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On mj-clean ring and strongly mj-clean ring

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Abstract: In this paper, we introduce the concepts of mj-clean and strongly mj-clean rings which are generalizations of j-clean ring and strongly j-clean ring, respectively. Let R be a ring with a nonzero identity and $m \geq 2$ a positive integer. We call the ring R as mj-clean if each element of R can be written as a sum of an m-potent and an element of $J(R)$ and also if these elements are commute, then we call R as strongly mj-clean ring. We examine the algebraic properties of these new concepts and show the effects of these structures on matrix rings, polynomial rings, power series and the transitions between them.

Key words: j-clean ring, strongly j-clean ring, mj-clean ring, strongly mj-clean ring

1. Introduction

Throughout this paper, we consider a ring R with nonzero identity $(0 \neq 1)$. The notations $J(R)$ and $U(R)$ will denote the Jacobson radical of R, which is the intersection of all maximal ideals of R, and the set of all unit elements of R, respectively. Let $x \in R$. Then, the set $ann_l(x)$ is defined as $\{r \in R \mid rx = 0\}$ and in a similar manner, $ann_r(x) = \{r \in R \mid xr \in 0\}$. Clean ring concept first appeared in 1977 by W. K. Nicholson in his work "Lifting idempotents and exchange rings" [10]. Elements that can be written as the sum of a unit and an idempotent are called clean elements of the ring. If all elements of a ring are clean, the ring is called a clean ring. Nicholson also introduced the strongly clean ring concept [11]. In this case, a ring is called a strongly clean ring if all elements of the ring can be written as a sum of a unit and an idempotent where these elements are commute. In [3], Camillo and Yu classified the relations among exchange rings, clean rings, potent rings, and semiperfect rings. In [7], Han and Nicholson examined matrix expansions of clean rings. Clean and strongly clean rings have been studied extensively in [3]-[8].

Since clean rings and strongly clean rings have a significant role in the theory of rings, there are several ways to generalize the concept of clean rings and strongly clean rings. One of these generalizations is [5] where Chen improved the definition of strongly J-clean ring by replacing a unit element with an element of the Jacobson radical and deeply searched the link amongst many structures that mentioned above. Additionally, Chen studied this concept on local ring and investigated it on matrix ring [6].

Let R be a ring, $m \geq 2$ a positive integer and $r \in R$. Then, r is called an m-potent element if $r^m = r$. Then, a ring is called an m-potent ring if all elements of it are m-potent. Purkait, Dutta and Kar recently introduced another generalizations of clean ring and strongly clean ring, which are called m-clean and

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strongly m-clean ring, respectively [12]. They improved these definitions by replacing idempotent element with m-potent element ($a^m = a$). For $m \geq 2$, a ring is called an m-clean ring if all elements of a ring can be written as the sum of a unit and an m-potent elements and also, if these elements are commute, then the ring is called a strongly m-clean ring. Also in [12] and [13], the structure of strong m-clean ring is characterized extensively in terms of m-semiperfect ring.

In this paper, our motivation is to introduce a new class which is generalization of (strongly) j-clean rings and has relation with (strongly) m-clean rings. So it is natural question to ask how would be a ring be whose elements can be written as a sum of an m-potent and an element of the Jacobson radical. In this paper, we assume $m \geq 2$. We call a ring as mj-clean if each element of R can be written as a sum of an m-potent and an element of $J(R)$ and also if these elements are commute, then we call R as strongly mj-clean ring.

In Section 2, first, it is shown that every (strongly) j-clean ring is actually a (strongly) mj-clean ring, and so (strongly) mj-clean rings are a generalization of j-clean rings. But the converse may not hold (see Example 2.2). With help of Lemma 2.3 and Theorem 2.4, it is obtained in Corollary 2.5 that if a ring R is strongly mj-clean, then $f^{m-1}Rf^{m-1}$ and $(1-f^{m-1})R(1-f^{m-1})$ are strongly mj-clean. Corollary 2.9 gives that if R is an m-local ring, then R is an m-semipotent ring if and only if R is an mj-clean ring. In Theorem 2.12, we get that each strongly mj-clean ring R with $charR = m$ is a strongly m-clean ring. In Theorem 2.13, Theorem 2.15 and Corollary 2.16, we give the relation between a ring R and the quotient ring $R/J(R)$ according to the concept of strongly mj-clean ring. Also, in Proposition 2.17, we investigate the concept of mj-clean ring on trivial extension $R(+)M$ of an R-module M. Finally, in Proposition 2.18, we study the concept of strongly mj-clean ring on power series and we show that any polynomial ring is not an mj-clean ring.

In Section 3, we study the concept of strongly mj-clean ring on the upper triangular matrix ring $T_n(R)$ and the matrix ring $M_2(R)$.

2. Characterization of (strongly) mj-clean rings

Definition 2.1 *Let R be a ring and $m \geq 2$ be a positive integer. $a \in R$ is called an (strongly) mj-clean element if it is the sum of an m-potent element and an element in $J(R)$ (that commute).*
A ring R is called an (strongly) mj-clean ring if each element of R is an (strongly) mj-clean element.

By the definition, one can easily seen that every (strongly) $2j$-clean ring is a (strongly) j-clean ring for $m = 2$ and also, every (strongly) j-clean ring is an (strongly) mj-clean ring. But in general mj-clean rings may not be a j-clean for $m \neq 2$. For this case, see the following example.

Example 2.2 *Let R be a local ring. Consider $2x2$ matrix ring $M_2(R)$ in which every matrix is a singular matrix. With help of Theorem 3.2 and Corollary 3.3, we can say that the matrix ring $M_2(R)$ is a strongly mj-clean ring if every matrix of $M_2(R)$ is a singular matrix. But it is not a j-clean ring because each matrix of $M_2(R)$ is similar to $\begin{pmatrix} 1+w_1 & k_2+w_2 \\ 0 & w_4 \end{pmatrix}$ where $\begin{pmatrix} 1 & k_2 \\ 0 & 0 \end{pmatrix}$ is an m-potent element of $M_2(R)$ and $w_1, w_2, w_4 \in J(R)$ by Corollary 3.3. However, it is not a j-clean element of $M_2(R)$ for $k_2 \neq 0$ and $w_2 \neq 0$ by [5, Corollary 5.3].*

Lemma 2.3 *Let R be a ring, $m \geq 2$ a positive integer and $a = g + w$ a strongly mj-clean element of R where g is an m-potent of R and $w \in J(R)$. Then $ann_l(a) \subseteq ann_l(g)$ and $ann_r(a) \subseteq ann_r(g)$.*

Proof Let $r \in ann_l(a)$. Then $ra = 0$ and thus, $r(g + w) = 0$ by assumption and so $rg + rw = 0$. We get $rg + rwg^{m-1} = 0$ by multiplying with g^{m-1}. Then $rg + rg^{m-1}w = 0$ as a is a strongly mj-clean element. Thus $rg(1 + g^{m-2}w) = 0$. Since $w \in J(R)$, then $1 + g^{m-2}w$ is a unit element of R. Therefore, $rg = 0$, which implies $r \in ann_l(g)$.
A similar argument shows $ann_r(a) \subseteq ann_r(g)$. □

Theorem 2.4 *Let f be an m-potent element of a ring R. Then $a \in f^{m-1}Rf^{m-1}$ is a strongly mj-clean element in R if and only if a is a strongly mj-clean element in $f^{m-1}Rf^{m-1}$. Moreover, $a \in (1-f^{m-1})R(1-f^{m-1})$ is a strongly mj-clean element in R if and only if a is a strongly mj-clean element in $(1-f^{m-1})R(1-f^{m-1})$.*

Proof Note that, $(f^{m-1})^2 = f^{m-1}$; namely, f^{m-1} is an idempotent element of R since f is m-potent and also $1 - f^{m-1}$ is an idempotent element of R since f^{m-1} is idempotent.
($\Leftarrow$): Let $a = g + w$ with $g^m = g \in f^{m-1}Rf^{m-1} \subseteq R$ and $w \in J(f^{m-1}Rf^{m-1})$. Since $f^{m-1}J(R)f^{m-1} = J(f^{m-1}Rf^{m-1}) \subseteq J(R)$, then $w \in J(R)$. Thus, $a \in f^{m-1}Rf^{m-1}$ is a strongly mj-clean element of R.
($\Rightarrow$): Assume that $a \in f^{m-1}Rf^{m-1}$ is a strongly mj-clean element of R. Let $a = g + w$ where $g^m = g \in R$ and $w \in J(R)$. It is clear that $1 - f^{m-1} \in ann_l(a) \cap ann_r(a)$. By Lemma 2.3, we have $1 - f^{m-1} \in ann_l(g) \cap ann_r(g)$. Then, we get $g(1 - f^{m-1}) = (1 - f^{m-1})g = 0$, which implies $g = gf^{m-1} = f^{m-1}g$. Similarly, we have $a = af^{m-1} = f^{m-1}a$. Since $a = g + w$, then $f^{m-1}af^{m-1} = f^{m-1}gf^{m-1} + f^{m-1}wf^{m-1}$. By the equation $a = af^{m-1} = f^{m-1}a$ and $a \in f^{m-1}Rf^{m-1}$, it can be easily seen that $a = f^{m-1}af^{m-1}$. Then $a = f^{m-1}gf^{m-1} + f^{m-1}wf^{m-1}$. Note that $(f^{m-1}gf^{m-1})^m = f^{m-1}gf^{m-1}$, which shows $f^{m-1}gf^{m-1}$ is an m-potent element of $f^{m-1}Rf^{m-1}$ and also $(f^{m-1}gf^{m-1})(f^{m-1}wf^{m-1}) = (f^{m-1}wf^{m-1})(f^{m-1}gf^{m-1})$. Therefore, a is a strongly mj-clean element in $f^{m-1}Rf^{m-1}$.
A similar argument shows that $a \in (1 - f^{m-1})R(1 - f^{m-1})$ is a strongly mj-clean element in R if and only if a is a strongly mj-clean element in $(1 - f^{m-1})R(1 - f^{m-1})$. □

By Theorem 2.4, we get the following corollary.

Corollary 2.5 *Let R be a ring and f an m-potent element of R. If R is a strongly mj-clean ring, then so is $f^{m-1}Rf^{m-1}$ and $(1 - f^{m-1})R(1 - f^{m-1})$.*

Proof It is obvious from the previous theorem. □

Proposition 2.6 *Let R has only trivial m-potent elements 0 and 1. Then, R is an mj-clean ring if and only if R is a local ring.*

Proof ($\Rightarrow$): Let R be an mj-clean ring. Let $x \in R$. By assumption, there exists $w \in J(R)$ such that either $x = 0 + w$ or $x = 1 + w$ since R is mj-clean. Thus, either $x \in J(R)$ or $x \in U(R)$. Therefore, R is a local ring.
($\Leftarrow$): Let R be a local ring. Then, every element of R is in either $U(R)$ or $J(R)$. Let $x \in R$. If $x \in J(R)$, the proof is done. Assume $x \in U(R)$. Since $x \notin J(R)$, then $1 - x \notin U(R)$. Thus, $1 - x \in J(R)$ since R is a local ring. Then, there is $w \in J(R)$ such that $1 - x = w$, that is, $x = 1 + (-w)$. Let $-w = w' \in J(R)$. Thus, $x = 1 + w'$ is an mj-clean element since 1 is m-potent. Therefore, R is an mj-clean ring. □

Definition 2.7 *[12, Definition 2.12] Let R be a ring. Then R is called an m-semipotent ring if for each ideal I of R such that $I \nsubseteq J(R)$ and there is $0 \neq x \in I$ with $x = x^m$.*

Proposition 2.8 *Let R has only trivial m-potent elements 0 and 1. Then R is an m-semipotent ring if and only if R is a local ring.*

Proof ($\Leftarrow$): Let $R = U(R) \cup J(R)$, namely, R be local. Thus, there is no ideal not contained by $J(R)$. By the definition of m-semipotent ring, the proof is straightforward.
($\Rightarrow$): Let R be an m-semipotent ring and $x \in R$. If $x \in U(R)$, then it is clear. Let $x \notin U(R)$. Assume that $x \notin J(R)$. Since x is a non unit, then $(x) \nsubseteq J(R)$. Since R is m-semipotent, then there is an element $0 \neq r \in (x)$ with $r = r^m$. Let $r = tx$ for some $t \in R$. By assumption, we get $r = 1$, and thus, $tx = 1$. Hence x is a unit element, which is a contradiction. Therefore, it must be $x \in J(R)$. □

As a result of Proposition 2.6 and Proposition 2.8, we give the following corollary without proof.

Corollary 2.9 *Let R has only trivial m-potent elements 0 and 1. Then, R is an m-semipotent ring if and only if R is an mj-clean ring.*

Let R be an m-potent ring. Since we can write each element x of R as $x = x + 0$ where $0 \in J(R)$, $x = x^m$ (and $0.x = x.0$), then R is a (strongly) mj-clean ring.

Proposition 2.10 *Let R be a ring. Then, R is an m-potent ring if and only if R is a (strongly) mj-clean ring and $J(R) = 0$.*

Proof ($\Rightarrow$): Let R be an m-potent ring. Then it is straightforward that R is an mj-clean ring since $0 \in J(R)$. Let $x \in J(R)$. By the assumption, $x^m = x$. Then, we can write $x(1 - x^{m-1}) = 0$. Since $1 - x^{m-1} \in U(R)$, then $x = 0$. Thus, $J(R) = 0$.
($\Leftarrow$): It is clear. □

Proposition 2.11 *Let R be a strongly mj-clean integral domain. Then, R is a local ring.*

Proof Let $x \in R$. Then, there are an m-potent element f of R and $w \in J(R)$ such that $x = f + w$. Let $f = 0$. Then $x = w \in J(R)$. Now, assume that $f \neq 0$. At first note that $f^m = f$ implies that $f(1 - f)(1 + f + ... + f^{m-2}) = 0$ and so by assumption, $(1 - f)(1 + f + ... + f^{m-2}) = 0$. Then, $1 - f^{m-1} = 0$. Thus, $f^{m-1} = 1$ and so $ff^{m-2} = 1$. Hence, we obtain $f \in U(R)$. Thus, $x = f + w \in U(R)$. Indeed, if $f + w$ is not a unit element, then there exists a maximal ideal M of R which contains $f + w$. Since $w \in J(R)$, then $w \in M$. Thus, $f \in M$ which contradicts with the fact that M is maximal. Therefore, $x = f + w$ must be in $U(R)$. Hence, R is a local ring. □

Theorem 2.12 *Let R be a ring and $charR = m$. If R is a strongly mj-clean, then R is a strongly m-clean ring.*

Proof Let $a \in R$. By our assumption, there exist an m-potent $f \in R$ and $w \in J(R)$ such that $a = f + w$ and $fw = wf$. Thus, $a = (f + 1) + (w - 1)$ where $w - 1 \in U(R)$ and $(f + 1)^m = f + 1$. Hence, a is an m-potent element of R. Therefore, R is a strongly m-clean ring. □

Theorem 2.13 *Let $h : R \to S$ be a surjective homomorphism. If R is a strongly mj-clean ring, then so is S.*

Proof Let $s \in S$. Since h is surjective, then there exists $r \in R$ such that $h(r) = s$ for each $s \in S$. Then, r is the sum of f and w where $f^m = f$, $w \in J(R)$ and $fw = wf$ since R is a strongly mj-clean. By [14], we get $h(J(R)) \subseteq J(S)$ and so $h(w) \in J(S)$ since $w \in J(R)$. We get $s = h(r) = h(f + w) = h(f) + h(w)$ where $h(f) = h(f^m) = (h(f))^m$ is an m-potent element of S and $h(f)h(w) = h(fw) = h(wf) = h(w)h(f)$. Thus, S is a strongly mj-clean ring. □

Definition 2.14 *[12, Definiton 2.11] Let I be an ideal of a ring R. Then it is said to be m-potents lift modulo I if whenever $x \in R$ and $x - x^m \in I$ imply that there is an m-potent $f \in R$ such that $f - x \in I$.*

Theorem 2.15 *Let I be a proper ideal of R with $I \subseteq J(R)$ and $charR = m$. Then, R is a strongly mj-clean ring if and only if R/I is a strongly mj-clean ring and m-potents lift modulo I.*

Proof Let I be a proper ideal of R with $I \subseteq J(R)$.
($\Rightarrow$): Consider the natural homomorphism $\pi : R \to R/I$, defined by $\pi(a) = a + I$ for each $a \in R$, is surjective. Thus R/I is a strongly mj-clean by Theorem 2.13. Let $r - r^m \in I \subseteq J(R)$ for some $r \in R$. Then $r = f + w$ where $f^m = f$, $w \in J(R)$ and $fw = wf$ since R is strongly mj-clean. We get that $r - r^m = f + w - (f + w)^m = f + w - (f^m + mf^{m-1}w + ... + mfw^{m-1} + w^m) = w - w^m \in I$ since $f^m = f$ and $charR = m$. Then $w \in I$ since $w - w^m = w(1 - w^{m-1}) \in I$ and $1 - w^{m-1}$ is a unit. Additionally, $r^m = (f + w)^m = f^m + mf^{m-1}w + ... + mfw^{m-1} + w^m = f + w^m$ by $charR = m$. By assumption, there exists $i \in I$ such that $r - r^m = i$. Since $r^m = f + w^m$ then $r - r^m = r - (f + w^m) = i$ and $f - r = -i - w^m \in I$ which shows m-potents lift modulo I.
($\Leftarrow$): Let R/I be strongly mj-clean ring and m-potents lift modulo I where I is a proper ideal of R with $I \subseteq J(R)$. Let $r \in R$. We will show that r is a strongly mj-clean element. Assume that $r \in I$. Then $r \in J(R)$ by our assumption. Since $r = 0 + r$, then r is strongly mj-clean. Now, assume that $r \notin I$. Let $r + I \in R/I$ such that $r + I = f + I + w + I$ where $(f + I)$ is an m-potent element of R and $w + I$ is in $J(R/I)$. Since $(f + I)^m = f^m + I = f + I$ and so $f - f^m \in I$, then there exists an m-potent g such that $g - f \in I$. Thus $g + I = f + I$. So $r + I = g + I + w + I$ and then $w + I = r - g + I$ where $r - g + I \in J(R)/I$, which yields that $r = g + r - g$ where g is m-potent and $r - g \in J(R)$. Thus, r is an mj-clean element in R. Therefore, R is strongly mj-clean ring. □

Corollary 2.16 *Let R be a ring with $charR = m$. Then, the following are equivalent:*

1. *R is a strongly mj-clean ring.*
2. *$R/J(R)$ is a strongly mj-clean ring and m-potents lift modulo $J(R)$.*
3. *$R/J(R)$ is a strongly mj-clean ring and R is a strongly m-clean ring.*

Proof (1) $\Leftrightarrow$ (2) is obvious from Theorem 2.13 and Theorem 2.15.
(1) $\Leftrightarrow$ (3) is clear from Theorem 2.12. □

Let M be an R-module. Then, the idealization $R(+)M = \{(a, m) : a \in R, \; m \in M\}$ is a commutative ring with componentwise addition and the multiplication $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in$

R; $m, m' \in M$ [9]. The Jacobson radical of $R(+)M$ is $J(R(+)M) = J(R)(+)M$ [2]. Let $m(R)$ be the set of all m-potent elements of R. The set of m-potent elements of $R(+)M$ are $m(R(+)M) = m(R)(+)0$ [2]. We determine the relation between $R(+)M$ and R according to the concept of mj-clean element.

Proposition 2.17 *Let M be an R-module. Then, $R(+)M$ is an mj-clean ring if and only if R is an mj-clean ring.*

Proof ($\Leftarrow$): Let R be an mj-clean ring and $(r, m) \in R(+)M$. Then, $r = f + w$ for an m-potent element of R and $w \in J(R)$ since $r \in R$. By the above explanation, $(f, 0)$ is an m-potent element of $R(+)M$ and $(w, m) \in J(R(+)M)$. Thus, $(r, m) = (f, 0) + (w, m)$ is an mj-clean element of $R(+)M$.
($\Rightarrow$): Let $R(+)M$ be an mj-clean ring and $r \in R$. Then, $(r, 0) \in R(+)M$ is an mj-clean element. Thus, $(r, 0)$ is written as a sum of an m-potent element $(f, 0)$ and an element (w, m) in $J(R(+)M) = J(R)(+)M$ such that $(r, 0) = (f, 0) + (w, m) = (f + w, m)$. Thus, $r = f + w$ where f is an m-potent element of R and $w \in J(R)$. Therefore, R is an mj-clean ring. □

Proposition 2.18 *Let R be a ring. Then, R is a strongly mj-clean ring if and only if $R[[X]]$ is a strongly mj-clean ring.*

Proof At first, note that $J(R[[X]]) = (J(R), X)$.
($\Rightarrow$): Let $g(X) = a_0 + a_1X + ... + a_nX^n + ... \in R[[X]]$. Since $a_0 \in R$, then $a_0 = f + w$ where f is an m-potent element in R and $w \in J(R)$. Thus $g(X) = f + (w + a_1X + ... + a_nX^n + ...)$ is an mj-clean element in $R[[X]]$ since $w + a_1X + ... + a_nx^n + ... \in J(R[[X]])$. Therefore, $R[[X]]$ is an mj-clean ring.
($\Leftarrow$): Consider the natural homomorphism $\pi : R[[X]] \to R$, defined by $\pi(f(X)) = f(0)$. Then, the rest follows from Theorem 2.13. □
But $R[X]$ is not an mj-clean ring. Since $J(R[X]) = 0$, then every element $r \in R[X]$ must be an m-potent for $R[X]$ to be mj-clean. But it is not possible.

3. Matrix rings

Proposition 3.1 *Let R be a ring and $T_n(R)$ a strongly mj-clean ring. Then R is a strongly mj-clean ring.*

Proof It is clear from Corollary 2.5 and [5, Lemma 4.1]. □

In the rest of paper, we assume that R is a local ring.

Theorem 3.2 *P is a strongly mj-clean element of $M_2(R)$ if and only if $P \in M_2(J(R))$ or P is similar to a matrix $\begin{pmatrix} 1+w_1 & k_2+w_2 \\ 0 & w_4 \end{pmatrix} = \begin{pmatrix} 1 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ 0 & w_4 \end{pmatrix}$ where $\begin{pmatrix} 1 & k_2 \\ 0 & 0 \end{pmatrix}$ is an m-potent element of $M_2(R)$ and $w_1, w_2, w_4 \in J(R)$.*

Proof ($\Leftarrow$): Assume that $P \in M_2(J(R))$. Then, the proof is done. Let P be similar to $\begin{pmatrix} 1+w_1 & k_2+w_2 \\ 0 & w_3 \end{pmatrix} = \begin{pmatrix} 1 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ 0 & w_4 \end{pmatrix}$ where $\begin{pmatrix} 1 & k_2 \\ 0 & 0 \end{pmatrix}$ is an m-potent element of $M_2(R)$ and $w_1, w_2, w_4 \in J(R)$.
Since $\begin{pmatrix} 1+w_1 & k_2+w_2 \\ 0 & w_4 \end{pmatrix}$ is a strongly mj-clean element, then P is a strongly mj-clean element.

($\Rightarrow$): Assume that P is a strongly mj-clean element of $M_2(R)$. Then, there are an m-potent element $F \in M_2(R)$ and $W \in J(M_2(R))$ with $P = F + W$ and $FW = WF$. Assume $F = 0$. Then the proof is done. Now, assume $F \neq 0$. Since $F^{m-1} \in M_2(R)$ is an idempotent element and by [5, Lemma 5.1], we have an element K of $GL_2(R)$ with $KF^{m-1}K^{-1} = diag(u_1, u_2)$, u_1, u_2 are idempotent elements of R. Thus, we get $u_1 = 1, u_2 = 0$ or $u_1 = 0, u_2 = 1$ since u_1, u_2 are idempotent elements of R. Without loss of generality, assume $KF^{m-1}K^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let us multiply the equation with $KFK^{-1} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$. Then, $KF^{m-1}K^{-1}KFK^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} KFK^{-1}$ and so $KF^{m}K^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} KFK^{-1} = \begin{pmatrix} k_1 & k_2 \\ 0 & 0 \end{pmatrix}$. Since F is an m-potent element, then $KFK^{-1} = \begin{pmatrix} k_1 & k_2 \\ 0 & 0 \end{pmatrix}$. Note that KFK^{-1} is an m-potent element, that is, $(KFK^{-1})^m = KF^mK^{-1} = KFK^{-1}$. Then $\begin{pmatrix} k_1^m & k_1^{m-1}k_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ 0 & 0 \end{pmatrix}$. We get $k_1 = 0$ or $k_1 = 1$. If $k_1 = 0$, then $k_2 = 0$ and so $KFK^{-1} = 0$. Thus, KPK^{-1} is in $J(M_2(R))$ and so P is similar to a KPK^{-1}. If $k_1 = 1$, then $KFK^{-1} = \begin{pmatrix} 1 & k_2 \\ 0 & 0 \end{pmatrix}$. Let $KWK^{-1} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$. Since $FW = WF$, then $KWK^{-1}KFK^{-1} = \begin{pmatrix} v_1 & k_2v_1 \\ v_3 & k_2v_3 \end{pmatrix} = KFK^{-1}KWK^{-1} = \begin{pmatrix} v_1 + k_2v_3 & v_2 + k_2v_4 \\ 0 & 0 \end{pmatrix}$. Then we obtain that $v_3 = 0$ and $k_2(v_1 - v_4) = v_2 \in J(R)$. Since R is a local ring, then $k_2 \in U(R)$ or $k_2 \in J(R)$. Assume $k_2 \in U(R)$ then $v_1 - v_4 = uv_2$ and so $KWK^{-1} = \begin{pmatrix} uv_2 + v_4 & v_2 \\ 0 & v_4 \end{pmatrix}$. Let $k_2 \in J(R)$ then $KWK^{-1} = \begin{pmatrix} v_1 & k_2(v_1 - v_4) \\ 0 & v_4 \end{pmatrix}$. Thus we get $KWK^{-1} = \begin{pmatrix} w_1 & w_2 \\ 0 & w_4 \end{pmatrix}$. Therefore, P is similar to a matrix $\begin{pmatrix} 1 + w_1 & k_2 + w_2 \\ 0 & w_4 \end{pmatrix}$. □

Let K be an element of $M_2(R)$. Then K is called singular matrix if K is not invertible. We get the following result of the previous theorem.

Corollary 3.3 *Let K be a singular matrix of $M_2(R)$. Then K is a strongly mj-clean if and only if K is similar to a matrix $\begin{pmatrix} 1 + w_1 & k_2 + w_2 \\ 0 & w_4 \end{pmatrix}$ where $\begin{pmatrix} 1 & k_2 \\ 0 & 0 \end{pmatrix}$ is a m-potent element of $M_2(R)$ and $w_1, w_2, w_4 \in J(R)$.*

Proof Let K be a singular matrix of $M_2(R)$. Then $K \notin M_2(J(R))$ [5]. Thus, the rest of the proof is clear from Theorem 3.2 and the definition of singular matrix. □

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