r^2}, \mathfrak{C}_3 = K = \frac{1}{r^3}.$$

Here, $H\mathfrak{C}_2 = K$, $H^2 = \mathfrak{C}_2$, and $H^3 = K$, i.e. the hypersurface (6.1) is the birotational umbilical hypersphere.

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