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On S-comultiplication modules

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Abstract: Let R be a commutative ring with $1 \neq 0$ and M be an R-module. Suppose that $S \subseteq R$ is a multiplicatively closed set of R. Recently Sevim et al. in [19] introduced the notion of an S-prime submodule which is a generalization of a prime submodule and used them to characterize certain classes of rings/modules such as prime submodules, simple modules, torsion free modules, S-Noetherian modules and etc. Afterwards, in [2], Anderson et al. defined the concepts of S-multiplication modules and S-cyclic modules which are S-versions of multiplication and cyclic modules and extended many results on multiplication and cyclic modules to S-multiplication and S-cyclic modules. Here, in this article, we introduce and study S-comultiplication modules which are the dual notion of S-multiplication module. We also characterize certain classes of rings/modules such as comultiplication modules, S-second submodules, S-prime ideals and S-cyclic modules in terms of S-comultiplication modules. Moreover, we prove S-version of the dual Nakayama's Lemma.

Key words: S-multiplication module, S-comultiplication module, S-prime submodule, S-second submodule

1. Introduction

Throughout this article we focus only on commutative rings with a unity and nonzero unital modules. R will always denote such a ring and M will denote such an R-module. This paper aims to introduce and study the concept of S-comultiplication modules which are both the dual notion of S-multiplication modules and a generalization of comultiplication modules. Sevim et al. in their paper [19] gave the concept of S-prime submodules and used them to characterize certain classes of rings/modules such as prime submodules, simple modules, torsion-free modules and S-Noetherian rings. A nonempty subset S of R is said to be a multiplicatively closed set (briefly, m.c.s.) of R if $0 \notin S, 1 \in S$ and $st \in S$ for each $s, t \in S$. From now on S will always denote a m.c.s. of R. Suppose that P is a submodule of M, K is a nonempty subset of M and J is an ideal of R. Then the residuals of P by K and J are defined as follows:

$$(P : K) = \{x \in R : xK \subseteq P\}$$

$$(P :_M J) = \{m \in M : Jm \subseteq P\}.$$

In particular, if $P = 0$, we sometimes use $ann(K)$ instead of $(0 : K)$. Recall from [19] that a submodule P of M is said to be S*-prime* if $(P : M) \cap S = \emptyset$ and there exists $s \in$S$-S$ such that $am \in P$ for some $a \in R$ and $m \in M$ implies either $sa \in (P : M)$ or $sm \in P$. In particular, an ideal I of R is said to be S*-prime* if I is an

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S-prime submodule of M. We note here that if $S \subseteq u(R)$, where $u(R)$ is the set of all units in R, the notions of an S-prime submodule and a prime submodule are the same.

Recall that an R-module M is said to be multiplication if each submodule N of M has the form $N = IM$ for some ideal I of R [12]. It is easy to note that M is a multiplication module if and only if $N = (N : M)M$ [16]. The authors in [16] showed that for a multiplication module M, a submodule N of M is prime if and only if $(N : M)$ is a prime ideal of R [16, Corollary 2.11].

The dual notion of prime submodule which is called a second submodule was first introduced and studied by S. Yassemi in [20]. Recall that a nonzero submodule P of M is said to be second if for each $a \in R$, either $aP = 0$ or $aP = P$. Note that if P is a second submodule of M, then $ann(P)$ is a prime ideal of R. For the last twenty years the dual notion of a prime submodule has attracted many researchers and it has been studied in many papers. See, for example, [5–7, 9, 13, 14]. Also, the notion of comultiplication module which is the dual notion of a multiplication module was first introduced by Ansari-Toroghy and Farshadifar in [8] and has been widely studied by many authors. See, for instance, [1, 10, 11, 15]. Recall from [8] that an R-module M is said to be comultiplication if each submodule N of M has the form $N = (0 :_M I)$ for some ideal I of R. Note that M is a comultiplication module if and only if $N = (0 :_M ann(N))$. for each submodule N of M.

Recently Anderson et al. in [2], introduced the notions of S-multiplication modules and S-cyclic modules, and they extended many properties of multiplication and cyclic modules to these two new classes of modules. They also showed that for S-multiplication modules, any submodule N of M is an S-prime submodule if and only if $(N : M)$ is an S-prime ideal of R [2, Proposition 4]. An R-module M is said to be S-multiplication if for each submodule N of M, there exist $s \in S$ and an ideal I of R such that $sN \subseteq IM \subseteq N$. Also M is said to be an S-cyclic module if there exists $s \in S$ such that $sM \subseteq Rm$ for some $m \in M$. They also showed that every S-cyclic module is an S-multiplication module and they characterized finitely generated multiplication modules in terms of S-cyclic modules (See, [2, Proposition 5] and [2, Proposition 8]).

Farshadifar in her paper [17] defined the dual notion of an S-prime submodule which is called an S-second submodule and investigated its many properties similar to second submodules. Recall that a submodule N of M is said to be an S-second if $ann(N) \cap S = \emptyset$ and there exists $s \in S$ such that either $saN = 0$ or $saN = sN$ for each $a \in R$. In particular, the author in [17] investigated the S-second submodules of comultiplication modules. Here we introduce S-comultiplication modules which are the dual notion of S-multiplication modules and investigate their many properties. Recall that an R-module M is said to be an S-comultiplication module if for each submodule N of M, there exist an $s \in S$ and an ideal I of R such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$.

Among other results in this paper, we chracterize certain classes of rings/modules such as comultiplication modules, S-second submodules, S-prime ideals and S-cyclic modules (See, Theorem 2.9, Theorem 2.14, Proposition 3.1, Theorem 3.4, Theorem 3.5, Theorem 3.6 and Theorem 4.6). Also, we prove the S-version of the Dual Nakayama's Lemma (See, Theorem 2.16).

2. S-comultiplication modules

Definition 2.1 *Let M be an R-module and $S \subseteq R$ be a m.c.s. of R. M is said to be an S-comultiplication module if for each submodule N of M, there exist an $s \in S$ and an ideal I of R such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$. In particular, a ring R is said to be an S-comultiplication ring if it is an S-comultiplication module over itself.*

Example 2.2 *Every R-module M with $ann(M) \cap S \neq \emptyset$ is trivially an S-comultiplication module.*

Example 2.3 ***(An S-comultiplication module that is not S-multiplication)*** *Let p be a prime number and consider the $\mathbb{Z}$-module*

$$E(p) = \{\alpha = \frac{m}{p^n} + \mathbb{Z} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}.$$

Then every submodule of $E(p)$ is of the form $G_t = \{\alpha = \frac{m}{p^t} + \mathbb{Z} : m \in \mathbb{Z}\}$ for some fixed $t \geq 0$. Take the multiplicatively closed set $S = \{1\}$. Note that $(G_t : E(p))E(p) = 0_{E(p)} \neq G_t$ for each $t \geq 1$. Then $E(p)$ is not an S-multiplication module. Now we will show that $E(p)$ is an S-comultiplication module. Let $t \geq 0$. Then it is easy to see that $(0 :_{E(p)} ann(G_t)) = (0 :_{E(p)} p^t\mathbb{Z}) = G_t$. Therefore $E(p)$ is an S-comultiplication module.

Example 2.4 *Every comultiplication module is also an S-comultiplication module. Also the converse is true provided that $S \subseteq u(R)$.*

Example 2.5 ***(An S-comultiplication module that is not comultiplication)*** *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$ and $S = reg(\mathbb{Z}) = \mathbb{Z} - \{0\}$. Now take the submodule $N = m\mathbb{Z}$, where $m \neq 0, \pm 1$. Then $(0 : ann(m\mathbb{Z})) = \mathbb{Z} \neq m\mathbb{Z}$ so that M is not a comultiplication module. Now take a submodule K of M. Then $K = k\mathbb{Z}$ for some $k \in \mathbb{Z}$. If $k = 0$, then choose $s = 1$ and note that $s(0 : ann(K)) = (0) = k\mathbb{Z}$. If $k \neq 0$, then choose $s = k$ and note that $s(0 : ann(K)) \subseteq k\mathbb{Z} = K \subseteq (0 : ann(K))$. Therefore M is an S-comultiplication module.*

Lemma 2.6 *Let M be an R-module. The following statements are equivalent.*

(i) M is an S-comultiplication module.

(ii) For each submodule N of M, there exists $s \in S$ such that $s(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$.

(iii) For each submodule K, N of M with $ann(K) \subseteq ann(N)$, there exists $s \in S$ such that $sN \subseteq K$.

Proof $(i) \Rightarrow (ii)$: Suppose that M is an S-comultiplication module and take a submodule N of M. Then by definition, there exist $s \in S$ and an ideal I of R such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$. Then note that $IN = (0)$ and so $I \subseteq ann(N)$. This gives that $s(0 :_M ann(N)) \subseteq s(0 :_M I) \subseteq N \subseteq (0 :_M ann(N))$ which completes the proof.

$(ii) \Rightarrow (iii)$: Suppose that $ann(K) \subseteq ann(N)$ for some submodules N, K of M. By (ii), there exist $s_1, s_2 \in S$ such that

$$s_1(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$$
$$s_2(0 :_M ann(K)) \subseteq K \subseteq (0 :_M ann(K)).$$

Since $ann(K) \subseteq ann(N)$, we have $(0 :_M ann(N)) \subseteq (0 :_M ann(K))$ and so

$$\begin{aligned} s_1 s_2(0 :_M ann(N)) &\subseteq s_2 N \subseteq s_2(0 :_M ann(N)) \\ &\subseteq s_2(0 :_M ann(K)) \subseteq K \end{aligned}$$

which completes the proof.

$(iii) \Rightarrow (ii)$: Suppose that (iii) holds. Let N be a submodule of M. Then it is clear that $ann(N) = ann(0 :_M ann(N))$. Then by (iii), there exists $s \in S$ such that $s(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$.

$(ii) \Rightarrow (i)$: It is clear. □

Let S be a m.c.s. of R. The saturation $S^\star$ of S is defined by $S^\star = \{x \in R : x|s \text{ for some } s \in S\}$. Also S is said to be a saturated m.c.s. of R if $S = S^\star$. Note that $S^\star$ is always a saturated m.c.s. of R containing S.

Proposition 2.7 *Let M be an R-module and S be a m.c.s. of R. The following assertions hold.*

(i) Let S_1 and S_2 be two m.c.s. of R and $S_1 \subseteq S_2$. If M is an S_1-comultiplication module, then M is also an S_2-comultiplication module.

(ii) M is an S-comultiplication module if and only if M is an $S^\star$-comultiplication module, where S^* is the saturation of S.

Proof (i): Clear.

(ii): Assume that M is an S-comultiplication module. Since $S \subseteq S^\star$, the result follows from the part (i).

Suppose M is an $S^\star$-comultiplication module. Take a submodule N of M. Since M is $S^\star$-comultiplication module, there exists $x \in S^\star$ such that $x(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$ by Lemma 2.6. Since $x \in S^\star$, there exists $s \in S$ such that $x|s$, that is, $s = rx$ for some $r \in R$. This implies that $s(0 :_M ann(N)) \subseteq x(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$. Thus M is an S-comultiplication module. □

Anderson and Dumitrescu, in 2002, defined the concept of S-Noetherian rings which is a generalization of Noetherian rings and they extended many properties of Noetherian rings to S-Noetherian rings. Recall from [4] that a submodule N of M is said to be an S-finite submodule if there exists a finitely generated submodule K of M such that $sN \subseteq K \subseteq N$. Also, M is said to be an S-Noetherian module if each submodule is S-finite. In particular, R is said to be an S-Neotherian ring if it is an S-Noetherian R-module.

Proposition 2.8 *Let R be an S-Noetherian ring and M be an S-comultiplication module. Then $S^{-1}M$ is a comultiplication module.*

Proof Let W be a submodule of $S^{-1}M$. Then, $W = S^{-1}N$ for some submodule N of M. Since M is an S-comultiplication module, there exists $s \in S$ such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$ for some ideal I of R. Then we get $S^{-1}(s(0 :_M I)) = S^{-1}((0 :_M I)) \subseteq S^{-1}N \subseteq S^{-1}((0 :_M I))$, that is, $S^{-1}N = S^{-1}((0 :_M I))$. Now we will show that $S^{-1}((0 :_M I)) = (0 :_{S^{-1}M} S^{-1}I)$. Let $\frac{m}{s'} \in S^{-1}((0 :_M I))$ where $m \in (0 :_M I)$ and $s' \in S$. Then we have $Im = (0)$ and so $(S^{-1}I)(\frac{m}{s'}) = (0)$. This implies that $\frac{m}{s'} \in (0 :_{S^{-1}M} S^{-1}I)$. For the converse, let $\frac{m}{s'} \in (0 :_{S^{-1}M} S^{-1}I)$. Then, we have $(S^{-1}I)(\frac{m}{s'}) = (0)$. This implies that for each $x \in I$, there exists $s'' \in S$ such that $s''xm = 0$. Since R is an S-Noetherian ring, I is S-finite. So, there exists $s^\star \in S$ and $a_1, a_2, \ldots, a_n \in I$ such that $s^\star I \subseteq (a_1, a_2, \ldots, a_n) \subseteq I$. As $\left(S^{-1}I\right)\left(\frac{m}{s'}\right) = (0)$ and $a_i \in I$, there exists $s_i \in S$ such that $s_i a_i m = 0$. Now, put $t = s_1 s_2 \cdots s_n s^\star \in S$. Then we have $t a_i m = 0$ for all a_i and so $tIm = 0$. Then we deduce $\frac{m}{s'} = \frac{tm}{ts'} \in S^{-1}((0 :_M I))$. Thus, $S^{-1}((0 :_M I)) = (0 :_{S^{-1}M} S^{-1}I)$ and so $W = S^{-1}N = (0 :_{S^{-1}M} S^{-1}I)$. Therefore, $S^{-1}M$ is a comultiplication module. □

Recall from [2] that a m.c.s. S of R is said to satisfy the maximal multiple condition if there exists $s \in S$ such that t divides s for each $t \in S$.

Theorem 2.9 *Let M be an R-module and S be a m.c.s. of R satisfying the maximal multiple condition. Then M is an S-comultiplication module if and only if $S^{-1}M$ is a comultiplication module.*

Proof $(\Rightarrow)$: Suppose that W is a submodule of $S^{-1}M$. Then $W = S^{-1}N$ for some submodule N of M. Since M is an S-comultiplication module, there exist $t' \in S$ and an ideal I of R such that $t'(0 :_M I) \subseteq N \subseteq (0 :_M I)$. This implies that $IN = (0)$ and so $S^{-1}(IN) = (S^{-1}I)(S^{-1}N) = 0$. Then we have $S^{-1}N \subseteq (0 :_{S^{-1}M} S^{-1}I)$. Let $\frac{m'}{s'} \in (0 :_{S^{-1}M} S^{-1}I)$. Then we get $\frac{a}{1}\frac{m'}{s'} = 0$ for each $a \in I$ and this yields that $uam' = 0$ for some $u \in S$. As S satisfies the maximal multiple condition, there exists $s \in S$ such that $u|s$ for each $u \in S$. This implies that $s = ux$ for some $x \in R$. Then we have $sam' = xuam' = 0$. Then we conclude that $Ism' = 0$ and so $sm' \in (0 :_M I)$. This yields that $t'sm' \in t'(0 :_M I) \subseteq N$ and so $\frac{m'}{s'} = \frac{t'sm'}{t'ss'} \in S^{-1}N$. Then we get $S^{-1}N = (0 :_{S^{-1}M} S^{-1}I)$ and so $S^{-1}M$ is a comultiplication module.

$(\Leftarrow)$: Suppose that $S^{-1}M$ is a comultiplication module. Let N be a submodule of M. Since $S^{-1}M$ is comultiplication, $S^{-1}N = (0 :_{S^{-1}M} S^{-1}I)$ for some ideal I of R. Then we have $(S^{-1}I)(S^{-1}N) = S^{-1}(IN) = 0$. Then for each $a \in I, m \in N$, we have $\frac{am}{1} = 0$ and thus $uam = 0$ for some $u \in S$. By the maximal multiple condition, there exists $s \in S$ such that $sam = 0$ and so $sIN = 0$. This implies that $N \subseteq (0 :_M sI)$. Now, let $m \in (0 :_M sI)$. Then $Ism = 0$ so it is easily seen that $(S^{-1}I)\frac{m}{1} = 0$. Then we conclude that $\frac{m}{1} \in (0 :_{S^{-1}M} S^{-1}I) = S^{-1}N$. Then there exists $x \in S$ such that $xm \in N$. Again by the maximal multiple condition, $sm \in N$. Then we have $s(0 :_M sI) \subseteq N \subseteq (0 :_M sI)$. Since sI is an ideal of R, M is an S-comultiplication module. □

Theorem 2.10 *Let $f : M \to M'$ be an R-homomorphism and $tKer(f) = (0)$ for some $t \in S$.*

(i) If M' is an S-comultiplication module, then M is an S-comultiplication module.

(ii) If f is an R-epimorphism and M is an S-comultiplication module, then M' is an S-comultiplication module.

Proof (i) Let N be a submodule of M. Since M' is an S-comultiplication module, there exist $s \in S$ and an ideal I of R such that $s(0 :_{M'} I) \subseteq f(N) \subseteq (0 :_{M'} I)$. Then we have $If(N) = f(IN) = 0$ and so $IN \subseteq Kerf$. Since $tKer(f) = 0$, we have $tIN = (0)$ and so $N \subseteq (0 :_M tI)$. Now we will show that $t^2s(0 :_M tI) \subseteq N \subseteq (0 :_M tI)$. Let $m \in (0 :_M tI)$. Then we have $tIm = 0$ and so $f(tIm) = tIf(m) = If(tm) = 0$. This implies that $f(tm) \in (0 :_{M'} I)$. Thus we have $sf(tm) = f(stm) \in s(0 :_{M'} I) \subseteq f(N)$ and so there exists $y \in N$ such that $f(stm) = f(y)$ and so $stm - y \in Ker(f)$. Thus we have $t(stm - y) = 0$ and so $t^2sm = tx$. Then we obtain

$$t^2s(0 :_M tI) \subseteq tN \subseteq N \subseteq (0 :_M tI).$$

Now put $t^2s = s' \in S$ and $J = tI$. Thus

$$s'(0 :_M J) \subseteq N \subseteq (0 :_M J).$$

Therefore M is an S-comultiplication module.

(ii) Let N' be a submodule of M'. Since M is an S-comultiplication module, there exist $s \in S$ and an ideal I of R such that

$$s(0 :_M I) \subseteq f^{-1}(N') \subseteq (0 :_M I).$$

This implies that $If^{-1}(N') = (0)$ and so $f(If^{-1}(N')) = IN' = (0)$ since f is surjective. Then, we have $N' \subseteq (0 :_{M'} I)$. On the other hand, we get $f(s(0 :_M I)) = sf((0 :_M I)) \subseteq f(f^{-1}(N')) = N'$. Now, let $m' \in (0 :_{M'} I)$. Then, $Im' = 0$. Since f is epimorphism, there exists $m \in M$ such that $m' = f(m)$. Then we have $Im' = If(m) = f(Im) = 0$ and so $Im \subseteq Kerf$. Since $tKer(f) = 0$, we have $tIm = (0)$ and so $tm \in (0 :_M I)$. Then we get $f(tm) = tf(m) = tm' \in f((0 :_M I))$. Thus we have $t(0 :_{M'} I) \subseteq f((0 :_M I))$ and hence $st(0 :_{M'} I) \subseteq sf((0 :_M I)) \subseteq N' \subseteq (0 :_{M'} I)$. Thus M' is an S-comultiplication module. □

As an immediate consequences of previous theorem, we give the following explicit results.

Corollary 2.11 *Let M be an R-module, N be a submodule of M and S be a m.c.s. of R. Then we have the following.*

(i) If M is an S-comultiplication module, then N is an S-comultiplication module.

(ii) If M is an S-comultiplication module and $tM \subseteq N$ for some $t \in S$, then M/N is an S-comultiplication R-module.

Proposition 2.12 *Let M_i be an R_i-module and S_i be a m.c.s. of R_i for each $i = 1, 2$. Suppose that $M = M_1 \times M_2$, $R = R_1 \times R_2$ and $S = S_1 \times S_2$. The following assertions are equivalent.*

(i) M is an S-comultiplication R-module.

(ii) M_1 is an S_1-comultiplication R_1-module and M_2 is an S_2-comultiplication R_2-module.

Proof $(i) \Rightarrow (ii)$: Assume that M is an S-comultiplication R-module. Take a submodule N_1 of M_1. Then $N_1 \times \{0\}$ is a submodule of M. Since M is an S-comultiplication module, there exist $s = (s_1, s_2) \in S_1 \times S_2$ and an ideal $J = I_1 \times I_2$ of R such that $(s_1, s_2)(0 :_M I_1 \times I_2) \subseteq N_1 \times \{0\} \subseteq (0 :_M I_1 \times I_2)$, where I_i is an ideal of R_i. Then we can easily get $s_1(0 :_{M_1} I_1) \subseteq N_1 \subseteq (0 :_{M_1} I_1)$ which shows that M_1 is an S_1-comultiplication module. Similarly, taking a submodule N_2 of M_2 and a submodule $\{0\} \times N_2$ of M, we can show that M_2 is an S_2-comultiplication module.

$(ii) \Rightarrow (i)$: Now assume that M_1 is an S_1-comultiplication module and M_2 is an S_2-comultiplication module. Let N be a submodule of M. Then we can write $N = N_1 \times N_2$ for some submodule N_i of M_i. Since M_1 is an S_1-comultiplication module,

$$s_1(0 :_{M_1} I_1) \subseteq N_1 \subseteq (0 :_{M_1} I_1)$$

for some ideal I_1 of R_1 and $s_1 \in S_1$. Since M_2 is an S_2-comultiplication module,

$$s_2(0 :_{M_2} I_2) \subseteq N_2 \subseteq (0 :_{M_2} I_2)$$

for some ideal I_2 of R_2 and $s_2 \in S_2$. Put $s = (s_1, s_2) \in S$. Then

$$\begin{aligned} s(0 :_M I_1 \times I_2) &= s_1(0 :_{M_1} I_1) \times s_2(0 :_{M_2} I_2) \\ &\subseteq N_1 \times N_2 \subseteq (0 :_{M_1} I_1) \times (0 :_{M_2} I_2) = (0 :_M I_1 \times I_2) \end{aligned}$$

where $I_1 \times I_2$ is an ideal of R and $(s_1, s_2) \in S$, as needed. □

Theorem 2.13 *Let $M = M_1 \times M_2 \times \cdots \times M_n$ be an $R = R_1 \times R_2 \times \cdots \times R_n$-module and $S = S_1 \times S_2 \times \cdots \times S_n$ be a m.c.s. of R where M_i are R_i-modules and S_i are m.c.s. of R_i for all $i \in \{1, 2, ..., n\}$, respectively. The following statements are equivalent.*

(i) M is an S-comultiplication R-module.

(ii) M_i is an S_i-comultiplication R_i-module for each $i = 1, 2, \ldots, n$.

Proof Here, induction can be applied on n. The statement is true when $n = 1$. If $n = 2$, result follows from Proposition 2.12. Assume that statements are equivalent for each $k < n$. We will show that it also holds for $k = n$. Now put $M' = M_1 \times M_2 \times \cdots \times M_{n-1}$, $R = R_1 \times R_2 \times \cdots \times R_{n-1}$ and $S = S_1 \times S_2 \times \cdots \times S_{n-1}$. Note that $M = M' \times M_n$, $R = R' \times R_n$ and $S = S' \times S_n$. Then by Proposition 2.12, M is an S-comultiplication R-module if and only if M' is an S'-comultiplication R'-module and M_n is an S_n-comultiplication R_n-module. The rest follows from the induction hypothesis. □

Let $\mathcal{P}$ be a prime ideal of R. Then we know that $S_{\mathcal{P}} = R - \mathcal{P}$ is a m.c.s. of R. If an R-module M is an $S_{\mathcal{P}}$-comultiplication module for a prime ideal $\mathcal{P}$ of R, then we say that M is a $\mathcal{P}$-comultiplication module. Now we will characterize comultiplication modules in terms of S-comultiplication modules.

Theorem 2.14 *Let M be an R-module. The following statements are equivalent.*

(i) M is a comultiplication module.

(ii) M is a $\mathcal{P}$-comultiplication module for each prime ideal $\mathcal{P}$ of R.

(iii) M is an $\mathcal{M}$-comultiplication module for each maximal ideal $\mathcal{M}$ of R.

(iv) M is an $\mathcal{M}$-comultiplication module for each maximal ideal $\mathcal{M}$ of R with $M_{\mathcal{M}} \neq 0_{\mathcal{M}}$.

Proof $(i) \Rightarrow (ii):$ Follows from Example 2.4.

$(ii) \Rightarrow (iii):$ Follows from the fact that every maximal ideal is prime.

$(iii) \Rightarrow (iv):$ Clear.

$(iv) \Rightarrow (i):$ Suppose that M is an $\mathcal{M}$-comultiplication module for each maximal ideal $\mathcal{M}$ of R with $M_{\mathcal{M}} \neq 0_{\mathcal{M}}$. Take a submodule N of M and a maximal ideal $\mathcal{M}$ of R. If $M_{\mathcal{M}} = 0_{\mathcal{M}}$, then clearly we have $N_{\mathcal{M}} = (0 :_M ann(N))_{\mathcal{M}}$. So assume that $M_{\mathcal{M}} \neq 0_{\mathcal{M}}$. Since M is an $\mathcal{M}$-comultiplication module, there exists $s_{\mathcal{M}} \notin \mathcal{M}$ such that $s_{\mathcal{M}}(0 :_M ann(N)) \subseteq N$. Then we have

$$(0 :_M ann(N))_{\mathcal{M}} = (s_{\mathcal{M}}(0 :_M ann(N)))_{\mathcal{M}} \subseteq N_{\mathcal{M}} \subseteq (0 :_M ann(N))_{\mathcal{M}}.$$

Thus we have $N_{\mathcal{M}} = (0 :_M ann(N))_{\mathcal{M}}$ for each maximal ideal $\mathcal{M}$ of R. Therefore, $N = (0 :_M ann(N))$ so that M is a comultiplication module. □

Now we shall give the S-version of dual Nakayama's Lemma for S-comultiplication modules. First, we need the following proposition.

Proposition 2.15 *Let M be an S-comultiplication R-module.*

(i) If I is an ideal of R with $(0 :_M I) = 0$, then there exists $s \in S$ such that $sM \subseteq IM$.

(ii) If I is an ideal of R with $(0 :_M I) = 0$, then for every element $m \in M$, there exists $s \in S$ and $a \in I$ such that $sm = am$.

(iii) If M is an S-finite R-module and I is an ideal of R with $(0 :_M I) = 0$, then there exist $s \in S$ and $a \in I$ such that $(s + a)M = 0$.

Proof $(i):$ Suppose that I is an ideal of R with $(0 :_M I) = 0$. Then we have $((0 :_M I) : M) = (0 : IM) = (0 : M)$. Then by Lemma 2.6 (iii), there exists $s \in S$ such that $sM \subseteq IM$.

(ii) : Suppose that I is an ideal of R with $(0:_M I)=0$. Then for any $m\in M$, we have $(0:Rm)=((0:_M I):Rm)=(0:Im)$. Again by Lemma 2.6 (iii), there exists $s\in S$ such that $sRm\subseteq Im$ and so $sm=am$ for some $a\in I$.

(iii) : Suppose that M is an S-finite R-module and I is an ideal of R with $(0:_M I)=0$. Then there exists $t\in S$ such that $tM\subseteq Rm_1+Rm_2+\cdots+Rm_n$ for some $m_1,m_2,\ldots,m_n\in M$. Since $(0:_M I)=0$, by (i), there exists $s\in S$ such that $sM\subseteq IM$. This implies that $stM\subseteq tIM=ItM\subseteq I(Rm_1+Rm_2+\cdots+Rm_n)=Im_1+Im_2+\cdots+Im_n$. Then for each $i=1,2,\ldots,n$ we have $stm_i=a_{i1}m_1+a_{i2}m_2+\cdots+a_{in}m_n$ and so $-a_{i1}m_1-a_{i2}m_2-\cdots+(st-a_{ii})m_i+\cdots-a_{in}m_n=0$. Now, let Δ be the following matrix

$$\begin{bmatrix} st-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & st-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & st-a_{nn} \end{bmatrix}_{n\times n}.$$

Then we have $|\Delta|\, m_i=0$ for each $i=1,2,\ldots,n$. Thus we obtain that $t\,|\Delta|\, M=0$. This implies that $t(s^nt^n+a)M=(s^nt^{n+1}+at)M=0$ for some $a\in I$. Now put $u=s^nt^{n+1}\in S$ and $b=at\in I$. Then we have $(u+b)M=0$ which completes the proof. □

Theorem 2.16 ***(S-dual Nakayama's Lemma)*** *Let M be an S-comultiplication module, where S is a m.c.s. of R satisfying the maximal multiple condition. Suppose that I is an ideal of R such that $tI\subseteq Jac(R)$ for some $t\in S$. If $(0:_M tI)=0$, then there exists $s\in S$ such that $sM=0$.*

Proof Suppose that S satisfies the maximal multiple condition. Then there exists $s\in S$ such that $t|s$ for each $t\in S$. Let I be an ideal of R with $tI\subseteq Jac(R)$ for some $t\in S$ and $(0:_M tI)=0$. Then for each $m\in M$, by Proposition 2.15 (ii), there exists $t'\in S$ such that $t'Rm\subseteq tIm$ and so $s^2t'Rm\subseteq s^2tIm\subseteq s^2Im$. Now, put $u=s^2t'$. By the maximal multiple condition we have $sRm\subseteq uRm\subseteq s^2Im$ and so $sm=s^2am$ for some $a\in R$. On the other hand, we note that $sI\subseteq tI\subseteq Jac(R)$. Thus we have $s(1-sa)m=0$. Since $sa\in Jac(R)$, we get $1-sa$ is unit and so $sm=0$. Thus we have $sM=0$. □

Corollary 2.17 ***(Dual Nakayama's Lemma)*** *Let M be a comultiplication module and I an ideal of R such that $I\subseteq Jac(R)$. If $(0:_M I)=0$, then $M=0$.*

Proof Take $S=\{1\}$ and apply Theorem 2.16. □

3. S-cyclic modules

In this section, we investigate the relations between S-comultiplication modules and S-cyclic modules.

Proposition 3.1 *Let M be an S-comultiplication R-module and N be a minimal ideal of R such that $(0:_M N)=0$. Then M is an S-cyclic module.*

Proof Choose a nonzero element m of M. Since M is an S-comultiplication module, there exist $s\in S$ and an ideal I of R such that $s(0:_M I)\subseteq Rm\subseteq(0:_M I)$. By the assumption $(0:_M N)=0$, we have

$$s((0:_M N):_M I)\subseteq Rm\subseteq((0:_M N):_M I)\Longrightarrow s(0:_M NI)\subseteq Rm\subseteq(0:_M NI).$$

Since $0 \subseteq NI \subseteq N$ and N is minimal ideal of R, either $NI = N$ or $NI = 0$. If the former case holds, we have $s(0 :_M N) \subseteq Rm \subseteq (0 :_M N)$. This means that $Rm = 0$, a contradiction. The second case implies $s(0 :_M 0) \subseteq Rm \subseteq (0 :_M 0)$. This means $sM \subseteq Rm \subseteq M$ proving that M is S-cyclic. □

Proposition 3.2 *Let M be an S-comultiplication module of R. Let $\{M_i\}$ be a collection of submodules of M with $\bigcap_i M_i = 0$. Then, for every submodule N of M there exists an $s \in S$ such that*

$$s\bigcap_i (N + M_i) \subseteq N \subseteq \bigcap_i (N + M_i).$$

Proof Let N be a submodule of M. Since M is an S-comultiplication module, we have $s(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$ for some $s \in S$. This implies $s(\bigcap_i M_i :_M ann(N)) \subseteq N \subseteq (\bigcap_i M_i :_M ann(N))$ since $\bigcap_i M_i = 0$. Then we obtain $s\bigcap_i (M_i :_M ann(N)) \subseteq N \subseteq \bigcap_i (M_i :_M ann(N))$. Thus

$$s\bigcap_i (N + M_i) \subseteq s\bigcap_i (M_i :_M ann(N)) \subseteq N \subseteq \bigcap_i (N + M_i).$$

□

Proposition 3.3 *Let M be an S-comultiplication module. Then for each submodule N of M and each ideal I of R with $N \subseteq s(0 :_M I)$ for some $s \in S$ there exists an ideal J of R such that $I \subseteq J$ and $s(0 :_M J) \subseteq N$.*

Proof Let N be a submodule of M. Since M is an S-comultiplication module, $s(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$ for some $s \in S$. So we obtain $s(0 :_M ann(N)) \subseteq N \subseteq s(0 :_M I)$. Taking $J = I + ann(N)$,

$$s(0 :_M J) = s(0 :_M I + ann(N)) \subseteq s(0 :_M I) \cap s(0 :_M ann(N)) \subseteq s(0 :_M ann(N)) \subseteq N.$$

□

Recall that an R-module M is said to be torsion-free if the set of torsion elements $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$ of M is zero. Also M is called a torsion module if $T(M) = M$. We refer the reader to [3] for more details on the torsion subset $T(M)$ of M.

Theorem 3.4 *Every S-comultiplication module is either S-cyclic or torsion.*

Proof Let M be an S-comultiplication module. Assume that M is not an S-cyclic module and $ann_R(m) = 0$ for some $m \in M$. Since Rm is a submodule of M and M is an S-comultiplication module, we have $s(0 :_M ann(m)) \subseteq Rm \subseteq (0 :_M ann(m))$. This gives $sM \subseteq Rm \subseteq M$ for some $s \in S$. This contradiction completes the proof. Hence, $ann(m) \neq 0$ for all $m \in M$ proving that M is a torsion module. □

Theorem 3.5 *Let R be an integral domain and M be an S-finite and S-comultiplication module. If sM is faithful for each $s \in S$, then M is an S-cyclic module.*

Proof Suppose that M is not an S-cyclic module. Then M is a torsion module by Theorem 3.4. Since M is an S-finite module, there exist $s \in S$ and $m_1, m_2, \ldots, m_n \in M$ such that $sM \subseteq Rm_1 + Rm_2 + \cdots + Rm_n$. This implies that $ann(Rm_1 + Rm_2 + \cdots + Rm_n) = \bigcap_{i=1}^{n} ann(m_i) \subseteq ann(sM) = 0$ since sM is faithful. Since

R is an integral domain, there exists $m_i \in M$ such that $ann(m_i) = 0$ which is a contradiction. Hence M is an S-cyclic module. □

Recall from [19] that an R-module M is said to be an S-torsion-free module if there exists $s \in S$ and so that whenever $am = 0$ for some $a \in R$ and $m \in M$, then either $sa = 0$ or $sm = 0$.

Theorem 3.6 *Every S-comultiplication S-torsion-free module is an S-cyclic module.*

Proof Let M be an S-comultiplication and S-torsion-free module. If $sM = 0$ for some $s \in S$, then M is an S-cyclic module. So assume that $sM \neq 0$ for each $s \in S$. Since M is an S-torsion-free module, there exists $t' \in S$ and such that whenever $am = 0$ for some $a \in R$ and $m \in M$, then either $t'a = 0$ or $t'm = 0$. Since $t'M \neq 0$, there exists $m \in M$ such that $t'm \neq 0$. As M is an S-comultiplication module, there exists $t \in S$ such that $t(0 :_M ann(m)) \subseteq Rm$. Since $ann(m)m = 0$ and M is S-torsion-free module, we conclude either $t'ann(m) = 0$ or $t'm = 0$. The second case is impossible. So we have $t'ann(m) = 0$ and so $t'M \subseteq (0 :_M ann(m))$. This implies that $t'tM \subseteq t(0 :_M ann(m)) \subseteq Rm$ where $t't \in S$, namely, M is an S-cyclic module. □

Let K be a nonzero submodule of M. K is said to be an S-minimal submodule if $L \subseteq K$ for some submodule of M, then there exists $s \in S$ such that $sK \subseteq L$.

Theorem 3.7 *Every S-comultiplication prime R-module M is S-minimal.*

Proof Let M be an S-comultiplication prime R-module. Assume that N is a submodule of M. Since M is prime, $ann(N) = ann(M)$. Also, $(0 :_M ann(N)) = (0 :_M ann(M))$. Since M is an S-comultiplication module, $s(0 :_M ann(N)) \subseteq N \subseteq (0 :_M ann(N))$ for some $s \in S$. Hence we get $s(0 :_M ann(M)) \subseteq N \subseteq (0 :_M ann(M))$ and it shows that $sM \subseteq N \subseteq M$. Therefore M is S-minimal. □

4. S-second submodules of S-comultiplication modules

This section is dedicated to the study of S-second submodules of S-comultiplication modules. Now, we need the following definition.

Definition 4.1 *Let M and M' be two R-modules and $f : M \to M'$ be an R-homomorphism.*

(i) If there exists $s \in S$ such that $f(m) = 0$ implies that $sm = 0$, then f is said to be an S-injective (or, just S-monic).

(ii) If there exists $s \in S$ such that $sM' \subseteq \operatorname{Im} f$, then f is said to be an S-epimorphism (or, just S-epic).

The following proposition is explicit. Let M be an R-module. An element $x \in R$ is called a zero divisor on M if there exists $0 \neq m \in M$ such that $xm = 0$, or equivalently, $ann_M(x) \neq (0)$. The set of all zero divisor elements of R on M is denoted by $z(M)$.

Proposition 4.2 *Let M and M' be two R-modules and $f : M \to M'$ be an R-homomorphism.*

(i) f is S-monic if and only if there exists $s \in S$ such that $sKer(f) = (0)$.

(ii) If f is monic, then f is S-monic for each m.c.s. S of R. The converse holds in case $S \subseteq R - z(M)$.

(iii) If f is epic, then f is S-epic for each m.c.s. S of R. The converse holds in case $S \subseteq u(R)$.

Recall from [19] that a submodule P of M with $(P : M) \cap S = \emptyset$ is said to be *S-prime* if there exists a fixed $s \in S$ such that whenever $am \in P$ for some $a \in R, m \in M$, then either $sa \in (P : M)$ or $sm \in P$. In particular, an ideal I of R is said to be *S-prime* if I is an S-prime submodule of M. We note here that Acraf and Hamed, in their paper [18], studied and investigated further properties of S-prime ideals. Now, we give the following required results which can be found in [19].

Proposition 4.3 *(i) ([19, Proposition 2.9]) If P is an S-prime submodule of M, then $(P : M)$ is an S-prime ideal of R.*

(ii) ([19, Lemma 2.16]) If P is an S-prime submodule of M, there exists a fixed $s \in S$ such that $(P :_M s') \subseteq (P :_M s)$ for each $s' \in S$.

(iii) ([19, Theorem 2.18]) P is an S-prime submodule of M if and only if $(P :_M s)$ is a prime submodule of M for some $s \in S$.

By the previous proposition, we deduce that P is an S-prime submodule if and only if there exists a fixed $s \in S$ such that $(P :_M s)$ is a prime submodule and $(P :_M s') \subseteq (P :_M s)$ for each $s' \in S$.

Sevim et al. in [19] gave many characterizations of S-prime submodules. Now we give a new characterization of S-prime submodules from another point of view.

Recall that a homomorphism $f : M \to M'$ is said to be *S-zero* if there exists $s \in S$ such that $sf(m) = 0$ for each $m \in M$, that is, $s \operatorname{Im} f = (0)$.

Proposition 4.4 *Let P be a submodule of M with $(P : M) \cap S = \emptyset$. The following statements are equivalent.*

(i) P is an S-prime submodule of M.

(ii) There exist a fixed $s \in S$ such that for any $a \in R$ and the homothety $M/P \xrightarrow{a\cdot} M/P$, either S-zero or S-injective with respect to $s \in S$.

Proof $(i) \Rightarrow (ii)$: Suppose that P is an S-prime submodule of M. Then there exists a fixed $s \in S$ such that $am \in P$ for some $a \in R, m \in M$ implies that $saM \subseteq P$ or $sm \in P$. Now, take $a \in R$ and assume that the homothety $M/P \xrightarrow{a\cdot} M/P$ is not S-injective with respect to $s \in S$. Then there exists $m \in M$ such that $a(m+P) = am+P = 0_{M/P}$ but $s(m+P) \neq 0_{M/P}$. This gives that $am \in P$ and $sm \notin P$. Since P is an S-prime submodule, we have $sa \in (P : M)$ and thus $sam' \in P$ for each $m' \in M$. Then we have $sa(m' + P) = 0_{M/P}$ for each $m' \in M$, that is, the homothety $M/P \xrightarrow{a\cdot} M/P$ is S-zero with respect to s.

$(ii) \Rightarrow (i)$: Suppose that (ii) holds. Let $am \in P$ for some $a \in R$ and $m \in M$. Assume that $sm \notin P$. Then we deduce the homothety $M/P \xrightarrow{a\cdot} M/P$ is not S-injective. Thus by (ii), $M/P \xrightarrow{a\cdot} M/P$ is S-zero with respect to $s \in S$, namely, $sa(m' + P) = 0_{M/P}$ for each $m' \in M$. This yields $sa \in (P : M)$. Therefore P is an S-prime submodule of M. □

It is well known that a submodule P of M is a prime submodule if and only if every homothety $M/P \xrightarrow{a\cdot} M/P$ is either injective or zero. This fact can be obtained by Propositon 4.4 by taking $S \subseteq u(R)$.

Recall from [17] that a submodule N of M with $ann(N) \cap S = \emptyset$ is said to be an S-second submodule if there exists $s \in S$ with $srN = 0$ or $srN = sN$ for each $r \in R$. Motivated by Proposition 4.4, we give a new characterization of S-second submodules from another point of view. Since the proof is similar to Proposition 4.4, we omit the proof.

Theorem 4.5 *Let N be a submodule of M with $ann(N) \cap S = \emptyset$. The following assertions are equivalent.*

(i) N is an S-second submodule.

(ii) There exists $s \in S$ such that for each $a \in R$, the homothety $N \overset{a}{\to} N$ is either S-zero or S-surjective with respect to $s \in S$.

(iii) There exists a fixed $s \in S$ so that for each $a \in R$, either $saN = 0$ or $sN \subseteq aN$.

The author in [17] proved that if N is an S-second submodule of M, then $ann(N)$ is an S-prime ideal of R and the converse holds under the assumption that M is comultiplication [17, Proposition 2.9]. Now we show that this fact is true even if M is an S-comultiplication module.

Theorem 4.6 *Let M be an S-comultiplication module. The following statements are equivalent.*

(i) N is an S-second sumodule of M.

(ii) $ann(N)$ is an S-prime ideal of R and there exists $s \in S$ such that $sN \subseteq s'N$ for each $s' \in S$.

Proof $(i) \Rightarrow (ii)$: The claim follows from [17, Proposition 2.9] and [17, Lemma 2.13].

$(ii) \Rightarrow (i)$: Suppose that $ann(N)$ is an S-prime ideal of R. Now we will show that N is an S-second submodule of M. To prove this take $a \in R$. Since $ann(N)$ is an S-prime ideal, by Proposition 4.3, there exists $s \in S$ such that $ann(sN)$ is a prime ideal and $ann(s'N) \subseteq ann(sN)$ for each $s' \in S$. Assume that $saN \neq (0)$. Now we shall show that $sN \subseteq aN$. Since M is an S-comultiplication module, there exist $s' \in S$ and an ideal I of R such that $s'(0 :_M I) \subseteq aN \subseteq (0 :_M I)$. This implies that $aI \subseteq ann(N)$. Since $ann(N)$ is an S-prime ideal, there exists $s \in S$ such that either $sa \in ann(N)$ or $sI \subseteq ann(N)$ by Proposition 4.3. The first case is impossible since $saN \neq (0)$. Thus we have $I \subseteq ann(sN)$. Then we have $s's(0 :_M ann(sN)) \subseteq s'(0 :_M I) \subseteq aN$. This implies that $s's^2N \subseteq s's(0 :_M ann(sN)) \subseteq aN$. Then by (ii), $sN \subseteq s's^2N \subseteq aN$. Then by Theorem 4.5 (iii) N is an S-second submodule of M. □

Theorem 4.7 *Let M be a comultiplication module. The following statements are equivalent.*

(i) N is a second submodule of M.

(ii) $ann(N)$ is a prime ideal of R.

Proof Take $S \subseteq u(R)$ and note that the concepts of S-comultiplication module and comultiplication modules are the same. On the other hand, the concepts of second submodule and S-second submodules are the same. The rest follows from Theorem 4.6. □

Theorem 4.8 *Let M be an S-comultiplication module and let N be an S-second submodule of M. If $N \subseteq N_1 + N_2 + \cdots + N_m$ for some submodules $N_1, N_2, \ldots, N_m$ of M, then there exists $s \in S$ such that $sN \subseteq N_i$ for some $1 \leq i \leq m$.*

Proof Suppose that N is an S-second submodule of an S-comultiplication module M. Suppose that $N \subseteq \sum_{i=1}^{m} N_i$ for some submodules $N_1, N_2, \ldots, N_m$ of M. Then we have $ann(\sum_{i=1}^{m} N_i) = \bigcap_{i=1}^{m} ann(N_i) \subseteq ann(N)$. Since N is an S-second submodule, by Theorem 4.6, $ann(N)$ is an S-prime ideal of R. Then by [19, Corollary 2.6], there exists $s \in S$ such that $sann(N_i) \subseteq ann(N)$ for some $1 \leq i \leq m$. This implies that $ann(N_i) \subseteq ann(sN)$. Then by Lemma 2.6 (iii), $stN \subseteq N_i$ for some $t \in S$ which completes the proof. □

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