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BILAL BILALOV

YUSUF ZEREN

SABINA SADIGOVA

ŞEYMA ÇETİN

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Solvability in the small of m–th order elliptic equations in weighted grand Sobolev spaces

Bilal BILALOV[1,*], **Yusuf ZEREN**[2], **Sabina SADIGOVA**[3,4] **Şeyma ÇETİN**[2]
[1]Department of Non-harmonic analysis, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
[2]Department of Mathematics, Yıldız Technical University, İstanbul, Turkey
[3]Laboratory of Mathematical Problems of Signal Processing, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
[4]Department of Mathematics, Khazar University, Baku, Azerbaijan

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Abstract: In this work we consider the Sobolev spaces generated by the norm of the power weighted grand Lebesgue spaces. It is considered m–th order elliptic equation with nonsmooth coefficients on bounded domain in R^n. This space is nonseparable and by using shift operator we define the separable subspace of it, in which infinitely differentiable functions are dense. The investigation needs to establish boundedness property of convolution regarding weighted grand Lebesgue spaces. Then on scheme of nonweighted case we establish solvability (strong sense) in the small of m–th order elliptic equations in power weighted grand Sobolev spaces. Note that in weighted spaces this question is considered for the first time in connection with certain mathematical difficulties.

Key words: Elliptic equation, solvability in small, weighted grand Lebesgue and Sobolev spaces

1. Introduction

Elliptic equations play an essential and key role in the theory of partial differential equations and remarkable monographs have been dedicated to the solvability problems of them (linear case) by various great mathematicians as I.G. Petrovski [48], O.A. Ladyzhenskaya, N.N.Uraltseva [38], L.Bers, F.John, F.Schechter [4], L.Hörmander [32], S.L.Sobolev [54], K.Moren [44], V.P.Mikhaylov [40], J.L.Lions, E.Magenes [39], K.Yosida [57], S.Mizohata [43], C.Miranda [42] and others. It should be noted that all these monographs deal with classical spaces such as continuous functions, Holder classes or Sobolev spaces. With appearance new spaces it is arised the question of investigation solvability problems of differential equations regarding to these spaces. Recently, interest has increasing in so-called nonstandart function spaces in the context of various problems of pure mathematics, mechanics and mathematical physics. To the set of such spaces we can include the Lebesgue spaces with variable summability index, Morrey spaces, grand Lebesgue spaces, Orlicz spaces, Lorentz spaces and etc. One can get more information in monographs [1, 21, 25, 31, 36, 37, 51, 52] concerning these spaces. The questions of mathematics were studied regarding these spaces in varying degrees. The problems of harmonic analysis and approximation theory have been relatively well studied in Lebesgue spaces with variable summability index and Morrey spaces (see e.g., [1, 3, 5–11, 15, 16, 21, 22, 25, 26, 30, 31, 33, 36, 37, 49, 51–53, 58]). The

*Correspondence:b_bilalov@mail.ru
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same problems have begun to be studied in grand Lebesgue spaces and important results have been obtained in this direction (see e.g., [21, 37, 58]). Along with this, it should be noted that the solvability problems of differential equations and boundary value problems for analytic functions have also begun to be studied in nonstandart spaces (see e.g., [12–14, 19, 20, 23, 28, 29, 37, 45–47, 51, 52, 55, 59]).

This article is devoted to this direction, namely here we consider the solvability questions in weighted grand Sobolev spaces for elliptic equations. This space is nonseparable and therefore we can consider boundary value problems for elliptic equations (including other partial differential equations) in two settings: 1) separable case and 2) nonseparable case. In separable case the smooth functions are dense in considered space (or subspace) and it allows us to use the classical scheme for the investigation. In nonseparable case of considered space the classical scheme is not applicable for the validity of many classical facts concerning corresponding Sobolev spaces and it required to find other methods of establishing. It should be noted that early this fact was noticed in works [60, 61] by V.V.Zhikov. Also note that the works [12–14, 37, 59] belong to the case 1) and the works [19, 20, 23, 28, 29, 45, 46, 51, 52, 55, 59] belong to the case 2).

In this work we consider the Sobolev spaces generated by the norm of the power weighted grand Lebesgue spaces. It is considered m−th order elliptic equation with nonsmooth coefficients on bounded domain in R^n. This space is nonseparable and by using shift operator we define the separable subspace of it, in which infinitely differentiable functions are dense. The investigation needs to establish boundedness property of convolution regarding weighted grand Lebesgue spaces. Then on scheme of nonweighted case we establish solvability (strong sense) in the small of m−th order elliptic equations in power weighted grand Sobolev spaces.

It is well known that many of the classical facts with respect to the convolution operator in weighted spaces are not true. This circumstance creates serious difficulties in the study of solvability questions in one sense or another of differential equations in weighted Sobolev spaces. In this paper, in the case of a concrete weight (i.e. a power weight), we propose ways to overcome these difficulties in studying the solvability in the small of elliptic equations in separable subspaces of weighted grand Lebesgue spaces.

It should be noted that similar questions regarding partial differential equations in weighted Lebesgue and variable Lebesgue spaces were considered in works [17, 18, 27].

2. Auxiliary facts and notation

We need some necessary standard notations and facts from work [12].

2.1. Standard notation

Z_+ will be the set of nonnegative integers. $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ will denote the open ball in R^n centered at x_0, where $|x| = \sqrt{x_1^2 + ... + x_n^2}$, $x = (x_1, ..., x_n)$. $\Omega_r(x_0) = \Omega \bigcap B_r(x_0)$, $B_r = B_r(0)$, $\Omega_r = \Omega_r(0)$. $|M|$ will stand for the Lebesgue measure of the set M; $\partial\Omega$ will be the boundary of the domain Ω; $\bar{\Omega} = \Omega \bigcup \partial\Omega$; $M_1 \Delta M_2$ will denote the symmetric difference between the sets M_1 and M_2; $diam\,\Omega$ will stand for the diameter of the set Ω; $\rho(x; M)$ will be the distance between x and the set M; and $\|T\|_{X \to Y}$ will denote the norm of the operator T, acting boundedly from X to Y. For $\forall\varepsilon \in (0, q-1)$ we will denote $q_\varepsilon = q - \varepsilon$. q' is conjugate to q number: $\frac{1}{q} + \frac{1}{q'} = 1$.

2.2. Elliptic operator of $m-$th order

Let $\Omega\subset R^n$ be some bounded domain with the rectifiable boundary $\partial\Omega$. We will use the notation of [4]. $\alpha=(\alpha_1,...,\alpha_n)$ will be the multiindex with the coordinates $\alpha_k\in Z_+, \forall k=\overline{1,n}$; $\partial_i=\frac{\partial}{x_i}$ will denote the differentiation operator, $\partial^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}$. For every $\xi=(\xi_1,...,\xi_n)$ we assume $\xi^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_n^{\alpha_n}$. Let L be an elliptic differential operator of m-th order

$$L=\sum_{|p|\leq m}a_p(x)\,\partial^p, \tag{2.1}$$

where $p=(p_1,...,p_n)$, $p_k\in Z_+$, $\forall k=\overline{1,n}$, $a_p(\cdot)\in L_\infty(\Omega)$ are real functions, i.e. the characteristic form

$$Q(x,\xi)=\sum_{|p|=m}a_p(x)\,\xi^p$$

is definite a.e. for $x\in\Omega$. It is known that in this case m is even. Let $m=2m'$, and assume without loss of generality that

$$(-1)^{m'}Q(x,\xi)>0,\ \forall\xi\neq 0,\ \text{a.e. } x\in\Omega.$$

Consider the elliptic operator L_0:

$$L_0=\sum_{|p|=m}a_p^0\,\partial^p, \tag{2.2}$$

with the constant coefficients a_p^0.

In what follows, by solution of the equation $Lu=f$ we mean a strong solution (see [4]). We will need the following classical result of [4].

Theorem 2.1 *[4] For an arbitrary m-th order elliptic operator L_0 of the form (2.2) with the constant coefficients, the function $J(x)$ can be constructed which has the following properties:*

i) If n is odd or if n is even and $n>m$, then

$$J(x)=\frac{\omega(x)}{|x|^{n-m}},$$

where $\omega(x)$ is a positive homogeneous function of degree zero (i.e. $\omega(tx)=\omega(x)$, $\forall t>0$). If n is even and $n\leq m$, then

$$J(x)=q(x)\log|x|+\frac{\omega(x)}{|x|^{n-m}},$$

where q is a homogeneous polynomial of degree $m-n$.

ii) The function $J(x)$ satisfies (in a generalized sense) the equation

$$L_0J(x)=\delta(x),$$

(δ is the Dirac function) so the following equality is true for every infinitely differentiable function $\varphi(\cdot)$ with compact support

$$\varphi(x)=\int[L_0\varphi(y)]\ J(x-y)\,dy=L_0\int\varphi(y)\,J(x-y)\,dy.$$

Let us consider the elliptic operator (2.1) and assign to it a "tangential operator"

$$L_{x_0} = \sum_{|p|=m} a_p\left(x_0\right) \partial^p, \tag{2.3}$$

at every point $x_0 \in \Omega$. Denote by $J_{x_0}(\cdot)$ the fundamental solution of the equation $L_{x_0}\varphi = 0$ in accordance with Theorem 2.1. The function $J_{x_0}(\cdot)$ is called a parametrics for the equation $L\varphi = 0$ with a singularity at the point x_0. Let

$$S_{x_0}\varphi = \psi(x) = \int J_{x_0}(x-y)\,\varphi(y)\,dy,$$

and assume

$$T_{x_0} = S_{x_0}\left(L_{x_0} - L\right). \tag{2.4}$$

In establishing the existence of the solution to the equation $Lu = f$, the following plays a significant role.

Lemma 2.2 *[4] If φ has compact support, then*

$$\varphi = T_{x_0}\varphi + S_{x_0}L\varphi,$$

and if

$$\varphi = T_{x_0}\varphi + S_{x_0}f,$$

then $L\varphi = f$.

2.3. Weighted grand Sobolev spaces $W^m_{q),\rho}(\Omega)$ and ${}_sW^m_{q),\rho}(\Omega)$

Firstly, let us define the grand Lebesgue space $L_{q)}(\Omega)$. Grand Lebesgue space $L_{q)}(\Omega)$, $1 < q < +\infty$, where $\Omega \subset R^n$ – bounded domain, is a Banach space of (Lebesgue) measurable functions f on Ω with norm

$$\|f\|_{L_{q)}(\Omega)} = \sup_{0<\varepsilon<q-1}\left(\varepsilon\int_\Omega |f|^{q-\varepsilon}\,dx\right)^{\frac{1}{q-\varepsilon}}.$$

The following continuous embeddings hold

$$L_q(\Omega) \subset L_{q)}(\Omega) \subset L_{q-\varepsilon_0}(\Omega),$$

where $\varepsilon_0 \in (0, q-1)$ is an arbitrary number. Let $\rho : \Omega \to [0, +\infty]$ be a weight function, i.e. ρ – measurable and $\left|\rho^{-1}\{0; +\infty\}\right| = 0$. The weighted case of $L_{q)}(\Omega)$ we define by norm

$$\|f\|_{L_{q),\rho}(\Omega)} = \sup_{0<\varepsilon<q-1}\left(\varepsilon\int_\Omega |f|^{q-\varepsilon}\,\rho dx\right)^{\frac{1}{q-\varepsilon}},$$

and corresponding space is denoted by $L_{q),\rho}(\Omega)$. Also the corresponding Sobolev space $W^m_{q),\rho}(\Omega)$ is defined by norm

$$\|f\|_{W^m_{q),\rho}(\Omega)} = \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L_{q),\rho}(\Omega)}.$$

Let us recall the definition of regular Borel measure.

Definition 2.3 *Let* $(M;\tau)$ *be a Hausdorff topological space, and let* $\mathscr{B}$ *be the* σ*-algebra of its Borel sets. A measure* $\mu:\mathscr{B}\to[0,\infty]$ *is called a regular Borel measure if it satisfies the following properties:*

i) $\mu(K)<\infty$ *for every compact set* K*;*

ii) If $A\in\mathscr{B}$*, then*

$$\mu(A)=\inf\{\mu(C): C \text{ is open and } B\subset C\};$$

iii) If C *is an open subset of* M*, then*

$$\mu(C)=\sup\{\mu(K): K \text{ is compact and } K\subset C\}.$$

In the sequel we will use the following well known result (see e.g., [2, p. 262]).

Theorem 2.4 *[2] Let* μ *be a regular Borel measure on a Hausdorf locally compact topological space* M*. Then the collection of all continuous functions with compact support is norm dense in* $L_p(\mu)$ *for every* $1\le p<\infty$*, where* $L_p(\mu)$ *is a Banach space of measurable functions on* M *with norm*

$$\|f\|_{L_p(\mu)}=\left(\int_M |f|^p\,d\mu\right)^{\frac{1}{p}}.$$

Below in this section we will assume that every function defined on Ω is extended by zero to $R^n\backslash\overline{\Omega}$. Let T_δ be a shift operator, i.e. $(T_\delta f)(x)=f(x+\delta)$, $\forall x\in\Omega$, where $\delta\in R^n$ is an arbitrary vector. Let

$${}_sL_{q),\rho}(\Omega)=\left\{f\in L_{q),\rho}(\Omega): \|T_\delta f-f\|_{L_{q),\rho}(\Omega)}\to 0,\ \delta\to 0\right\}.$$

With the norm $\|\cdot\|_{q),\rho}$ the space ${}_sL_{q),\rho}(\Omega)$ becomes a Banach space (i.e. the subspace of $L_{q),\rho}(\Omega)$), and moreover, the following continuous embeddings hold

$$L_{q,\rho}(\Omega)\subset {}_s L_{q),\rho}(\Omega)\subset L_{q),\rho}(\Omega)\subset L_{q_\varepsilon,\rho},\ \ q_\varepsilon=q-\varepsilon,$$

for $\forall\varepsilon\in(0,\ q-1)$, if $\rho\in L_1(\Omega)$.

For more information about this and other facts one can see, e.g., works [34, 50, 56]. Assume

$$W_{q)}(\Omega)=\bigcup_{\varepsilon\in(0,\ q-1)} L_{q'_\varepsilon}(\Omega).$$

It is valid the following easy proved.

Lemma 2.5 *Let* $\rho^{-1}\in W_{q)}(\Omega)$*. Then the following continuous embedding* $L_{q),\rho}(\Omega)\subset L_1(\Omega)$ *is true.*

Proof Let $\varepsilon_0\in(0,\ q-1)$ such that $\rho\in L_{q'_{\varepsilon_0}}(\Omega)$. Applying the Holder inequality we have

$$\int_\Omega |f|\,dx=\int_\Omega |f\rho|\,\rho^{-1}dx\le \varepsilon_0^{\frac{1}{q_\varepsilon}}\|f\rho\|_{L_{q_{\varepsilon_0}}(\Omega)}\,\varepsilon_0^{-\frac{1}{q_{\varepsilon_0}}}\left\|\rho^{-1}\right\|_{L_{q'_{\varepsilon_0}}(\Omega)}\le c_{\varepsilon_0}\|f\|_{L_{q),\rho}(\Omega)}.$$

Lemma is proved. □

Completely analogously to the nonweighted case it is proved the following lemma on density of the differentiable functions in ${}_sL_{q),\rho}(\Omega)$.

Lemma 2.6 *Let* $\rho \in L_1(\Omega)$. *Then* $\overline{C_0^\infty}(\Omega) = {}_s L_{q),\rho}(\Omega)$ *(the closure is taken regarding the norm* $\|\cdot\|_{q),\rho}$*).*

Firstly, let us prove the Minkowski inequality for $L_{q),\rho}(\Omega)$.

Proposition 2.7 *(Minkowski inequality) Let* $(X; X_\sigma; \mu)$ *be a measurable space with a* σ *– additive measure* $\mu(\cdot)$ *on a* σ *– algebra* X_σ *of subsets* X *and let* $\rho(\cdot)$ *be a weight function on* Ω, $1 < q < +\infty$. *Then*

$$\left\| \int_X F(x;\cdot)\, d\mu(x) \right\|_{L_{q),\rho}(\Omega)} \leq \int_X \|F(x;\cdot)\|_{L_{q),\rho}(\Omega)}\, d\mu(x). \tag{2.5}$$

Proof Let $\varepsilon \in (0, q-1)$ be an arbitrary number. By using the Minkowski inequality for integrals in $L_{q_\varepsilon,\rho}(\Omega)$, $q_\varepsilon = q - \varepsilon$, we have

$$\left\| \int_X F(x;\cdot)\, d\mu(x) \right\|_{L_{q_\varepsilon,\rho}(\Omega)} \leq \int_X \|F(x;\cdot)\|_{L_{q_\varepsilon,\rho}(\Omega)}\, d\mu(x).$$

Consequently

$$\varepsilon^{\frac{1}{q_\varepsilon}} \left\| \int_X F(x;\cdot)\, d\mu(x) \right\|_{L_{q_\varepsilon,\rho}(\Omega)} \leq$$

$$\leq \int_X \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q_\varepsilon}} \|F(x;\cdot)\|_{L_{q_\varepsilon,\rho}(\Omega)}\, d\mu(x) = \int_X \|F(x;\cdot)\|_{L_{q),\rho}(\Omega)}\, d\mu(x).$$

It immediately follows from here the inequality (2.5).

Proposition is proved. □

Using inequality (2.5) we can prove the Lemma 2.6.

Proof of Lemma 2.6 Let $\omega_\varepsilon(\cdot)$ be an ε – cap

$$\omega_\varepsilon(x) = \begin{cases} c_\varepsilon \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}\right), & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon, \end{cases}$$

where c_ε is a constant s.t.

$$\int_{R^n} \omega_\varepsilon(x)\, dx = 1.$$

Let $f \in {}_sL_{q),\rho}(\Omega)$. Consider the convolution

$$f_\varepsilon(x) = (f * \omega_\varepsilon)(x) = \int_{R^n} f(x-y)\, \omega_\varepsilon(y)\, dy.$$

It is evident that $f_\varepsilon \in C^\infty\left(\overline{\Omega}\right)$. We have

$$\|f_\varepsilon - f\|_{L_{q),\rho}(\Omega)} = \left\| \int_{R^n} \omega_\varepsilon(y)\left(f(\cdot - y) - f(\cdot)\right) dy \right\|_{L_{q),\rho}(\Omega)} \leq$$

$$\leq \int_{R^n} \omega_\varepsilon(y) \|f(\cdot - y) - f(\cdot)\|_{L_{q),\rho}(\Omega)}\, dy \leq \sup_{|y|<\varepsilon} \|f(\cdot - y) - f(\cdot)\|_{L_{q),\rho}(\Omega)} \to 0,\ \varepsilon \to 0.$$

On the other hand, since $f_\varepsilon \in L_{q,\rho}(\Omega)$, it is evident that, if $C_0^\infty(\Omega)$ is dense in $L_{q,\rho}(\Omega)$, then $\exists \{g_n\} \subset C_0^\infty(\Omega)$:

$$\|f_\varepsilon - g_n\|_{L_{q),\rho}(\Omega)} \to 0, \ n \to \infty.$$

Consequently

$$\|f - g_n\|_{L_{q),\rho}(\Omega)} \leq \|f_\varepsilon - g_n\|_{L_{q),\rho}(\Omega)} + \|f_\varepsilon - f\|_{L_{q),\rho}(\Omega)}.$$

Since $\exists c > 0$:

$$\|\varphi\|_{L_{q),\rho}(\Omega)} \leq c\,\|\varphi\|_{L_{q),\rho}(\Omega)}, \ \forall \varphi \in L_{q),\rho}(\Omega),$$

We have

$$\|f_\varepsilon - g_n\|_{L_{q),\rho}(\Omega)} \leq c\,\|f_\varepsilon - g_n\|_{L_{q),\rho}(\Omega)} \to 0, \ n \to \infty.$$

It is not hard to see that the measure $\mu(E) = \int_E \rho dx$ on Ω under the condition $\rho \in L_1(\Omega)$ is regular Borel measure. Then it follows from Theorem 2.4 that C_0^∞ is dense in $L_{q,\rho}(\Omega)$. As a result it follows the density of $C_0^\infty(\Omega)$ in ${}_sL_{q),\rho}(\Omega)$.

Lemma is proved.

The following separable weighted Sobolev space is also defined

$${}_sW_{q)}^m(\Omega) = \left\{ f \in W_{q)}^m(\Omega) : \|T_\delta f - f\|_{W_{q)}^m(\Omega)} \to 0, \ \delta \to 0 \right\}.$$

In obtaining main results we need the Makenhoupt class of weights $A_q(\Omega)$. We will say that $\rho \in A_q(\Omega)$ if the weight ρ satisfies the following condition

$$\sup_{0<\varepsilon<q-1} \frac{1}{|E|} \left\| \rho^{\frac{1}{q}} \chi_E \right\|_{L_q(\Omega)} \left\| \rho^{-\frac{1}{q}} \chi_E \right\|_{L_{q'}(\Omega)} < +\infty.$$

Consider the following singular kernel

$$k(x) = \frac{\omega(x)}{|x|^n},$$

where $\omega(x)$ is a positive homogeneous function of degree zero, which is infinitely differentiable and satisfies

$$\int_{|x|=1} \omega(x)\, d\sigma = 0,$$

$d\sigma$ being a surface element on the unit sphere. Denote by K the corresponding singular integral

$$(Kf)(x) = k * f(x) = \int_\Omega f(y)\, k(x-y)\, dy.$$

The following theorem is proved in [37].

Theorem 2.8 *[37] The singular operator* $K \in \left[L_{q),\rho}(\Omega)\right] \Leftrightarrow \rho \in A_q(\Omega)$.

Let us prove the following.

Theorem 2.9 *Let* $\rho \in A_q(\Omega)$, $1 < q < +\infty$. *Then* $K \in \left[{}_sL_{q),\rho}(\Omega)\right]$.

Proof Let $\rho \in A_q(\Omega) \Rightarrow K \in \left[L_{q),\rho}(\Omega)\right]$. It suffices to show that

$$\|T_\delta Kf - Kf\|_{L_{q),\rho}(\Omega)} \to 0,\ \delta \to 0,\ \forall f \in {}_sL_{q),\rho}(\Omega). \tag{2.6}$$

Taking into account $f(x) = 0,\ \forall x \in R^n \backslash \overline{\Omega}$, we have

$$(T_\delta Kf)(x) = (Kf)(x+\delta) = \int_{R^n} k(x+\delta-y) f(y)\, dy =$$

$$= \int_{R^n} k(x-y) f(y+\delta)\, dy = (KT_\delta f)(x).$$

Then it follows from (2.6) that

$$\|T_\delta Kf - Kf\|_{L_{q),\rho}(\Omega)} = \|K(T_\delta f - f)\|_{L_{q),\rho}(\Omega)} \le \|K\|_{\left[L_{q),\rho}(\Omega)\right]} \|T_\delta f - f\|_{L_{q),\rho}(\Omega)} \to 0,\ \delta \to 0.$$

Theorem is proved. □

In the following we need some facts about boundedness of convolution in weighted Lebesgue spaces. Let

$$\|f\|_{L_{p,\rho}(\Omega)} = \left(\int_\Omega |f(x)^p|\, \rho(x)\, dx\right)^{\frac{1}{p}},\ 1 \le p < \infty.$$

Corresponding weighted Lebesgue space is denoted by $L_{p,\rho}(\Omega)$. The case $\Omega = R^n$ is denoted as $L_{p,\rho}$ and accept notation $L(p;\alpha)$ for $\rho(x) = |x|^{\alpha p}$. Let us consider the convolution of two functions on R^n:

$$(f * g)(x) = \int_{R^n} f(x-y)\, g(y)\, dy,\ x \in R^n.$$

Consistent use will be made of expressions of the form $X * Y \subset Z$ involving function spaces X, Y and Z. These indicate that whenever $f \in X,\ g \in Y$, then $f * g \in Z$ with

$$\|f * g\|_Z \le K \|f\|_X \|g\|_Y,$$

the positive constant K is independent of f and g. It is valid the following theorem, proved in [35].

Theorem 2.10 *[35]*

$$L(p_1;\alpha) * L(p_2;\beta) \subset L(p_3;-\gamma),$$

provided:

i) $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{\alpha+\beta+\gamma}{n} - 1;\ 1 < p_1, p_2, p_3 < \infty;\ \frac{1}{p_3} \le \frac{1}{p_1} + \frac{1}{p_2};$

ii) $\alpha < n\left(1 - \frac{1}{p_1}\right);\ \beta < n\left(1 - \frac{1}{p_2}\right);\ \gamma < \frac{n}{p_3};$

iii) $\alpha + \beta,\ \beta + \gamma,\ \gamma + \alpha \ge 0.$

Let $r>0$ be some positive fixed number and let us consider the following integral

$$I(x)=\int_{|y|<r}|x-y|^{m-n-|p|}\,|\psi(y)|\,dy,$$

where $p:|p|<m$ is some multiindex. Let us try to apply the Theorem 2.10 to these integral. We will use the following well known formula for $\alpha_0<n$:

$$\int_{|x-y|<r}\frac{dy}{|x-y|^{\alpha_0}}=\frac{|B_1|}{n-\alpha_0}r^{n-\alpha_0},\ \forall x\in R^n,$$

where $|B_1|$ is a volume of a unit ball B_1 in R^n (see, e.g., [41, p. 19]). Let us define

$$f(x)=\begin{cases}|x|^{m-n-|p|}\ , & |x|<r,\\ 0\ , & |x|\geq r,\end{cases}$$

$$g(x)=\begin{cases}\psi(x)\ , & |x|<r,\\ 0\ , & |x|\geq r.\end{cases}$$

It is evident that $\mathrm{supp}\,(\mathrm{f}*\mathrm{g})\subset \mathrm{B}_{2\mathrm{r}}$ and $I(x)=(f*g)(x)$, $\forall x\in B_r$. Let $q=1+\delta$, where $\delta>0$ is sufficiently small number and assume $\alpha_1=n-m+|p|$. It is evident that $n-m\leq\alpha_1\leq n-1$. Consider

$$\|f\|^{p_2}_{L(p_2;\beta)}=\int_{R^n}|f|^{p_2}\,|x|^{p_2\beta}\,dx=\int_{|x|<r}\frac{dx}{|x|^{(\alpha_1-\beta)p_2}}=C_{\beta;p_2}r^{n-(\alpha_1-\beta)p_2},$$

where $C_{\beta;p_2}=\frac{|B_1|}{n-(\alpha_1-\beta)p_2}$. Applying Theorem 2.10 to the integral $I(\cdot)$ we have

$$\|I(\cdot)\|_{L(p_3;-\gamma)}\leq C\,\|g\|_{L(p_1;\alpha)}\,\|f\|_{L(p_2;\beta)}\,. \tag{2.7}$$

Let us choose parameters $p_k;\alpha;\beta;\gamma,\ k=\overline{1,3}$; according to the conditions *i) – iii)* of Theorem 2.10. Let $\alpha=-\gamma; p_3=q; p_1=q-\varepsilon_0$, where $\varepsilon_0\in(0,q-1)$ and let $q>1$ be some number. It is evident that $\frac{1}{q}\leq\frac{1}{q-\varepsilon_0}+\frac{1}{p_2}$. Let $\beta=n\left(1-\frac{1}{p_2}\right)-\delta_0,\ \delta_0>0$ is sufficiently small number. Consequently

$$\frac{1}{q}=\frac{1}{q-\varepsilon_0}+\frac{1}{p_2}+1-\frac{1}{p_2}-\frac{\delta_0}{n}-1=\frac{1}{q-\varepsilon_0}-\frac{\delta_0}{n}.$$

Therefore we choose ε_0 according to these relations, i.e. $\varepsilon_0=\frac{\delta_0q^2}{n+\delta_0q}$. It is not hard to check that for sufficiently small $\delta_0>0$ it holds $\varepsilon_0\in(0,q-1)$. For parameter α we obtain the following conditions

$$\begin{aligned}&\gamma<\tfrac{n}{q}\Rightarrow\alpha>-\tfrac{n}{q};\\ &\alpha<n\left(1-\tfrac{1}{p_1}\right)\Rightarrow\alpha<n\left(1-\tfrac{1}{q-\varepsilon_0}\right);\\ &\alpha\geq-\beta\Rightarrow\alpha\geq-n\left(1-\tfrac{1}{p_2}\right)+\delta_0;\\ &\beta+\gamma\geq0\Rightarrow-\alpha\geq-\beta\Rightarrow\alpha\leq\beta\Rightarrow\alpha\leq n\left(1-\tfrac{1}{p_2}\right)-\delta_0\,.\end{aligned}$$

Let us take $p_2=q-\varepsilon_0$. Then as a result we obtain the following condition for α:

$$A_{n;q}<\alpha\leq n\left(1-\frac{1}{q-\varepsilon_0}\right)-\delta_0,$$

where $A_{n;q}=\max\left\{-\frac{n}{q};-n\left(1-\frac{1}{q-\varepsilon_0}\right)+\delta_0\right\}$. Summing all this conditions we have

$$\varepsilon_0=\frac{\delta_0 q^2}{n+\delta_0 q};\qquad A_{n;q}<\alpha\leq n\left(1-\frac{1}{q-\varepsilon_0}\right)-\delta_0, \tag{2.8}$$

where $\delta_0 : 0<\delta_0<<+\infty$ is some number. Consequently, it follows from relations (2.8) the validity of conditions *i) – iii)* of Theorem 2.10 regarding to parameters α,β and γ. Consider

$$n-(\alpha_1-\beta)\,p_2=n-\left(n-m+|p|-n\left(1-\frac{1}{p_2}\right)+\delta_0\right)p_2=(m-|p|+\delta_0)\,p_2.$$

It is evident that if $|p|<m$, then $(m-|p|+\delta_0)\,p_2>0$ for $0<\delta_0<1$. Taking into account the inequality (2.7) we have

$$\|I(\cdot)\|_{L(q;\alpha)}\leq c\,\|f\|_{L(p_2;\beta)}\,\|g\|_{L(q-\varepsilon_0;\alpha)}\leq cr^{m-|p|+\delta_0}\,\|g\|_{L_{q),\alpha}(B_r)}\,. \tag{2.9}$$

It follows from condition (2.8) that $\rho(x)=|x|^{\alpha q}\in L_1(B_r)$. Then from the following chain of continuous embeddings

$$L_{q,\rho}(\Omega)\subset L_{q),\rho}(\Omega)\subset L_{q-\varepsilon_0,\rho}(\Omega)\,,$$

for $\forall\varepsilon_0\in(0,q-1)$, we get

$$\|I(\cdot)\|_{L_{q),\rho}(r)}\leq C_r\,\|I(\cdot)\|_{L_{q,\rho}(r)}\,,$$

where the constant C_r depends only on $r>0$. If $r\leq r_0$ then we can take C_r independently of r. In fact

$$\|I(\cdot)\|_{L_{q),\rho}(r)}=\sup_{0<\varepsilon<q-1}\left(\varepsilon\int_{B_r}|I(\cdot)|^{q-\varepsilon}\rho dx\right)^{\frac{1}{q-\varepsilon}}\leq$$

$$\leq\left|\alpha_\varepsilon=\frac{q}{q-\varepsilon}\right|\leq\sup_{0<\varepsilon<q-1}\left(\varepsilon\int_{B_r}|I(\cdot)|^{q}\rho dx\right)^{q}\left(\int_{B_r}\rho dx\right)^{\frac{1}{\alpha'_\varepsilon(q-\varepsilon)}},$$

where

$$\frac{1}{\alpha_\varepsilon}+\frac{1}{\alpha'_\varepsilon}=1\Rightarrow\frac{1}{\alpha'_\varepsilon}=\frac{\varepsilon}{q}\Rightarrow\frac{1}{\alpha'_\varepsilon(q-\varepsilon)}=\frac{\varepsilon}{q(q-\varepsilon)}\leq\frac{q-1}{q^2}.$$

As a result we obtain

$$\|I(\cdot)\|_{L_{q),\rho}(r)}\leq C_{r_0}\,\|I(\cdot)\|_{L_{q,\rho}(r)}\,,$$

where $C_{r_0}=(q-1)^{\frac{1}{q}}\left(\int_{B_{r_0}}\rho dx\right)^{\frac{q-1}{q^2}}$.

Therefore it is valid the following.

Lemma 2.11 *Let $\rho(x)=|x|^{\alpha q}$, $A_{n;q}=\max\left\{-\frac{n}{q};-n\left(1-\frac{1}{q-\varepsilon_0}\right)+\delta_0\right\}$, where $\varepsilon_0=\frac{\delta_0 q^2}{n+\delta_0 q}$ and $\delta>0$ be sufficiently small number. If it holds*

$$A_{n;q}<\alpha\leq n\left(1-\frac{1}{q-\varepsilon_0}\right)-\delta_0,$$

then for $\forall r_0>0$, there exists $C_{r_0}>0:\forall r\in(0,r_0)$, such that it is valid

$$\|I(\cdot)\|_{L_{q),\rho}(r)}\leq C_{r_0}r^{m-|p|+\delta_0}\,\|\psi\|_{L_{q),\rho}(r)}\,.$$

The validity of lemma immediately follows from inequality (2.9) and expression for $g(\cdot)$.

3. The space $N^m_{q),\rho}(\Omega)$ Main Lemma

In ${}_sW^m_{q),\rho}(\Omega)$ along with norm $\|\cdot\|_{W^m_{q),\rho}(\Omega)}$ let us consider the following norm

$$\|f\|_{N^m_{q),\rho}(\Omega)} = \sum_{|p|\le m} d_\Omega^{|p|-\frac{n}{q}} \|\partial^p f\|_{L_{q),\rho}(\Omega)},$$

where $d_\Omega = diam\ \Omega$ and the corresponding space we will denote by $N^m_{q),\rho}(\Omega)$. Accept $N^0_{q),\rho}(\Omega) = N_{q),\rho}(\Omega)$. It is not difficult to see that the norms $\|\cdot\|_{W^m_{q),\rho}(\Omega)}$ and $\|\cdot\|_{N^m_{q),\rho}(\Omega)}$ are equivalent, and therefore the collection of functions of spaces $W^m_{q),\rho}(\Omega)$ and $N^m_{q),\rho}(\Omega)$ coincide. In addition, assume

$${}_sN^m_{q),\rho}(\Omega) = \left\{ f \in N^m_{q,\rho}(\Omega) : \|T_\delta f - f\|_{N^m_{q,\rho}(\Omega)} \to 0, \delta \to 0 \right\}.$$

Accept the following.

Definition 3.1 *We will say that the operator L has the property $P_{x_0})$ if its coefficients satisfy the conditions: i) $a_p \in L_\infty(B_r(x_0))$, $\forall |p| \le m$, for some $r > 0$; ii) $\exists r > 0$: for $|p| = m$ the coefficient $a_p(\cdot)$ coincides a.e. in $B_r(x_0)$ with some function bounded and continuous at the point $x_0 \in \Omega$.*

It is absolutely clear that if $a_p \in C(\Omega)$, $\forall |p| \le m$, then L has the property $P_{x_0})$ for $\forall x_0 \in \Omega$.

Let us consider the m-th order elliptic operator L with the coefficients $a_p(x)$ defined by (2.1), and the corresponding operator T_{x_0} defined by (2.4). Denote the operators S_{x_0}, L_{x_0} and T_{x_0}, corresponding to the point $x_0 = 0$, by S_0, L_0 and T_0, respectively. Let us prove the following key lemma.

Main Lemma. *Let m–th order elliptic operator L have the property $P_{x_0})$ at the point $x_0 \in \Omega$. Let $\varphi \in N^m_{q)}(B_r(x_0))$ and φ vanish in a neighbourhood of $|x - x_0| = r$. Let the weight $\rho(x) = |x|^{\alpha q}$ satisfy all conditions of Lemma 2.11. Then it is valid*

$$\|T_{x_0}\varphi\|_{N^m_{q),\rho}(B_r(x_0))} \le \sigma(r) \|\varphi\|_{N^m_{q),\rho}(B_r(x_0))},$$

where the function $\sigma(r) \to 0$, $r \to 0$, depends only on the ellipticity constant L_{x_0}, on the coefficients of L.

Proof We will follow the scheme of the work [12]. By the value of the function $a_p(\cdot)$ at the point $x_0 = 0$ for $|p| = m$ we will mean the value of the corresponding function from the property $P_{x_0})$ continuous at this point. For simplicity, we assume that $n \ge 3$ and is odd, with $r < 1$. Following [4], we assume

$$\psi = (L_0 - L)\varphi = \psi_1 + \psi_2,$$

$$\psi_1(x) = \sum_{|p|=m} (a_p(0) - a_p(x)) \partial^p \varphi(x) = \sum_{|p|=m} b_p(x) \partial^p \varphi(x),$$

where $b_p(x) = a_p(0) - a_p(x)$ and $\psi_2(x) = -\sum_{|p|<m} a_p(x) \partial^p \varphi(x)$. Obviously, $b_p(0) = 0$, and consequently, $\operatorname*{supvrai}_{|x|<r} |b_p(x)| = \bar{\bar{o}}(1)$, $r \to 0$. It is obvious that

$$\|\psi_1\|_{L_{q),\rho}(r)} \le \bar{\bar{o}}(1) \sum_{|p|=m} \|\partial^p \varphi\|_{L_{q),\rho}(r)}, \ \rho \to 0.$$

Let $\chi = T_0\varphi$. Considering the expression for T_0, we obtain

$$\chi = S_0 (L_0 - L) \varphi = S_0\psi = \int_{B_r} J_0 (x-y) \psi (y) dy.$$

For $|p| < m$ we have

$$\partial^p \chi (x) = \int_{B_r} \partial_x^p J_0 (x-y) \psi (y) dy.$$

In this case, the following estimate is valid for the derivatives $\partial^p J_0$:

$$|\partial^p J_0 (x)| \le c |x|^{m-n-|p|}.$$

Then for $\partial^p \chi$ we obtain

$$|\partial^p \chi (x)| \le c \int_{|y|<r} |x-y|^{m-n-|p|} |\psi (y)| dy. \tag{3.1}$$

In the sequel we will use the following well-known formula for $\alpha < n$:

$$\int_{|x-y|<r} \frac{dy}{|x-y|^{\alpha}} = \frac{|B_1| r^{n-\alpha}}{n-\alpha}, \ \forall x \in R^n, \tag{3.2}$$

where $|B_1|$ is a volume of a unit ball B_1 in R^n (see, e.g., [41, p. 19]. Let $\varepsilon \in (0, q-1)$ be some number. Apply Theorem 2.10 to the integral

$$I (x) = \int_{|y|<r} |x-y|^{m-n-|p|} |\psi (y)| dy.$$

Taking into account the Lemma 2.11 for $\|\partial^p \chi\|_{L_{q),\rho}(r)}$ we obtain the estimate

$$\|\partial^p \chi\|_{L_{q),\rho}(r)} \le c_{r_0} r^{m-|p|+\delta_0} \|\psi\|_{L_{q),\rho}(r)},$$

$\forall r \in (0, r_{.0})$, where $\delta_0 > 0$ – sufficiently small number and the constant c_{r_0} is independent of r. It follows directly from the above inequality that

$$r^{|p|} \|\partial^p \chi\|_{L_{q),\rho}(r)} \le C_{r_0} r^m \|\psi\|_{L_{q),\rho}(r)}, \ \forall p : |p| < m.$$

Taking into account the estimate for $\|\psi_1\|_{L_{q)}(r)}$, we obtain

$$r^{|p|} \|\partial^p \chi\|_{L_{q),\rho}(r)} \le c r^m \left(\|\psi_1\|_{L_{q),\rho}(r)} + \|\psi_2\|_{L_{q),\rho}(r)} \right) \le$$

$$\le c r^m \left(\bar{\bar{o}} (1) \sum_{|p|=m} \|\partial^p \varphi\|_{L_{q),\rho}(r)} + \sum_{|p|<m} \|\partial^p \varphi\|_{L_{q),\rho}(r)} \right).$$

$$r^{|p|-\frac{n}{q}} \|\partial^p \chi\|_{L_{q),\rho}(r)} \le$$

$$\leq c\left(\bar{\bar{o}}(1)\sum_{|p|=m} r^{m-\frac{n}{q}}\left\|\partial^p\varphi\right\|_{L_{q),\rho}(r)} + r^{m-1}\sum_{|p|<m} r^{|p|-\frac{n}{q}}\left\|\partial^p\varphi\right\|_{L_{q),\rho}(r)}\right) =$$

$$= \bar{\bar{o}}(1)\left\|\varphi\right\|_{N_{q),\rho}(r)}\,,\ r\to 0.$$

Thus, the following relation is true

$$r^{|p|-\frac{n}{q}}\left\|\partial^p\chi\right\|_{L_{q),\rho}(r)} \leq \bar{\bar{o}}(1)\left\|\varphi\right\|_{N_{q),\rho}(r)}\,,\ r\to 0\,,\ \forall\,|p|<m. \tag{3.3}$$

Let us estimate $\left\|\partial^p\chi\right\|_{L_{q),\rho}(r)}$ for $|p|=m$. In this case we have

$$\partial^p\chi(x) = \int_{B_r}\partial^p J_0(x-y)\,\psi(y)\,dy + c\psi(x)\,, \tag{3.4}$$

(see [4, p. 235]). Then $\partial^p J_0(x)$ is a singular kernel. From conditions of Lemma 2.11 on parameter α it follows that $\rho\in A_q(r)$, where $\rho(x)=|x|^{\alpha\,q}$. Concerning this fact one can see e.g., the work [24]. Then from Theorem 2.8 it follows that $K\in\left[L_{q),\rho}(r)\right]$ Applying Theorem 2.8 from (3.4) we obtain

$$\left\|\partial^p\chi\right\|_{L_{q),\rho}(r)} \leq c\left\|\psi\right\|_{L_{q),\rho}(r)}\,,\ \ |p|=m.$$

Consequently

$$r^{m-\frac{n}{q}}\left\|\partial^p\chi\right\|_{L_{q),\rho}(r)} \leq$$

$$\leq c\left(\bar{\bar{o}}(1)\sum_{|p|=m} r^{m-\frac{n}{q}}\left\|\partial^p\varphi\right\|_{L_{q),\rho}(r)} + r\sum_{|p|<m} r^{|p|-\frac{n}{q}}\left\|\partial^p\varphi\right\|_{L_{q),\rho}(r)}\right) =$$

$$= \bar{\bar{o}}(1)\left\|\varphi\right\|_{N_{q),\rho}(r)}\,,\ r\to 0\,.$$

Then, taking into account (3.3), as a result we have

$$\left\|\chi\right\|_{N_{q),\rho}(r)} = \left\|T_0\varphi\right\|_{N_{q),\rho}(r)} \leq \bar{\bar{o}}(1)\left\|\varphi\right\|_{N_{q),\rho}(r)}\,,\ r\to 0\,.$$

The lemma is proved. □

4. Local existence theorem

First let us prove that the elliptic operator L is bounded from $N^m_{q),\rho}(\Omega)$ to ${}_sL_{q),\rho}(\Omega)$. Namely, it is valid the following.

Lemma 4.1 *Let the coefficients $a_p(\cdot)$ of the elliptic operator L satisfy the condition $a_p(\cdot)\in L_\infty(\Omega)$, $\forall p:$ $|p|\leq m$, and the weight $\rho(x)=|x|^{\alpha q}$ satisfy all conditions of Lemma 2.11. Then $L\in\left[N^m_{q),\rho}(\Omega);{}_sN_{q),\rho}(\Omega)\right]$.*

Proof This lemma is proved completely analogously to the Lemma 4.1 of the work [12]. For completeness we will give full proof. It suffices to prove that $f \in {}_sN_{q),\rho}(\Omega)$ and $\varphi \in L_\infty(\Omega)$ imply $\varphi f \in {}_sN_{q),\rho}(\Omega)$. It is absolutely clear that $\varphi f \in N_{q),\rho}(\Omega)$. Therefore it suffices to prove the relation

$$\|T_\delta(\varphi f) - \varphi f\|_{N_{q),\rho}(\Omega)} \to 0\,,\ \delta \to 0.$$

We have

$$\|\varphi(\cdot+\delta)\, f(\cdot+\delta) - \varphi(\cdot)\, f(\cdot)\|_{N_{q),\rho}(\Omega)} \le \|\varphi(\cdot+\delta)\,(f(\cdot+\delta) - f(\cdot))\|_{N_{q),\rho}(\Omega)} +$$

$$+\|(\varphi(\cdot+\delta) - \varphi(\cdot))\, f(\cdot)\|_{N_{q),\rho}(\Omega)} = \Delta_\delta^{(1)} + \Delta_\delta^{(2)},$$

where $\Delta_\delta^{(1)}$ and $\Delta_\delta^{(2)}$ are the corresponding terms on the right-hand side of this inequality. For $\Delta_\delta^{(1)}$ we have

$$\Delta_\delta^{(1)} \le \|\varphi\|_\infty \|f(\cdot+\delta) - f(\cdot)\|_{N_{q),\rho}(\Omega)} \to 0\,,\ \delta \to 0.$$

Further, let $\varepsilon > 0$ be an arbitrary number. As $\overline{C_0^\infty(\Omega)} = {}_sN_{q),\rho}(\Omega)$ (the closure in $N_{q),\rho}(\Omega)$), it is clear that $\exists g \in C_0^\infty(\Omega)$:

$$\|f - g\|_{N_{q),\rho}(\Omega)} < \varepsilon.$$

Consequently, for $\Delta_\delta^{(2)}$ we have

$$\Delta_\delta^{(2)} = \|(\varphi(\cdot+\delta) - \varphi(\cdot))\,(f(\cdot) - g(\cdot) + g(\cdot))\|_{N_{q),\rho}(\Omega)} \le 2\,\|\varphi\|_\infty\, \varepsilon + \Delta_\delta^{(3)},$$

where

$$\Delta_\delta^{(3)} = \|(\varphi(\cdot+\delta) - \varphi(\cdot))\, g(\cdot)\|_{N_{q),\rho}(\Omega)}\,.$$

To complete the proof, it remains to show that $\Delta_\delta^{(3)} \to 0\,,\ \delta \to 0$. It is absolutely clear that $\varphi \in N_{q),\rho}(\Omega)$. Then from the continuous embedding $L_q(\Omega) \subset L_{q),\rho}(\Omega)$ we have

$$\Delta_\delta^{(3)} \le \|g\|_\infty \|\varphi(\cdot+\delta) - \varphi(\cdot)\|_{N_{q),\rho}(\Omega)} \le c\,\|g\|_\infty \|\varphi(\cdot+\delta) - \varphi(\cdot)\|_{N_{q),\rho}(\Omega)} \to 0\,,\ \delta \to 0,$$

where $c > 0$ is a constant independent of δ. The lemma is proved. □

Previous obtained results allow us analogously to the work [12] to prove the following existence theorem on solvability in the small.

Theorem 4.2 *Let L be a m–th order elliptic operator which has the property $P_{x_0})$ at some point $x_0 \in \Omega$ and weight $\rho(x) = |x|^{\alpha q}$, $1 < q < +\infty$, satisfy all conditions of Lemma 2.11. For $\forall f \in {}_sN_{q),\rho}(\Omega)$ for sufficiently small $r > 0$, there exists a solution of the equation $Lu = f$ belonging to the class $N_{q),\rho}(B_r(x_0))$.*

Proof Again this theorem is proved analogously to the Theorem 4.2 of work [12]. For the convenience of the reader we give full proof. Without loss of generality, we will assume that $x_0 = 0$ (and so $0 \in \Omega$). Let us show that $L_0 S_0 \varphi = \varphi$, $\forall \varphi \in {}_sN_{q),\rho}(r)$. In fact, let $\varphi \in {}_sN_{q),\rho}(r)$ be an arbitrary function. By Lemma 2.2, we have

$$\exists\{\varphi_k\} \subset C_0^\infty(r) : \|\varphi_k - \varphi\|_{N_{q),\rho}(r)} \to 0,\ k \to \infty.$$

By Theorem 2.1, $L_0 S_0 \varphi_k = \varphi_k$, $\forall k \in N$. Therefore it suffices to prove that $L_0 S_0$ acts boundedly in ${}_sN_{q),\rho}(r)$. We have

$$L_0 S_0 \varphi_k = L_0 \int_{B_r} J_0(x-y)\varphi_k(y)\,dy.$$

Taking into account that L_0 is a homogeneous differential operator of m-th order, by differentiation formula (3.4), for $|p| = m$ we have

$$(L_0 S_0 \varphi_k)(x) = \int_{B_r} L_0 J_0(x-y)\varphi_k(y)\,dy + const\,\varphi_k(x).$$

As $L_0 J_0(x)$ is a singular kernel, from Theorem 2.9 it follows that $L_0 S_0$ is bounded for the functions from $C_0^\infty(r)$ to ${}_sN_{q),\rho}(r)$, and $\overline{C_0^\infty(r)} = {}_sN_{q),\rho}(r)$ implies its boundedness in ${}_sN_{q),\rho}(r)$. Thus, $L_0 S_0 = I_{{}_sN_{q),\rho}(r)}$ in ${}_sN_{q),\rho}(r)$, where $I_{{}_sN_{q),\rho}(r)}$ is a unit operator in ${}_sN_{q),\rho}(r)$. We have

$$L_0 T_0 = L_0 S_0 (L_0 - L) = I_{{}_sN_{q),\rho}(r)} (L_0 - L) = L_0 - L.$$

Using this expression, the equation $Lu = f$ can be rewritten as follows:

$$L_0 u - L_0 T_0 u = f \Rightarrow L_0 \left(I_{N_{q),\rho}} - T_0\right) u = f,$$

where $I_{N_{q),\rho}}$ is a unit operator in $N_{q),\rho}(r)$. Hence we obtain

$$\left(I_{N_{q),\rho}} - T_0\right) u = S_0 f.$$

By Main Lemma, we have $\|T_0\|_{N_{q),\rho}\to N_{q),\rho}} = \bar{\bar{o}}(1)$, $r \to 0$. Therefore, for sufficiently small r we have $\|T_0\|_{N_{q),\rho}\to N_{q),\rho}} < 1$. Then, the operator $\left(I_{N_{q),\rho}} - T_0\right)$ is boundedly invertible in $N_{q),\rho}(r)$ and, by Lemma 2.2, the function

$$u = \left(I_{N_{q),\rho}} - T_0\right)^{-1} S_0 f$$

is a solution of the equation $Lu = f$.

The theorem is proved. □

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