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Analyzing bifurcation, stability, and chaos control for a discrete-time prey-predator model with Allee effect

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Abstract: In this paper, the qualitative behavior of a discrete-time prey-predator model with Allee effect in prey population is discussed. Firstly, the existence of the fixed points and their topological classification are analyzed algebraically. Then, the conditions of existence for both period-doubling and Neimark–Sacker bifurcations arising from coexistence fixed point with the help of the center manifold theorem and bifurcation theory are investigated. OGY feedback control method is implemented to control chaos in the proposed model due to the emergence of bifurcations. Finally, numerical simulations are performed to support the theoretical findings.

Key words: Prey-predator model, stability analysis, Allee effect, period-doubling bifurcation, Neimark–Sacker bifurcation, Chaos control

1. Introduction

Prey-predator relationship is a very important phenomenon that occurs in nature. Mathematical modeling of prey-predator interactions and their analysis have been considered by ecologists and biologists for the last few decades. The classic prey-predator model is the Lotka–Volterra model, which was formulated and proposed by Lotka in the United States in 1925 and Volterra in Italy in 1926 [18, 28]. It describes the dynamics of biological systems in which two species interact. After that, several attempts have been made to generalize and extend this model.

In population dynamics, when the population density is very low, there is a positive correlation between the population unit growth rate and the population density. This phenomenon can be called the Allee effect [6, 25, 26], starting with Allee's research [1]. Factors such as mating difficulty, mating depression, food problem, and protection from a predator are considered Allee effect. The Allee effect is classified according to the density-dependent properties at low density. When the population density is low, a strong Allee effect will appear and when the proliferation rate is positive and increases, the Allee effect will be weak. Allee effect is observed in many natural species; for example, in plants, insects, marine invertebrates, birds, and mammals. Consequently, analysis of systems involving Allee effect has gained lots of importance in problems associated with various fields such as conservation biology [5, 11], sustainable harvesting [17], pest control, biological control [10], population management [2], biological invasions [3, 4, 20, 21, 29], metapopulation dynamics [30, 32], interacting species [7, 9, 16, 27, 31]. Therefore, studies on Allee effect have received more and more attention from both

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mathematicians and ecologists [12–15, 22, 23].

In [24], the authors have considered the following discrete-time prey-predator population model:

$$\begin{aligned} x_{t+1} &= \delta x_t (1 - x_t) - x_t y_t \\ y_{t+1} &= y_t (1-\alpha) + \beta x_t y_t \end{aligned} \tag{1.1}$$

where x_t and y_t represent the number of prey and predator population, respectively. The parameter δ is the intrinsic growth rate of the prey populations with carrying ability one in the absence of predator. The death rate of predator is denoted by α, and β denotes the growth rate of predator in the presence of the prey. The parameters α, β, and δ are positive values. They have analyzed dynamics of the system such as the existence of nonnegative fixed points and local stability of the fixed points. They have shown that the system goes through period-doubling bifurcation and Neimark–Sacker bifurcation about axial and positive equilibrium states with prey growth rate as the bifurcation factor.

We modified the system (1.1) with weak Allee effect in prey as follows:

$$\begin{aligned} x_{t+1} &= \delta x_t (1 - x_t) - x_t y_t \frac{x_t}{x_t + \theta} \\ y_{t+1} &= y_t (1-\alpha) + \beta x_t y_t \end{aligned} \tag{1.2}$$

where $\frac{x}{x+\theta}$ is the Allee effect function, $\theta > 0$ is Allee constant, α, β, and δ parameters have the same meaning as in system (1.1).

We summarize this paper as follows. We introduce a discrete-time prey-predator model with Allee effect on prey population in Section 1. The existence of biologically feasible equilibria, and conditions for local asymptotic stability of these fixed points are investigated in Section 2. In Section 3.1, we show that interior fixed point of system (1.2) undergoes period-doubling bifurcation whenever growth parameter δ of the prey population is taken as the bifurcation parameter. In Section 3.2, it is proven that system (1.2) undergoes Neimark–Sacker bifurcation around its interior equilibrium point. The OGY method based on state feedback control for chaos control is introduced in Section 4. Lastly, numerical simulations are presented in Section 5 to illustrate our theoretical discussion.

2. Existence of the fixed points

In this section, we will investigate the existence of the positive fixed points of the system (1.2) and analyze the stability of these fixed points.

Definiton 2.1. A point (x^*, y^*) is called fixed point of the system (1.2), when it satisfies the following system:

$$\begin{aligned} x^* &= \delta x^* (1 - x^*) - x^* y^* \frac{x^*}{x^* + \theta} \\ y^* &= y^* (1-\alpha) + \beta x^* y^* \end{aligned} \tag{2.1}$$

Lemma 2.2

$i)$ The system (1.2) has an always trivial fixed point $E_1 = (0,0)$.

$ii)$ The system (1.2) has a fixed point $E_2 = (\frac{\delta - 1}{\delta}, 0)$ if $\delta > 1$.

$iii)$ The system (1.2) has a unique nontrivial positive fixed point

$E_3 = \left(\frac{\alpha}{\beta}, \frac{(\alpha + \theta\beta)(\delta(\beta - \alpha) - \beta)}{\beta\alpha}\right)$ if $\delta(\beta - \alpha) > \beta$.

Now, we will give topological classification of the fixed points of the system (1.2). The Jacobian matrix J of the system (1.2) evaluated at any point (x, y) is given by

$$J(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.2}$$

where

$$a_{11} = \delta(1 - 2x) - \frac{xy(x + 2\theta)}{(x + \theta)^2},$$

$$a_{12} = -\frac{x^2}{x + \theta}, \quad a_{21} = \beta y, \quad a_{22} = 1 - \alpha + \beta x,$$

and the characteristic equation of the Jacobian matrix $J(x, y)$ can be expressed as

$$F(\lambda) = \lambda^2 - trJ(x,y)\lambda + \det J(x,y) = 0, \tag{2.3}$$

where $trJ(x,y) = a_{11} + a_{22}$ and $\det J(x,y) = a_{11}a_{22} - a_{12}a_{21}$.

Definition 2.3. Assume that λ_1 and λ_2 are roots of the characteristic equation (2.3). Then the fixed point of the system (1.2) is called

$i)$ sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$,

$ii)$ source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$,

$iii)$ saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$,

$iv)$ nonhyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Let us start the trivial fixed point $E_1 = (0,0)$. Computing the Jacobian matrix at the fixed point E_1

$$J(E_1) = \begin{pmatrix} \delta & 0 \\ 0 & 1 - \alpha \end{pmatrix} \tag{2.4}$$

is obtained. Thus, the characteristic equation and eigenvalues are as follows:

$$\begin{aligned} F(\lambda) &= \lambda^2 - (\delta + 1 - \alpha)\lambda + \delta(1 - \alpha) = 0, \\ \lambda_1 &= \delta, \quad \lambda_2 = 1 - \alpha. \end{aligned} \tag{2.5}$$

According to Definition 2.3, we can get the following lemma:

Lemma 2.4. For the fixed point $E_1 = (0,0)$ the following topological classification holds:
i) E_1 is a sink if $0 < \delta < 1$ and $0 < \alpha < 2$,

ii) E_1 is a source if $\delta > 1$ and $\alpha > 2$,

iii) E_1 is a saddle if $(\delta > 1$ and $0 < \alpha < 2)$ or $(0 < \delta < 1$ and $\alpha > 2)$,

iv) E_1 is a nonhyperbolic if $\delta = 1$ or $\alpha = 2$.

Next, the Jacobian matrix evaluated at fixed point $E_2 = (\frac{\delta - 1}{\delta}, 0)$ is given by

$$J(E_2) = \begin{pmatrix} 2-\delta & -\dfrac{(\delta-1)^2}{\delta(\delta-1+\delta\theta)} \\ 0 & \dfrac{(1-\alpha+\beta)\delta-\beta}{\delta} \end{pmatrix}.$$

Thus, the characteristic equation and eigenvalues are as follows:

$$\begin{aligned} F(\lambda) &= \lambda^2 - \frac{[(3-\alpha+\beta-\delta)\delta-\beta]}{\delta}\lambda + \frac{(\delta-2)[(\alpha-1-\beta)\delta+\beta]}{\delta} = 0, \\ \lambda_1 &= 2-\delta, \ \lambda_2 = \frac{(1-\alpha+\beta)\delta-\beta}{\delta} = 1 - V, \end{aligned} \tag{2.6}$$

where $V = \alpha - \beta + \frac{\beta}{\delta}$. Similarly to the fixed point E_1, we can get the following lemma:

Lemma 2.5. For the fixed point $E_2 = (\frac{\delta-1}{\delta}, 0)$ the following topological classification holds:
i) E_2 is a sink if $1 < \delta < 3$ and $0 < V < 2$,

ii) E_2 is a source if $\delta > 3$ and $V > 2$ or $V < 0$,

iii) E_2 is a saddle if $(1 < \delta < 3$ and $V < 0$ or $V > 2)$ or $(\delta > 3$ and $0 < V < 2)$,

iv) E_2 is a nonhyperbolic if $\delta = 1, 3$ or $V = 2$.

Finally, the Jacobian matrix evaluated at the last fixed point

$$E_3 = \left(\frac{\alpha}{\beta}, \frac{(\alpha+\theta\beta)(\delta(\beta-\alpha)-\beta)}{\beta\alpha}\right)$$

as follows:

$$J(E_3) = \begin{pmatrix} \dfrac{-\left(\alpha^2+\theta\beta^2\right)\delta+\beta(\alpha+2\theta\beta)}{\beta(\alpha+\theta\beta)} & -\dfrac{\alpha^2}{\beta(\alpha+\theta\beta)} \\ \dfrac{(-\alpha^2+(1-\theta)\beta\alpha+\theta\beta^2)\delta-(\alpha+\theta\beta)\beta}{\alpha} & 1 \end{pmatrix} \tag{2.7}$$

Moreover, the characteristic polynomial of $J(E_3)$ is given by:

$$F(\lambda) = \lambda^2 + \left[\frac{(-2\beta + \delta\alpha)\,\alpha + (\delta - 3)\,\theta\beta^2}{\beta\,(\alpha + \theta\beta)}\right]\lambda \tag{2.8}$$

$$-\frac{[(1-\alpha)\delta - 2 + \alpha]\,\theta\beta^2 + [((\theta - 1)\alpha\delta - 1 + \alpha)\,\beta + (1 + \alpha)\alpha\delta]\,\alpha}{\beta(\alpha + \theta\beta)}.$$

Then, by simple computations it follows that

$$F(1) = \frac{(\delta\,(\beta - \alpha) - \beta)\alpha}{\beta}, \tag{2.9}$$

$$F(-1) = K\delta + \frac{[(6 - \alpha)\theta\beta + (4 - \alpha)\alpha]\,\beta}{\beta(\alpha + \theta\beta)}, \tag{2.10}$$

$$F(0) - 1 = S\delta - \frac{[\theta\beta\,(\alpha - 2) + \alpha\,(\alpha - 1)]\,\beta}{\beta\,(\alpha + \theta\beta)} - 1, \tag{2.11}$$

where $K = -\frac{(2-\alpha)\theta\beta^2 + (\theta-1)\alpha^2\beta + 2\alpha^2 + \alpha^3}{\beta(\alpha+\theta\beta)}$, $S = -\frac{\alpha^2 + \theta\beta^2 - \alpha^2\beta - \alpha\theta\beta^2 + \alpha^3 + \alpha^2\theta\beta}{\beta(\alpha+\theta\beta)}$.

Now, we have the following result:

Lemma 2.6. Assume that $F(\lambda) = \lambda^2 - A\lambda + B = 0$ and $F(1) > 0$ with λ_1 and λ_2 are roots of the characteristic equation $F(\lambda) = 0$. Then the following results hold:

i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $F(0) < 1$,

ii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $F(0) > 1$,

iii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if and only if $F(-1) < 0$,

iv) $|\lambda_1| = 1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $F(0) \neq \pm 1$,

v) λ_1 and λ_2 are conjugate complex numbers with $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if $A^2 - 4B < 0$ and $F(0) = 1$.

Before analyzing dynamics of the fixed point E_3, we define δ_1 as a root of $F(-1) = 0$, δ_2 as a root of $\det(J) - 1 = 0$, and δ_3 as a root of $F(1) = 0$, so one can apply Lemma 2.6 to prove the following results.

Lemma 2.7. Assume that $\delta > \delta_3$ and $\delta\,(\beta - \alpha) > \beta$ then for the coexistence fixed point E_3 of the system (1.2), the following holds :

i) E_3 is a sink if the following conditions hold:

i1) $K < 0, S > 0$ and $\delta_3 < \delta < \min\{\delta_1, \delta_2\}$,

i2) $K < 0, S < 0$ and $\max\{\delta_2, \delta_3\} < \delta < \delta_1$,

$ii)$ E_3 is a source if the following conditions hold:

$ii1)$ $K < 0, S > 0$ and $\max\{\delta_2, \delta_3\} < \delta < \delta_1$,

$ii2)$ $K < 0, S < 0$ and $\delta_3 < \delta < \min\{\delta_1, \delta_2\}$,

$iii)$ E_3 is a saddle if the following condition holds:

$K < 0, S > 0$ or $K < 0, S < 0$ and $\max\{\delta_1, \delta_3\} < \delta$,

$iv)$ Assume that λ_1 and λ_2 are roots of $F(\lambda) = \lambda^2 - A\lambda + B = 0$ then $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $K < 0, S < 0,$ $\delta = \delta_1$ and

$$\theta \neq -\frac{\alpha^2 [5\alpha + 4(1-\beta)]}{\beta(5\alpha^2 - 4\alpha\beta + 4\beta)}, -\frac{\alpha}{\beta},$$

$v)$ The roots of $F(\lambda) = \lambda^2 - A\lambda + B = 0$ are conjugate complex numbers with $|\lambda_{1,2}| = 1$ if and only if $K < 0,$ $S > 0,$ $\delta = \delta_2$ and

$$\begin{aligned} &\left[(4\beta + \delta)\alpha^2 + 4\beta^2\alpha(1 - \delta(1 - 2\theta)) + \theta\beta^2((1-\delta)(8\beta - 2\delta) + 4\delta\theta\beta)\right]\alpha^2 \\ &< (1-\delta)\theta^2\beta^4(4\alpha + \delta - 1). \end{aligned}$$

where

$$\delta_1 = \frac{[(6-\alpha)\theta\beta + (4-\alpha)\alpha]\beta}{(2-\alpha)\theta\beta^2 + (\theta-1)\alpha^2\beta + 2\alpha^2 + \alpha^3}, \tag{2.12}$$

$$\delta_2 = \frac{\beta\left(\alpha^2 + \alpha\theta\beta - \theta\beta\right)}{\beta(\alpha^2 + \alpha\theta\beta - \theta\beta) - \alpha^2(\alpha + \theta\beta + 1)}, \tag{2.13}$$

$$\delta_3 = \frac{\beta}{\beta - \alpha},$$

$$K = -\frac{(2-\alpha)\theta\beta^2 + (\theta-1)\alpha^2\beta + 2\alpha^2 + \alpha^3}{\beta(\alpha + \theta\beta)}, \tag{2.14}$$

$$S = -\frac{\alpha^2 + \theta\beta^2 - \alpha^2\beta - \alpha\theta\beta^2 + \alpha^3 + \alpha^2\theta\beta}{\beta(\alpha + \theta\beta)}.$$

3. Bifurcation analysis

In this section, we will analyze the period-doubling bifurcation and Neimark–Sacker bifurcation of the system (1.2) at the coexistence fixed point. By using the center manifold theorem and bifurcation theory [8], we will obtain existence conditions for period-doubling bifurcation and Neimark–Sacker bifurcation.

3.1. Period-doubling bifurcation

Suppose that the following condition holds:

$$\begin{aligned}&\left[(4\beta+\delta)\,\alpha^2+4\beta^2\alpha\,(1-\delta\,(1-2\theta))+\theta\beta^2\,((1-\delta)\,(8\beta-2\delta)+4\delta\theta\beta)\right]\alpha^2\\&>(1-\delta)\,\theta^2\beta^4\,(4\alpha+\delta-1)\,.\end{aligned} \tag{3.1}$$

Then let λ_1 and λ_2 be distinct real roots of the equation (2.8). Also, we suppose that

$$\delta=\frac{\left[(6-\alpha)\theta\beta+(4-\alpha)\alpha\right]\beta}{(2-\alpha)\theta\beta^2+(\theta-1)\alpha^2\beta+2\alpha^2+\alpha^3}. \tag{3.2}$$

Therefore, roots of the equation (2.8) are $\lambda_1=-1$ and

$$\lambda_2=\frac{(\alpha+\theta\beta)\left(4\alpha^2-3\alpha\beta\right)+2(\alpha^2+\theta\beta^2)}{(2-\alpha)\theta\beta^2+\left[(\theta-1)\beta+2+\alpha\right]\alpha^2}.$$

Moreover, $|\lambda_2|\neq 1$ under the following condition:

$$\begin{aligned}\frac{(\alpha+\theta\beta)\left(4\alpha^2-3\alpha\beta\right)+2(\alpha^2+\theta\beta^2)}{(2-\alpha)\theta\beta^2+\left[(\theta-1)\beta+2+\alpha\right]\alpha^2}&\neq\pm 1,\\(2-\alpha)\theta\beta^2+\left[(\theta-1)\beta+2+\alpha\right]\alpha^2&\neq 0.\end{aligned} \tag{3.3}$$

Let us consider the term Ω_{FB} as follows:

$$\Omega_{FB}=\left\{(\alpha,\beta,\delta,\theta)\in\mathbb{R}_+^4:K<0,S<0,\quad (3.1),(3.2),\text{ and }(3.3)\text{ are satisfied}\right\}.$$

To discuss the period-doubling bifurcation for the system (1.2) at its unique positive fixed point E_3, we take δ as bifurcation parameter. Then, variation of parameters α,β,δ, and θ in small neighborhood of Ω_{FB} gives the emergence of period-doubling bifurcation. Also, we set

$$\delta=\delta_1=\frac{\left[(6-\alpha)\theta\beta+(4-\alpha)\alpha\right]\beta}{(2-\alpha)\theta\beta^2+(\theta-1)\alpha^2\beta+2\alpha^2+\alpha^3}. \tag{3.4}$$

Then for $(\alpha,\beta,\delta_1,\theta)\in\Omega_{FB}$, the system (1.2) can be expressed by

$$\begin{pmatrix}X\\Y\end{pmatrix}\to\begin{pmatrix}\delta X(1-X)-XY\dfrac{X}{X+\theta}\\Y(1-\alpha)+\beta XY\end{pmatrix}. \tag{3.5}$$

Let $\widetilde{\delta}_1$ be a small bifurcation parameter such that $\left|\widetilde{\delta}_1\right|\ll 1$, then we can give corresponding perturbed mapping of (3.5) as follows:

$$\begin{pmatrix}X\\Y\end{pmatrix}\to\begin{pmatrix}\left(\delta+\widetilde{\delta}_1\right)X(1-X)-XY\dfrac{X}{X+\theta}\\Y(1-\alpha)+\beta XY\end{pmatrix} \tag{3.6}$$

Now, we apply the transformations

$$x = X - \frac{\alpha}{\beta}$$
$$y = Y - \frac{(\alpha+\theta\beta)\left(\left(\delta+\widetilde{\delta}_1\right)(\beta-\alpha)-\beta\right)}{\beta\alpha}$$

then from the map (3.6), we get

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1\left(x,y,\widetilde{\delta}_1\right) \\ g_2\left(x,y,\widetilde{\delta}_1\right) \end{pmatrix} \tag{3.7}$$

where

$$g_1\left(x,y,\widetilde{\delta}_1\right) = a_{13}x^2 + a_{14}xy + b_1x^3 + b_2x^2y + d_1x\widetilde{\delta}_1 + d_2x^2\widetilde{\delta}_1 + O\left(\left(|x|+|y|+|\widetilde{\delta}_1|\right)^4\right),$$
$$g_2\left(x,y,\widetilde{\delta}_1\right) = a_{23}xy + O\left(\left(|x|+|y|+|\widetilde{\delta}_1|\right)^4\right),$$

$$\begin{aligned}
a_{11} &= \frac{-\left(\alpha^2+\theta\beta^2\right)\delta+\beta(\alpha+2\theta\beta)}{\beta(\alpha+\theta\beta)}, a_{12} = -\frac{\alpha^2}{\beta(\alpha+\theta\beta)} \\
a_{21} &= \frac{(-\alpha^2+(1-\theta)\beta\alpha+\theta\beta^2)\delta-(\alpha+\theta\beta)\beta}{\alpha},\ a_{22}=1,\ a_{23}=\beta, \\
a_{13} &= \frac{\beta^3\theta^2(1-\delta)-\alpha^2\delta(\alpha+2\beta\theta)}{\alpha(\alpha+\beta\theta)^2},\ a_{14} = -\frac{(\alpha+2\beta\theta)\alpha}{(\alpha+\beta\theta)^2}, \\
b_1 &= \frac{\beta^3\theta^2(\beta+\delta\alpha-\beta\delta)}{\alpha(\alpha+\beta\theta)^3},\ b_2 = -\frac{\beta^3\theta^2}{(\alpha+\beta\theta)^3}, \\
d_1 &= 1-\frac{2\alpha}{\beta},\ d_2 = -1.
\end{aligned} \tag{3.8}$$

Now, we use the following translation to convert the coefficient matrix in map (3.7) into normal form

$$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.9}$$

where

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1-a_{11} & \lambda_2-a_{11} \end{pmatrix}$$

is an invertible matrix. From (3.7) and (3.9), we get

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_3\left(u,v,\widetilde{\delta}_1\right) \\ g_4\left(u,v,\widetilde{\delta}_1\right) \end{pmatrix}, \tag{3.10}$$

$$
\begin{aligned}
g_3\left(u,v,\widetilde{\delta}_1\right) &= -\frac{(\lambda_2+a_{11})\,b_1}{a_{12}(\lambda_2+1)}x^3-\frac{(a_{11}-\lambda_2)\,a_{14}}{a_{12}(\lambda_2+1)}x^2y-\frac{(a_{11}-\lambda_2)\left(a_{13}+d_2\widetilde{\delta}_1\right)}{a_{12}(\lambda_2+1)}x^2\\
&\quad -\left(\frac{(a_{11}-\lambda_2)\,b_2-a_{23}a_{12}}{a_{12}(\lambda_2+1)}\right)xy-\frac{(a_{11}-\lambda_2)\,d_1}{a_{12}(\lambda_2+1)}x\widetilde{\delta}_1+O\left(\left(|u|+|v|+|\widetilde{\delta}_1|\right)^4\right),\\
g_4\left(u,v,\widetilde{\delta}_1\right) &= \frac{(1+a_{11})\,b_1}{a_{12}(\lambda_2+1)}x^3+\frac{(1+a_{11})\,a_{14}}{a_{12}(\lambda_2+1)}x^2y+\frac{(1+a_{11})\left(a_{13}+d_2\widetilde{\delta}_1\right)}{a_{12}(\lambda_2+1)}x^2\\
&\quad +\left(\frac{(1+a_{11})\,b_2}{a_{12}(\lambda_2+1)}+\frac{a_{23}}{\lambda_2+1}\right)xy+\frac{(1+a_{11})\,d_1}{a_{12}(\lambda_2+1)}x\widetilde{\delta}_1+O\left(\left(|u|+|v|+|\widetilde{\delta}_1|\right)^4\right),
\end{aligned}
$$

$$x=a_{12}(u+v)\,,y=-(1+a_{11})\,u+(\lambda_2-a_{11})\,v.$$

In order to apply the center manifold therem, let $W^c(0,0,0)$ be the center manifold of (3.10) evaluated at $(0,0)$ in a small neighborhood of $\widetilde{\delta}_1=0,$ then $W^c(0,0,0)$ can be approximated as follows:

$$W^c(0,0,0)=\left\{\left(u,v,\widetilde{\delta}_1\right)\in\mathbb{R}^3: v=m_1u^2+m_2u\widetilde{\delta}_1+m_3\widetilde{\delta}_1^2+O\left(\left(|u|+|\widetilde{\delta}_1|\right)^3\right)\right\},$$

where

$$
\begin{aligned}
m_1 &= \frac{(1+a_{11})\left[b_2(1+a_{11})+a_{12}(a_{23}-a_{13})\right]}{\lambda_2^2-1},\\
m_2 &= -\frac{(1+a_{11})\,d_1}{(\lambda_2+1)^2},\ m_3=0.
\end{aligned}
$$

Therefore, the map restricted to the center manifold $W^c(0,0,0)$ is given by

$$F: u\to -u+k_1u^2+k_2u\widetilde{\delta}_1+k_3u^2\widetilde{\delta}_1+k_4u\widetilde{\delta}_1^2+k_5u^3+O\left(\left(|u|+|\widetilde{\delta}_1|\right)^4\right),$$

$$
\begin{aligned}
k_1 &= -\frac{(a_{11}-\lambda_2)\,a_{12}a_{13}}{\lambda_2+1} + \left(\frac{(a_{11}-\lambda_2)\,b_2 + a_{23}a_{12}}{\lambda_2+1}\right)(1+a_{11}),\\
k_2 &= \frac{(\lambda_2-a_{11})\,d_1}{\lambda_2+1},\\
k_3 &= \frac{((a_{11}-\lambda_2)\,b_2 - a_{23}a_{12})\,(\lambda_2-2a_{11}-1)\,(1+a_{11})\,d_1}{(\lambda_2+1)^3}\\
&\quad + \frac{2\,(a_{11}-\lambda_2)\,a_{12}a_{13}\,(1+a_{11})\,d_1}{(\lambda_2+1)^3} - \frac{(a_{11}-\lambda_2)\,(d_1m_1+a_{12}d_2)}{\lambda_2+1},\\
k_4 &= \frac{(a_{11}-\lambda_2)\,d_1^2\,(1+a_{11})}{(\lambda_2+1)^3},\\
k_5 &= -\frac{((a_{11}-\lambda_2)\,b_2 - a_{23}a_{12})}{\lambda_2+1}\,(\lambda_2-2a_{11}-1)\,m_1\\
&\quad + \frac{(a_{11}-\lambda_2)\,a_{12}(2a_{13}m_1 - 1 - a_{11} + a_{12}b_1)}{\lambda_2+1}.
\end{aligned}
$$

Next, we define the following two nonzero real numbers:

$$
l_1 = \left(\frac{\partial^2 g_3}{\partial u \partial \widetilde{\delta}_1} + \frac{1}{2}\frac{\partial F}{\partial \widetilde{\delta}_1}\frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = \frac{(a_{11}-\lambda_2)\,d_1}{\lambda_2+1} + k_1k_2,
$$

$$
l_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = k_1^2 + k_5.
$$

Therefore, we have the following result about the period-doubling bifurcation of the system (1.2).

Theorem 3.1 Assume that $l_2 \neq 0$, then the system (1.2) undergoes period-doubling bifurcation at the unique point E_3, when parameter δ varies in small neighborhood of δ_1. Furthermore, if $l_2 > 0$, then the period-two orbits that bifurcate from positive fixed point E_3 are stable, and if $l_2 < 0$,then these orbits are unstable.

3.2. Neimark–Sacker bifurcation

We will investigate the existence and direction of Neimark–Sacker bifurcation for unique positive steady-state E_3 of system (1.2). From Lemma 2.7, it is established that E_3 is nonhyperbolic fixed point where the Jacobian matrix evaluated at E_3 has pair of complex conjugate eigenvalues with modulus one if the following condition is satisfied:

$$
\begin{aligned}
\Delta(\alpha,\beta,\delta,\theta) &= \left[(4\beta+\delta)\,\alpha^2 + 4\beta^2\alpha\,(1-\delta\,(1-2\theta)) + \theta\beta^2\,((1-\delta)\,(8\beta-2\delta)+4\delta\theta\beta)\right]\alpha^2\\
&\quad - (1-\delta)\,\theta^2\beta^4\,(4\alpha+\delta-1) < 0.
\end{aligned}
$$

Assume that

$$
\Omega_{NSB} = \left\{ \begin{array}{c} (\alpha,\beta,\delta,\theta) \in \mathbb{R}_+^4 : K<0, S>0, \Delta(\alpha,\beta,\delta,\theta)<0, \\ \delta = \frac{\beta\left(\alpha^2+\alpha\theta\beta-\theta\beta\right)}{\beta(\alpha^2+\alpha\theta\beta-\theta\beta)-\alpha^2(\alpha+\theta\beta+1)} \end{array} \right\}.
$$

In order to discuss Neimark–Sacker bifurcation for positive fixed point E_3 of the system (1.2), we take δ as a bifurcation parameter. Then, variation of parameters α, β, δ, and θ in small neighborhood of Ω_{NSB} yields Neimark–Sacker bifurcation. Assume that $(\alpha, \beta, \delta_2, \theta) \in \Omega_{NSB}$, then the system (1.2) can be expressed by the following two-dimensional map:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \longrightarrow \begin{pmatrix} \delta_2 X(1-X) - XY\dfrac{X}{X+\theta} \\ Y(1-\alpha) + \beta XY \end{pmatrix}. \tag{3.11}$$

Let $\widetilde{\delta}_2$ denote the bifurcation parameter such that $\left|\widetilde{\delta}_2\right| \ll 1$, then we can give corresponding perturbed mapping of (3.11) as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \longrightarrow \begin{pmatrix} \left(\delta_2 + \widetilde{\delta}_2\right) X(1-X) - XY\dfrac{X}{X+\theta} \\ Y(1-\alpha) + \beta XY \end{pmatrix}. \tag{3.12}$$

Now, we apply the transformations $x = X - x^*$ and $y = Y - y^*$,

where $x^* = \frac{\alpha}{\beta}$ and $y^* = \dfrac{(\alpha+\theta\beta)\left(\left(\delta+\widetilde{\delta}_1\right)(\beta-\alpha)-\beta\right)}{\beta\alpha}$, then from the map (3.12), we get

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \tag{3.13}$$

where

$$\begin{aligned} f_1(x,y) &= m_{13}x^2 + m_{14}xy + m_{15}x^3 + m_{16}x^2y + O((|x|+|y|)^4), \\ f_2(x,y) &= m_{23}xy + O((|x|+|y|)^4), \end{aligned}$$

$$\begin{aligned} m_{11} &= \frac{-\left(\alpha^2+\theta\beta^2\right)(\delta_2+\widetilde{\delta}_2) + \beta(\alpha+2\theta\beta)}{\beta(\alpha+\theta\beta)}, m_{12} = -\frac{\alpha^2}{\beta(\alpha+\theta\beta)}, \\ m_{21} &= \frac{(-\alpha^2+(1-\theta)\beta\alpha+\theta\beta^2)(\delta_2+\widetilde{\delta}_2) - (\alpha+\theta\beta)\beta}{\alpha}, m_{22} = 1, \\ m_{13} &= \frac{-(\delta_2+\widetilde{\delta}_2)\left[-\theta^3+3\theta^2x^*+3{x^*}^2\theta+{x^*}^3\right]+y^*\theta^2}{(x^*+\theta)^3}, \\ m_{14} &= -\frac{x^*(2\theta+x^*)}{(x^*+\theta)^2}, m_{15} = \frac{y^*\theta^2}{(x^*+\theta)^4}, m_{16} = \frac{-\theta^2}{(x^*+\theta)^3}, m_{23} = \beta. \end{aligned}$$

The characteristic equation of Jacobian matrix of linearized system of (3.13) is evaluated at the fixed point $(0,0)$ as follows:

$$\lambda^2 - A(\widetilde{\delta}_2)\lambda + B(\widetilde{\delta}_2) = 0, \tag{3.14}$$

where

$$A(\widetilde{\delta}_2) = \frac{\beta\left(-3\theta\beta - 2\alpha\right) + (\delta_2 + \widetilde{\delta}_2)\left[(\alpha^2 + \theta\beta^2)\right]}{\beta\left(\alpha + \theta\beta\right)},$$

$$B(\widetilde{\delta}_2) = \frac{(\delta_2 + \widetilde{\delta}_2) - \alpha^2\left[1 + (\beta + \beta\theta + \alpha^2 + \theta - \alpha\theta\beta^2)\beta^2\right]}{\beta\left(\alpha + \theta\beta\right)} - \frac{\alpha\beta(\theta\beta^2 - 1) - \theta\beta^2(2 - \alpha\beta^2)}{\beta\left(\alpha + \theta\beta\right)}.$$

Since $(\alpha, \delta_2, \theta, \beta) \in \Omega_{NSB}$, the complex conjugate roots of (3.14) are given by

$$\lambda_1 = \frac{A(\widetilde{\delta}_2) - i\sqrt{4B\left(\widetilde{\delta}_2\right) - \left(A(\widetilde{\delta}_2)\right)^2}}{2},$$

and

$$\lambda_2 = \frac{A(\widetilde{\delta}_2) + i\sqrt{4B\left(\widetilde{\delta}_2\right) - \left(A(\widetilde{\delta}_2)\right)^2}}{2}.$$

Then we have

$$|\lambda_1| = |\lambda_2| = \sqrt{-\frac{(\alpha^2 + \theta\beta^2 - \alpha\beta^4\theta - \alpha^2\beta^3 + \alpha^2\beta^3\theta + \alpha^3\beta^2)\left(\delta_2 + \widetilde{\delta}_2\right)}{\beta\left(\alpha + \theta\beta\right)} - \frac{-\alpha\beta - 2\theta\beta^2 + \alpha\beta^4\theta + \alpha^2\beta^2}{\beta\left(\alpha + \theta\beta\right)}}$$

and

$$\left(\frac{d|\lambda_2|}{d\widetilde{\delta}_2}\right)_{\widetilde{\delta}_2=0} = \left(\frac{d|\lambda_1|}{d\widetilde{\delta}_2}\right)_{\widetilde{\delta}_2=0} = -\frac{-\alpha\beta^4\theta + (-1+\theta)\alpha^2\beta^3 + (\theta + \alpha^3)\beta^2 + \alpha^2}{2(\alpha + \beta\theta)\beta} \neq 0.$$

Also we get $-2 < A\left(0\right) < 2$ since of $(\alpha, \beta, \delta_2, \theta) \in \Omega_{NSB}$. On the other hand, we have $A(0) = \frac{\left(-3\theta\beta^2 - 2\alpha\beta + \delta_2\alpha^2 + \delta_2\theta\beta^2\right)}{\beta\left(\alpha + \theta\beta\right)}$. $A(0) \neq 0, \ -1,$that is

$$\delta_2 \neq \frac{3\theta\beta^2 + 2\alpha\beta}{\alpha^2 + \theta\beta^2}, \ \frac{2\theta\beta^2 + \alpha\beta}{\alpha^2 + \theta\beta^2}. \tag{3.15}$$

Conditions (3.15) together $(\alpha, \beta, \delta, \theta) \in \Omega_{NSB}$ make sure that $A(0) \neq 0, -1$ and in a result we have $\lambda_1^m, \lambda_2^m \neq 1$ for all $m = 1, 2, 3, 4$ at $\widetilde{\delta}_2 = 0$. Hence, roots of (3.14) do not lie in the intersection of the unit circle with the coordinate axes $\widetilde{\delta}_2 = 0$. In order to obtain the normal form of (3.13) at $\widetilde{\delta}_2 = 0$, assuming that $\kappa = \frac{A(0)}{2}$, $\omega = \frac{\sqrt{4B(0) - (A(0))^2}}{2}$.

Moreover, we consider the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} m_{12} & 0 \\ \kappa - m_{11} & -\omega \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.16}$$

Under transformation (3.16), the normal form of (3.13) can be written as:

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} \kappa & -\omega \\ \omega & \kappa \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \widetilde{f}(u,v) \\ \widetilde{g}(u,v) \end{pmatrix}, \tag{3.17}$$

$$\begin{aligned}
\widetilde{f}(u,v) &= \frac{m_{15}x^3}{m_{12}} + \left(\frac{m_{16}y}{m_{12}} + \frac{m_{13}}{m_{12}}\right)x^2 + \frac{m_{14}xy}{m_{12}} + O((|u|+|v|)^4), \\
\widetilde{g}(u,v) &= \frac{(\kappa - m_{11})(m_{13}x^2 + m_{14}xy + m_{15}x^3 + m_{16}x^2y)}{m_{12}\omega} \\
&\quad - \frac{m_{23}x^2 + m_{24}xy + m_{25}x^3 + m_{26}x^2y}{\omega} + O((|u|+|v|)^4),
\end{aligned}$$

$$x = m_{12}u \text{ and } y = (\kappa - m_{11})u - \omega v.$$

Next, we consider the following real number:

$$L = \left(\left[-Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\zeta_{20}\zeta_{11}\right) - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + Re(\lambda_2\zeta_{21})\right]\right)_{\delta_2=0},$$

where

$$\begin{aligned}
\zeta_{20} &= \frac{1}{8}\left[\widetilde{f}_{uu} - \widetilde{f}_{uv} + 2\widetilde{g}_{uv} + i(\widetilde{g}_{uu} - \widetilde{g}_{vv} - 2\widetilde{f}_{uv})\right], \\
\zeta_{11} &= \frac{1}{4}\left[\widetilde{f}_{uu} + \widetilde{f}_{uv} + i(\widetilde{g}_{uu} - \widetilde{g}_{vv})\right], \\
\zeta_{02} &= \frac{1}{8}\left[\widetilde{f}_{uu} - \widetilde{f}_{uv} - 2\widetilde{g}_{uv} + i(\widetilde{g}_{uu} - \widetilde{g}_{vv} + 2\widetilde{f}_{uv})\right], \\
\zeta_{21} &= \frac{1}{16}\left[\widetilde{f}_{uuu} + \widetilde{f}_{uvv} + \widetilde{g}_{uuv} + \widetilde{g}_{vvv} + i(\widetilde{g}_{uuu} + \widetilde{g}_{uvv} - \widetilde{f}_{uuv} - \widetilde{f}_{vvv})\right].
\end{aligned}$$

Furthermore, the partial derivaties of $\widetilde{f}$ and $\widetilde{g}$ evaluated at $\widetilde{\delta}_2 = 0$ are given by:

$$\begin{aligned}
\widetilde{f}_{uu} &= 2m_{12}m_{13} + 2m_{14}\kappa - 2m_{14}m_{11}, \ \widetilde{f}_{vv} = 0, \\
\widetilde{g}_{uv} &= -m_{14}(\kappa - m_{11}) + m_{24}m_{12}, \\
\widetilde{g}_{uu} &= \frac{(\kappa - m_{11})(2m_{13}m_{12}^2 + 2m_{14}(\kappa - m_{11})m_{12}}{m_{12}\omega} - \frac{2m_{23}m_{12}^2 + 2m_{24}m_{12}(\kappa - m_{11})}{\omega}, \\
\widetilde{f}_{uv} &= -m_{14}\omega\ ,\ \widetilde{g}_{vv} = 0,\ \widetilde{f}_{uuu} = 6m_{12}^2m_{15} + 6m_{12}m_{16}(\kappa - m_{11}),\ \widetilde{f}_{uvv} = 0, \\
\widetilde{g}_{uuv} &= -2m_{12}m_{16}(\kappa - m_{11}) + 2m_{26}m_{12}^2, \\
\widetilde{g}_{uuu} &= \frac{(\kappa - m_{11})(6m_{12}^3m_{15} + 6m_{16}m_{12}^2(\kappa - m_{11}))}{m_{12}\omega} - \frac{6m_{25}m_{12}^3 + 6m_{26}m_{12}^2(\kappa - m_{11})}{\omega}, \\
\widetilde{g}_{vvv} &= \widetilde{g}_{uvv} = \widetilde{f}_{vvv} = 0, \widetilde{f}_{uuv} = -2m_{12}m_{16}\omega.
\end{aligned}$$

Now we have the following result that gives parametric conditions for the existence and direction of Neimark–Sacker bifurcation for positive fixed point of system (1.2).

Teorem 4.1 Suppose that (3.13) holds and $L \neq 0$, then system (1.2) endures Neimark–Sacker bifurcation at its unique positive steady-state E_3 when the bifurcation parameter varies in a small neighborhood of $\delta_2 = \frac{\beta\left(\alpha^2+\alpha\theta\beta-\theta\beta\right)}{\beta(\alpha^2+\alpha\theta\beta-\theta\beta)-\alpha^2(\alpha+\theta\beta+1)}$. Furthermore, if $L < 0$, then an attracting invariant closed curve bifurcates from the fixed point for $\delta > \delta_2$, and if $L > 0$, then a repelling invariant closed curve bifurcates from the fixed point for $\delta < \delta_2$.

4. Chaos control

Chaos is the general name for nonlinear dynamical systems that behave noise-like. Chaotic behavior is examined in chemistry, physics, ecology, biology, chemical engineering, telecommunications, etc. Furthermore, the practical methods related to chaos control can be implemented in various areas such as communications, physics laboratories, biochemistry, turbulence, and cardiology. Chaos is indecomposable, is highly dependent on the initial condition, and consists of a large number of periodic points and orbits. Because of this, the solution of a chaotic system is difficult to predict, which calls for a way to control it. The control algorithm of Ott, Grebogi, and Yorke (OGY, [19]) manages to do this. The proposed methodology is known as the OGY method. OGY is a discrete control algorithm, perturbing the system at discrete moments in time. The OGY method is an important and effective method.

In this section, we analyze the OGY method based on state feedback control for chaos control. To apply this technique to model (1.2), we can rewrite this system as follows:

$$\begin{aligned} x_{n+1} &= \delta x_n (1 - x_n) - x_n y_n \frac{x_n}{x_n + \theta} = f(x_n, y_n, \delta) \\ y_{n+1} &= y_n(1-\alpha) + \beta x_n y_n = g(x_n, y_n, \delta) \end{aligned} \tag{4.1}$$

where δ denotes the parameter of chaos control. Moreover, it is assumed that $|\delta - \delta_0| < \varepsilon$ where $\varepsilon > 0$ and δ_0 represents the nominal parameter lies in the chaotic regions. Then we assume that (x^*, y^*) is an interior unstable fixed point of system (1.2) and is located in some chaotic regions. To move the unstable fixed point towards a stable one, the system (1.2) is linearized in the neighborhood of the unstable fixed point (x^*, y^*) as follows:

$$\begin{bmatrix} x_{n+1} - x^* \\ y_{n+1} - y^* \end{bmatrix} \approx A \begin{bmatrix} x_n - x^* \\ y_n - y^* \end{bmatrix} + B\left[\delta - \delta_0\right], \tag{4.2}$$

where

$$\begin{aligned} A &= \begin{bmatrix} \frac{\partial f(x_n,y_n,\delta_0)}{\partial x_n} & \frac{\partial f(x_n,y_n,\delta_0)}{\partial y_n} \\ \frac{\partial g(x_n,y_n,\delta_0)}{\partial x_n} & \frac{\partial g(x_n,y_n,\delta_0)}{\partial y_n} \end{bmatrix} \\ &= \begin{bmatrix} -\dfrac{\theta\beta^2(\delta-2)+\alpha(\delta\alpha-\beta)}{\beta(\alpha+\theta\beta)} & -\dfrac{\alpha^2}{\beta(\alpha+\theta\beta)} \\ -\dfrac{\theta\beta^2(1-\delta)-\delta\alpha(\beta-\alpha-\theta\beta)+\alpha\beta}{\beta(\alpha+\theta\beta)} & 1 \end{bmatrix} \end{aligned}$$

and

$$B=\left[\begin{array}{c}\frac{\partial f(x_n,y_n,\delta_0)}{\partial s}\\ \frac{\partial g(x_n,y_n,\delta_0)}{\partial s}\end{array}\right]=\left[\begin{array}{c}x^*-(x^*)^2\\ 0\end{array}\right].$$

Next, we define the following controllability matrix for the system (4.1):

$$C=[B:AB]=\left[\begin{array}{cc}x^*-(x^*)^2 & -\frac{\left[\theta\beta^2(\delta-2)+\alpha(\delta\alpha-\beta)\right](\beta-\alpha)\alpha}{\beta^3(\alpha+\theta\beta)}\\ 0 & -\frac{\left[\theta\beta^2(1-\delta)-\delta\alpha(\beta-\alpha-\theta\beta)+\alpha\beta\right](\beta-\alpha)}{\beta^2}\end{array}\right]. \tag{4.3}$$

Then it is easy to see that rank of C is 2. Now suppose that

$$[\delta-\delta_0]=-R\left[\begin{array}{c}x_n-x^*\\ y_n-y^*\end{array}\right],$$

where $R=[p_1\ p_2]$, then system (4.2) can be written as:

$$\left[\begin{array}{c}x_{n+1}-x^*\\ y_{n+1}-y^*\end{array}\right]\approx[A-BR]\left[\begin{array}{c}x_n-x^*\\ y_n-y^*\end{array}\right]. \tag{4.4}$$

Furthermore, the fixed point (x^*,y^*) is locally asymptotically stable if and only if both eigenvalues of the matrix $A-BR$ lie in an open unit disk. The Jacobian matrix $A-BR$ of the controlled system (4.4) can be written as follows:

$$A-BR=\left[\begin{array}{cc}-\frac{\theta\beta^2(\delta-2)+\alpha(\delta\alpha-\beta)}{\beta(\alpha+\theta\beta)}-\frac{\alpha(\beta-\alpha)}{\beta^2}p_1 & -\frac{\alpha^2}{\beta(\alpha+\theta\beta)}\frac{\alpha(\beta-\alpha)}{\beta^2}p_2\\ -\frac{\theta\beta^2(1-\delta)-\delta\alpha(\beta-\alpha-\theta\beta)+\alpha\beta}{\beta(\alpha+\theta\beta)} & 1\end{array}\right].$$

The characteristic equation of the Jacobian matrix $A-BR$ is given by

$$\begin{aligned}P(\lambda) \;=\;& \lambda^2+\frac{\left[\left(\alpha^2+\theta^2\beta^2\right)\delta\beta-\beta^2(2\alpha+3\theta\beta)+\alpha(\beta-\alpha)(\alpha+\theta\beta)p_1\right]}{\beta^2(\alpha+\theta\beta)}\lambda\\ &+\frac{\left(p_2\beta^4-2p_2\alpha\beta^3+p_2\alpha^2\beta^2\right)\theta^2\delta+\left[2p_2\alpha^3\beta-(4p_2+1)\alpha^2\beta^2\right]\theta\delta}{\beta^2(\alpha+\theta\beta)}\\ &+\frac{\left[(1+2p_2)\beta^3\alpha-\beta^3\right]\theta\delta+\left(p_2\alpha^4-(1+2p_2)\beta^3\alpha+(p_2\beta+\beta-1)\alpha^2\beta\right)\delta}{\beta^2(\alpha+\theta\beta)}\\ &+\frac{(\alpha-\beta)p_2\beta^3\theta^2+\left[(p_1+2p_2\beta)\beta\alpha^2-(2p_2\beta+p_1+\beta)\beta^2\alpha+2\beta^3\right]\theta}{\beta^2(\alpha+\theta\beta)}\\ &+\frac{(p_1+p_2\beta)\alpha^3-(p_2\beta+p_1+\beta)\beta\alpha^2+\beta^2\alpha}{\beta^2(\alpha+\theta\beta)}.\end{aligned} \tag{4.5}$$

Let λ_1 and λ_2 be the eigenvalues of the characteristic equation (4.5), then we get

$$\lambda_1+\lambda_2=\frac{\alpha(\alpha-\beta)}{\beta^2}p_1-\frac{\theta\beta^2(\delta-3)+\alpha(\delta\alpha-2\beta)}{\beta(\alpha+\theta\beta)},$$

and

$$\begin{aligned}\lambda_1\lambda_2 &= \frac{\left[\alpha^3+(\theta-1)\beta\alpha^2-\alpha\beta^2\theta\right]}{\beta^2(\alpha+\theta\beta)}p_1 \\ &+ + \frac{\left[\alpha^4\delta+(1-2(1-\theta)\alpha)\beta\alpha^3+\left((1+\theta^2-4\theta)\delta-1+2\theta\right)\beta^2\alpha^2\right]}{\beta^2(\alpha+\theta\beta)}p_2 \\ &+ \frac{\left(2(1-\theta)\theta\delta-2\theta+\theta^2\right)\beta^3\alpha+(\delta-1)\theta^2\beta^4}{\beta^2(\alpha+\theta\beta)}p_2 \\ &+ \frac{-\alpha^3\beta\delta+\left[((1-\theta)\delta-1)\beta^2-\delta\beta\right]\alpha^2+\left[(\delta-1)\theta\beta+1\right]\alpha\beta^2+(2-\delta)\theta\beta^3}{\beta^2(\alpha+\theta\beta)}.\end{aligned} \tag{4.6}$$

Then in order to obtain the lines of marginal stability, we must solve the equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These restrictions make sure that $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Using $\lambda_1\lambda_2 = 1$ in equation (4.6) then,

$$\begin{aligned}L_1 &= \frac{\left[\alpha^3+(\theta-1)\beta\alpha^2-\alpha\beta^2\theta\right]}{\beta^2(\alpha+\theta\beta)}p_1 \\ &+ \frac{\left[\alpha^4\delta+(1-2(1-\theta)\alpha)\beta\alpha^3+\left((1+\theta^2-4\theta)\delta-1+2\theta\right)\beta^2\alpha^2\right]}{\beta^2(\alpha+\theta\beta)}p_2 \\ &+ \frac{\left(2(1-\theta)\theta\delta-2\theta+\theta^2\right)\beta^3\alpha+(\delta-1)\theta^2\beta^4}{\beta^2(\alpha+\theta\beta)}p_2 \\ &+ \frac{-\alpha^3\beta\delta+\left[((1-\theta)\delta-1)\beta^2-\delta\beta\right]\alpha^2}{\beta^2(\alpha+\theta\beta)} \\ &+ \frac{\left[(\delta-1)\theta\beta+1\right]\alpha\beta^2+(2-\delta)\theta\beta^3}{\beta^2(\alpha+\theta\beta)} \\ &= 0.\end{aligned}$$

Moreover, suppose that $\lambda_1 = 1$, then

$$L_2 = \left[\frac{\delta\alpha^3}{\beta^2}+\frac{(1-2\delta+\delta\theta)\alpha^2}{\beta}+(\theta-2\delta\theta+\delta-1)\alpha+(\delta-1)\theta\beta\right]p_2-\frac{\delta\alpha^2}{\beta}+(\delta-1)\alpha = 0.$$

Finally, suppose that $\lambda_1 = -1$, then

$$\begin{aligned}L_3 &= \frac{\left[\alpha^3+(\theta-1)\beta\alpha^2-\alpha\beta^2\theta\right]}{\beta^2(\alpha+\theta\beta)}p_1 \\ &+ \frac{\left[\alpha^4\delta+(1-2(1-\theta)\alpha)\beta\alpha^3+\left((1+\theta^2-4\theta)\delta-1+2\theta\right)\beta^2\alpha^2\right]}{\beta^2(\alpha+\theta\beta)}p_2 \\ &+ \frac{\left(2(1-\theta)\theta\delta-2\theta+\theta^2\right)\beta^3\alpha+(\delta-1)\theta^2\beta^4}{\beta^2(\alpha+\theta\beta)}p_2 \\ &+ \frac{-\delta\beta\left(\alpha^2+\theta\beta^2+\alpha^2\beta\theta-\alpha^3-\beta-\beta^2\alpha\delta+\alpha\beta\right)+\theta\beta^3(2-\alpha)}{\beta^2(\alpha+\theta\beta)} \\ &+ 2-\frac{\alpha^2\delta-\alpha\beta(\delta-2)\theta\beta^2}{\beta(\alpha+\theta\beta)} \\ &= 0.\end{aligned}$$

Then, stable eigenvalues lie within the triangular region in p_1p_2 plane bounded by the straight lines L_1, L_2, L_3 for particular parametric values.

5. Numerical simulations

In this section, we will give two numerical examples to support the theoretical finding in former sections. We use Maple and Matlab programs for numerical simulations.

Example 5.1. We take $\alpha = 0.1$, $\beta = 0.4, \theta = 0.9, \delta \in [3.4, 4.2]$ and initial condition $(x_0, y_0) = (0.24, 7)$, then the system (1.2) undergoes period-doubling bifurcation as the bifurcation parameter δ varies in small neighborhood of $\delta = 3.418082937$. At $(\alpha, \beta, \theta, \delta) = (0.1, 0, 4, 0, 9, 3.418082937)$, the system (1.2) has unique positive fixed point $(0.25, 7.192386133)$. The characteristic equation of the Jacobian matrix of the system (1.2) is given by:

$$\lambda^2 + 0.07817811033\lambda - 0.9218218902 = 0. \tag{5.1}$$

The eigenvalues are obtained as $\lambda_1 = -1, \lambda_2 = 0.9218218899$ with $|\lambda_2| \neq 1$.
Therefore, $(\alpha, \beta, \theta, \delta) = (0.1, 0, 4, 0, 9, 3.418082937) \in \Omega_{FB}$. The bifurcation diagrams of the system (1.2) are shown in Figure 1. From Figures 1a and 1b, we observe that the positive fixed point $(0.25, 7.192386133)$ of the system (1.2) is stable for $\delta < 3.418082937$ and loses its stability through a period-doubling bifurcation for $\delta = 3.418082937$ and $\delta > 3.418082937$. There is a period-doubling cascade in orbits of periods- 2,4,8,16 and nonperiodic oscillations as parameter δ varies. Maximum Lyapunov exponents which exhibit the existence of periodic orbits and the chaotic behavior are plotted in Figure 1c. It is shown that maximum Lyapunov values are sometimes negative and sometimes positive. The positive Lyapunov exponent values support the existence of chaotic oscillations in the nonlinear systems.

Example 5.2 For the parameter values $\alpha = 1.5, \beta = 3.25, \theta = 0.2, \delta \in [6.4, 7.5]$ and initial condition $(x_0, y_0) = (0.45, 3.55)$, the positive fixed point of the system (1.2) is evaluated as $(0.4615384615, 3.607804878)$. Then the system (1.2) endures Neimark–Sacker bifurcation as $\delta_{NS} = 6.531707317$. The characteristic equation of the Jacobian matrix of the system (1.2) is obtained as follows:

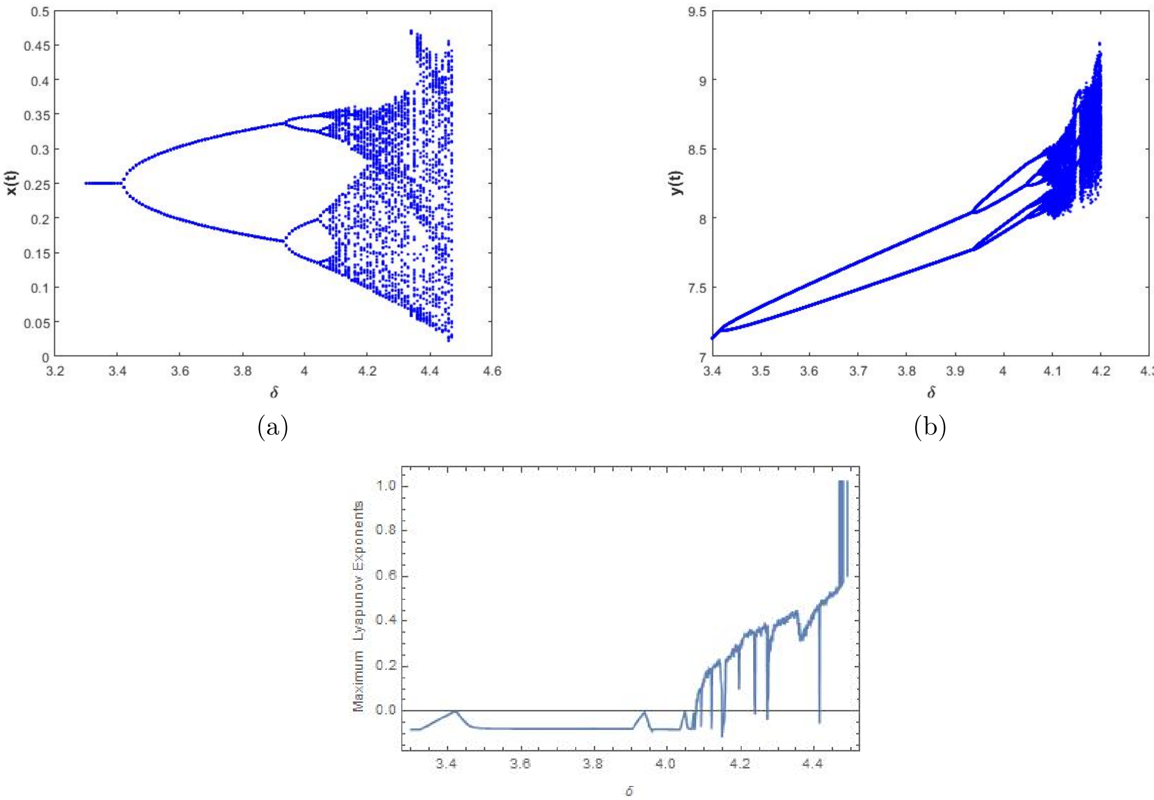
$$\lambda^2 + 1.775609754\lambda + 1.000000000 = 0. \tag{5.2}$$

We can get the roots of the characteristic equation (5.2) as $\lambda_{1,2} = -0.887804877 \pm 0.4602200565i$ with $|\lambda_{1,2}| = 1$. Then $(\alpha, \beta, \theta, \delta) = (1.5, 3.25, 0, 2, 6.531707317) \in \Omega_{NSB}$. The corresponding bifurcation diagrams and maximum Lyapunov exponents (MLE) are plotted in Figure 2. The bifurcation diagrams in Figures 2a and 2b show that the sthe stability of $(0.4615384615, 3.607804878)$ coexistence fixed point happens for $\delta < 6.531707317$

and loses its stability at $\delta = 6.531707317$ and attracting invariant curve appears if $\delta > 6.531707317$. Maximum

Lyapunov exponents are numerically computed to confirm the existence of the chaotic sets in Figure 2c.

To apply the OGY feedback control method for system (1.2), we take $(\alpha, \beta, \theta, \delta_0) = (1.5, 3.25, 0, 2, 6.6)$. System (1.2) has the unstable fixed point $(0.4615384615, 3.660512821)$. Then we give the following controlled system corresponding to these parametric values:



(a) (b)

(c)

Figure 1. Bifurcation diagrams and MLE for the system (1.2) for values of system (1.2) for values of $\alpha = 0.1, \beta = 0.4, \theta = 0.9, \delta \in [3.4, 4.2]$ and initial condition $(x_0, y_0) = (0.24, 7)$.
a. Bifurcation diagram for x_t **b.** Bifurcation diagram for y_t. **c.** Maximum Lyapunov exponents

$$
\begin{aligned}
x_{t+1} &= (6.6 - p_1(x - 0.4615384615) - p_2(y - 3.660512821))x_t(1 - x_t) - x_t y_t \frac{x_t}{x_t + \theta} \\
y_{t+1} &= y_t(1 - \alpha) + \beta x_t y_t
\end{aligned}
\tag{5.3}
$$

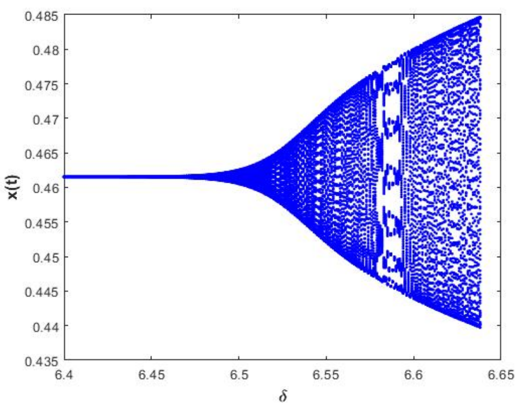
where $K = [\rho_1 \quad \rho_2]$ is a matrix. We have

$$
A = \begin{bmatrix} -2.818246869 & -0.3220035778 \\ 11.89666667 & 1 \end{bmatrix},
$$

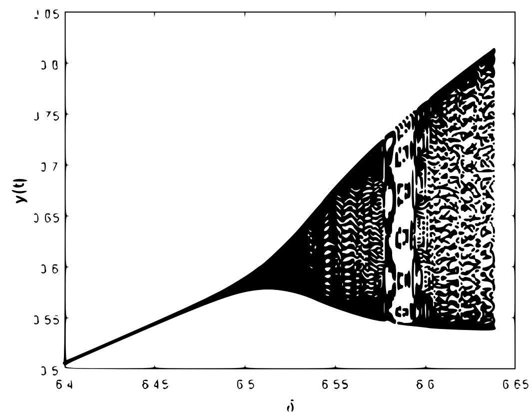
$$
B = \begin{bmatrix} 0.2485207100 \\ 0 \end{bmatrix},
$$

and

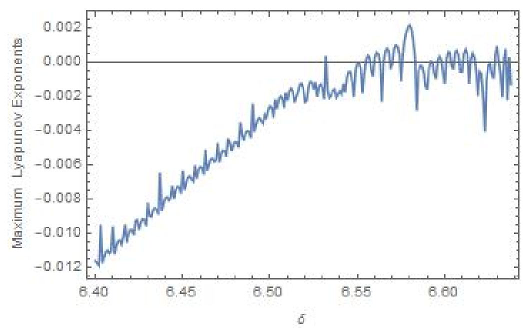
$$
\begin{aligned}
C &= [B : AB] \\
&= \begin{bmatrix} 0.2485207100 & -0.7003927128 \\ 0 & 2.956568047 \end{bmatrix}.
\end{aligned}
$$



(a)



(b)



(c)

Figure 2. Bifurcation diagrams and MLE for the system (1.2) for values of $\alpha = 1.5, \beta = 3.25, \theta = 0.2, \delta \in [6.4, 7.5]$ and initial condition $(x_0, y_0) = (0.45, 3.55)$.
a. Bifurcation diagram for x_t **b.** Bifurcation diagram for y_t. **c.** Maximum Lyapunov exponents

Then it is easy to check that rank of C matrix is 2. Therefore, the system (5.3) is controllable. Then, Jacobian matrix $A - BK$ of the controlled system (5.3) is given by

$$A - BK = \begin{bmatrix} -2.818246869 - 0.2485207100p_1 & -0.3220035778 - .2485207100p_2 \\ 11.89666667 & 1 \end{bmatrix}$$

Moreover, the lines L_1, L_2, and L_3 for marginal stability are given by:

$$\begin{aligned} L_1 &= 0.01252236300 - 0.2485207100p_1 + 2.956568047p_2 = 0, \\ L_2 &= 3.830769232 + 2.956568047p_2 = 0, \\ &\text{and} \\ L_3 &= 0.1942754940 - .4970414200p_1 + 2.956568047p_2 = 0. \end{aligned}$$

Then, the stable triangular region bounded by marginal lines L_1, L_2, and L_3 for the controlled system (5.3) is shown in Figure 3.

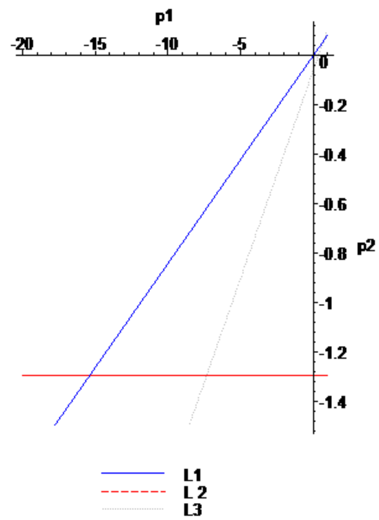


Figure 3. Triangular stability region bounded by L_1, L_2, and L_3 for the controlled system (5.3).

6. Conclusion

In this study, we consider a discrete-time prey-predator model with Allee effect on the prey population. We investigate the existence of the fixed points and their local asymptotic stabilities, the presence of period-doubling, and Neimark–Sacker bifurcation for the fixed point in system (1.2) in order to support the complexity. It is analyzed that the system (1.2) has three fixed points E_1, E_2, E_3. Topological classification of these fixed points of the system (1.1) has been obtained by using linearization technique. We show that if $0 < \alpha < \beta$, then system (1.2) has a unique coexistence fixed point E_3. It is proven that the system (1.2) undergoes both period-doubling bifurcation and Neimark–Sacker bifurcation with the help of center manifold theorem and bifurcation theory. Obtained theoretical results are supported by some figures such as bifurcation diagrams and maximum Lyapunov exponents. Under influence of Neimark–Sacker bifurcation, the system (1.2) produces unstable invariant closed curves. Also, when the system (1.2) undergoes period-doubling bifurcation, we observe that there is a period-doubling cascade in orbits of periods- 2,4,8,16 and nonperiodic oscillations as parameter δ varies. It is well known that the existence or nonexistence of chaotic solutions for a dynamical system is determined by calculating Lyapunov exponent. Generally, a positive Lyapunov exponent is considered to be one of the characteristics which imply the existence of chaos. That is, when the system has a positive largest Lyapunov exponent, then the system exhibits chaotic dynamics. We show that Lyapunov exponent values are sometimes negative and sometimes positive. Lyapunov exponent values larger than 0 confirm the existences of the chaos, quasi-periodic orbits, and periodic orbits in the chaotic region. Bifurcation and fluctuating behaviors of the system (1.2) are controlled through the utilization of chaos control strategies. We reveal that stability can be rebuilt through the OGY feedback control method which is based on feedback control methodology for an extensive range of parameters.

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References

[1] Allee WC. Animal Aggregations: A study in General Sociology, University of Chicago Press, Illinois, Ill, USA, 1931.

[2] Berec L, Angulo E, Counchamp F. Multiple Allee effects and population management. Trends in Ecology and Evolution 2006, 22 (4): 185-191. doi. 10.1016/j.tree.2006.12.002

[3] Caz MT, Hastings A. Allee effects in biological invasions. Ecology Letters 2005; 8 : 895-908. doi. 10.1111/j.1461-0248.2005.00787.x

[4] Chen LL, Lin ZS. The effect of habitat destruction on metapopulations with the Allee-like effect: a study case of Yancheng in Jiangsu Province. China Ecological Modelling 2008; 213 (3): 356-364. doi. 10.1016/j.ecolmodel.2007.12.016

[5] Correigh MG. Habitat selection reduces extinction of populations subject to Allee effects. Theoretical Population Biology 2003; 64: 1-10. doi: 10.1016/s0040-5809(03)00025-x

[6] Courchamp F, Clutton-Brock T, Grenfell B. Inverse density dependence and the Allee effect. Trends in Ecology and Evolution 1999; 14(10): 405-410. doi.10.1016/S0169-5347(99)01683-3.

[7] David SB, Maurice WS, Berec L. How predator functional responses and Allee effects in prey affect the paradox of enrichment and population collapses. Theorotical Population Biology 2007; 72: 136-147.

[8] Din Q, Donchev T, Kolev D. Stability, bifurcation analysis and chaos control in chlorine dioxide-iodine-maionic acid reaction, MATCH Communications in Mathematical and in Computer Chemistry 2018; 79: 577-606.

[9] Ferdy JB, Molofsky J. Allee effect, spatial structure and species coexistance. Journal of Theoretical Biology 2002; 217: 413-427. doi. 10.1006/jtbi.2002.3051

[10] Hopper KR, Roush RT. Mate finding, dispersal, number released, and the success of biological-control introductions. Ecological Entomology 1993; 18: 321–331.

[11] Işık S. A study of stability and bifurcation analysis in discrete-time predator–prey system involving the Allee effect., International Journal of Biomathematics 2019; 12(1): 1950011. doi. 10.1142/S1793524519500116

[12] Kangalgil F, Seval I. Controlling chaos and Neimark-sacker bifurcation in a discrete-time predator-prey system. Hacettepe Journal of Mathematics and Statistics 2020; 49 (5): 1761-1776. doi.10.15672/hujms.531024

[13] Kangalgil F, Gümüşboğa F. Period-doubling bifurcation and stability in a two dimensional discrete prey-predator model with Allee effect and immigration parameter on prey. Cumhuriyet Science Journal 2022; 43 (1): 88-97. doi.org/10.17776/csj.1026330

[14] Kangalgil F. Neimark–Sacker bifurcation and stability analysis of a discrete-time prey–predator model with Allee effect in prey. Advances in Difference Equations 2019; 2019 (1): 1-12. doi.org/10.1186/s13662-019-2039-y

[15] Kangalgil F. Flip bifurcation and stability in a discrete-time prey-predator model with Allee Effect. Cumhuriyet Science Journal 2019; 40 (1): 141-149. doi.org/10.17776/csj.509898

[16] Kent A, Doncaster CP, Sluckin T. Consequences for predators of rescue and Allee effects on prey. Ecological Modelling 2003; 162: 233-245. doi. 10.1016/S0304-3800(02)00343-5

[17] Lin ZS, Li BL. The maximum sustainable yield of Allee dynamic system. Ecological Modelling 2002; 154: 1-7.

[18] Lotka AJ. Elements of Mathematical Biology, Williams & Wilkins, Baltimore.1925.

[19] Ott E, Grebogi C, Yorke JA. Controlling chaos. Physical Review Letters 1990; 4 (11): 1196-1199.

[20] Patric CT, Stefanie LW, Derek MJ, Bjørnstad ON, Andrew ML. Invasion speed is affected by geographical variation in the strength of Allee effects. Ecology Letters 2007; 10: 36-43. doi. 10.1111/j.1461-0248.2006.00991.x

[21] Regina CA, Simone AD, Michel ISC. A numerical model to solve single-species invasion problems with Allee effects. Ecological Modelling 2006; 192: 601-617. doi. 10.1016/j.ecolmodel.2005.07.018

[22] Saifuddin M, Biswas S, Samanta S, Sarkar S, Chattopadhyay J. Complex dynamics of an eco-epidemiological model with different competition coefficients and weak Allee in the predator. Chaos Solitons Fractals 2016; 91: 270-285. doi. 10.1016/j.chaos.2016.06.009

[23] Sasmal SK, Chattopadhyay J. An eco-epidemiological system with infected prey and predator subject to the weak Allee effect. Mathematical Biosciences 2013; 246 (2): 260-271.

[24] Selvam AGM, Jacintha M, Dhineshbabu R. Bifurcation analysis and chaotic behaviour in discrete – Time Predator-Prey System. International Journal of Computational Engineering Research 2019; 9 (4): 1-9.

[25] Stephens PA, Sutherland WJ. Freckleton RP. What is the Allee effect? Oikos 1999; 87 (1): 185-190.

[26] Stephens PA, Sutherland WJ. Consequences of the Allee effect for behaviour, ecology and conservation. Trends in Ecology and Evolution 1999; 14 (10): 401-405.

[27] Van KT, De Roos AM, Persson L. Bistability and an Allee effect as emergent consequeences of stage-specific predation. Journal of Theoretical Biology 2005; 237: 67-74.

[28] Volterra V. Variazioni e fluttuazioni del numero di' individui in specie animali conviventi. Memoria della Reale Accademia Nazionale dei Lincei 1926; 2: 31-113.

[29] Wang MH, Kot M. Speeds of invasion in a model with strong or weak Allee effects. Mathematical Biosciences 2001; 171: 83-97. doi. 10.1016/S0025-5564(01)00048-7

[30] Zhou SR, Liu CZ, Wang G. The competitive dynamics of metapopulations subject to the Allee-like effect. Theorotical Population Biology. 2004; 65 (1): 29-37.

[31] Zhou SR, Liu CZ, Wan G. The Stability of predator-prey systems subject to the Allee effects. Theorotical Population Biology 2005; 67: 23-31. doi. 10.1016/j.tpb.2004.06.007

[32] Zhou SR, Wang G. Allee-like effects in metapopulation dynamics. Mathematical Biosciences 2004; 189: 103-113. doi. 10.1016/j.mbs.2003.06.001