Turkish Journal of Mathematics

Volume 46
Number 5 *SI-2*

Article 16

1-1-2022

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1864 – 1870
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doi:10.55730/1300-0098.3237

A variant of Rosset's approach to the Amitsur-Levitzki theorem and some $\mathbb{Z}_2$-graded identities of $\mathrm{M}_n(E)$

Dedicated to the 70th anniversary of Vesselin Drensky

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Received: 22.10.2021 • **Accepted/Published Online:** 19.01.2022 • **Final Version:** 20.06.2022

Abstract: In the spirit of Rosset's proof of the Amitsur-Levitzki theorem, we show how the standard identiy (for matrices over a commutative base ring) and the addition of external Grassmann variables can be used to derive a certain $\mathbb{Z}_2$-graded polynomial identity of $\mathrm{M}_n(E)$.

Key words: The full matrix algebra over the infinite dimensional Grassmann algebra, the Amitsur-Levitzki theorem on $n \times n$ matrices

1. Introduction

An algebra R means a not necessarily commutative unitary algebra over a commutative ring C (or over a field K), and the notation for the full $n \times n$ matrix algebra over R is $\mathrm{M}_n(R)$.

In case of $\mathrm{char}(K) = 0$, Kemer's pioneering work (see [9], [10]) on the T-ideals of associative algebras (leading to the solution of the Specht problem) revealed the importance of the identities satisfied by $\mathrm{M}_n(E)$ and $\mathrm{M}_{n,d}(E)$, where

$$E = K\langle v_1, v_2, ..., v_i, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j\rangle = E_0 \oplus E_1 \tag{1.1}$$

is the naturally $\mathbb{Z}_2$-graded Grassmann (exterior) algebra generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i\geq 1}$. The K-subspace E_0 generated by 1 and the monomials in the v_i's of even length and E_1 is the K-subspace generated by the monomials in the v_i's of odd length. We note that E_0 is a commutative subalgebra of E and E is Lie nilpotent of index 2.

Let $K\langle x_1, x_2, \ldots, x_i, \ldots\rangle$ denote the free associative K-algebra generated by the infinite sequence $x_1, x_2, \ldots, x_i, \ldots$ of noncommuting indeterminates. The prime T-ideals of this K-algebra are exactly the T-ideals of the identities satisfied by $\mathrm{M}_n(K)$ for $n \geq 1$ (see [2]). The T-prime (or verbally prime) T-ideals are the prime T-ideals plus the T-ideals of the identities of $\mathrm{M}_n(E)$ for $n \geq 1$ and of $\mathrm{M}_{n,d}(E)$ for $1 \leq d \leq n-1$, where $\mathrm{M}_{n,d}(E)$ is the K-subalgebra of $\mathrm{M}_n(E)$ consisting of the so-called (n,d)-supermatrices with two diagonal E_0 blocks of sizes $d \times d$ and $(n-d) \times (n-d)$ and with two E_1 blocks of sizes $d \times (n-d)$ and $(n-d) \times d$. Another remarkable result is that any T-ideal contains the T-ideal of the identities satisfied by $\mathrm{M}_n(E)$ for sufficiently large n (see p. 20 in [10]).

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2010 *AMS Mathematics Subject Classification:* 16R10,16R40,16W50,15A24,15A75,15B33

The above mentioned three classes of T-prime (verbally prime) PI-algebras serve as basic building blocks in Kemer's theory, where $\mathbb{Z}_2$-graded identities also play an important role. Since the appearance of [9] and [10] considerable efforts have been concentrated on the study of the various algebraic properties of $\mathrm{M}_n(E)$ and $\mathrm{M}_{n,d}(E)$, see [1, 4–8, 11, 14–16].

The aim of the present note is to present a certain $\mathbb{Z}_2$-graded polynomial identity of the $\mathbb{Z}_2$-graded full matrix algebra $\mathrm{M}_n(E) = \mathrm{M}_n(E_0) \oplus \mathrm{M}_n(E_1)$. There is a possibility to derive the mentioned identity by using the Amitsur-Levitzki standard identity (see [3])

$$S_{2n}(x_1, x_2, \ldots, x_{2n}) = \sum_{\pi \in \mathrm{Sym}\{1,2,\ldots,2n\}} \mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(2n)} = 0 \tag{1.2}$$

of degree $2n$ (for $n \times n$ matrices over a commutative base ring). In this case, the addition of external Grassmann variables to E is essential. The ingenious idea of using (additional) Grassmann variables in an enviroment without Grassmann algebras first appeared in Rosset's short proof of the Amitsur-Levitzki theorem (see [12]). The use of a single additional Grassmann variable (out of E) in the study of $\mathrm{M}_n(E)$ appears in a certain companion matrix construction (see [13]) providing a Cayley-Hamilton identity for a matrix $A \in \mathrm{M}_n(E)$ of degree n^2 (an entirely different treatment in [14] provided a similar CH identity of the same degree). Our present work can be considered as a variation on Rosset's original theme. One of the referees caused a surprise by providing a different approach to derive the same $\mathbb{Z}_2$-graded polynomial identity of $\mathrm{M}_n(E)$ based on the use of the $*$-transform of a $\mathbb{Z}_2$-graded polynomial and the so-called Grassmann envelope. The authors decided to keep their original proof and to present the mentioned short proof of the referee at the end of the paper.

2. A $\mathbb{Z}_2$-graded identity of $\mathrm{M}_n(E)$

The Grassmann algebra

$$E = K \langle v_1, v_2, ..., v_i, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \le i \le j \rangle = K \langle V \rangle \tag{2.1}$$

generated by (the countably) infinite set $V = \{v_1, v_2, \ldots, v_t, \ldots\}$ of anticommuting indeterminates can naturally be extended as

$$F = K \langle V \cup W \rangle = K \langle v_1, v_2, \ldots, v_t, \ldots, w_1, w_2, \ldots, w_t, \ldots \rangle \tag{2.2}$$

by using a bigger set $V \cup W$ of anticommuting generators, where

$$W = \{w_1, w_2, \ldots, w_t, \ldots\} \text{ and } V \cap W = \varnothing. \tag{2.3}$$

Now we have $v_i v_j + v_j v_i = 0$, $w_i w_j + w_j w_i = 0$ for all $1 \le i \le j$ and $v_i w_j + w_j v_i = 0$ for all $1 \le i, j$. The Grassmann algebra

$$G = K \langle w_1, w_2, ..., w_i, ... \mid w_i w_j + w_j w_i = 0 \text{ for all } 1 \le i \le j \rangle = K \langle W \rangle \tag{2.4}$$

generated by W is also a sub K-algebra of F. Since the cardinalities of V, W and $V \cup W$ are all equal to $\aleph_0$, the K-algebras E, G and F are isomorphic.

A $\mathbb{Z}_2$-graded K-algebra R is a pair (R_0, R_1), where R_0 and R_1 are K-subspaces of R such that $R = R_0 \oplus R_1$ is a direct sum and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \{0, 1\}$, where $i + j$ is taken modulo 2. A $\mathbb{Z}_2$-graded identity of $R = R_0 \oplus R_1$ is of the form

$$h(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_k) = 0, \tag{2.5}$$

where $h(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_k)$ is in the free polynomial K-algebra generated by the noncommuting indeterminates $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_k$. We only require that

$$h(r_1, r_2, \ldots, r_m, r'_1, r'_2, \ldots, r'_k) = 0 \tag{2.6}$$

for all substitutions such that $r_1, r_2, \ldots, r_m \in R_0$ and $r'_1, r'_2, \ldots, r'_k \in R_1$.

Thus, $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_k\}$ are called the sets of even and odd variables (indeterminates) in h, respectively.

For a vector $\overrightarrow{i} = (i_1, i_2, \ldots, i_k)$ with strictly increasing integer coordinates $1 \le i_1 < i_2 < \cdots < i_k \le 2n$ take

$$\Pi(\overrightarrow{i}) = \{\pi \in \mathrm{Sym}\{1, 2, \ldots, 2n\} \mid \pi(i_1), \pi(i_2), \ldots, \pi(i_k) \in \{1, 2, \ldots, k\}\}$$

and consider the complementary vector $\underset{\rightarrow}{i} = (j_1, j_2, \ldots, j_{2n-k})$ with $\{j_1, j_2, \ldots, j_{2n-k}\} = \{1, 2, \ldots, 2n\} \setminus \{i_1, i_2, \ldots, i_k\}$ and $1 \le j_1 < j_2 < \cdots < j_{2n-k} \le 2n$. Now

$$\tau(\overrightarrow{i}) = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2n \\ i_1 & i_2 & & i_k & j_1 & j_2 & & j_{2n-k} \end{pmatrix} \tag{2.7}$$

defines a permutation in $\mathrm{Sym}\{1, 2, \ldots, 2n\}$. We need two more permutations

$$\pi(\overrightarrow{i}) \in \mathrm{Sym}\{1, 2, \ldots, k\} \text{ and } \pi(\underset{\rightarrow}{i}) \in \mathrm{Sym}\{k+1, k+2, \ldots, 2n\} \tag{2.8}$$

which are determined by $\pi \in \Pi(\overrightarrow{i})$ as follows:

$$\pi(\overrightarrow{i}) = \begin{pmatrix} 1 & 2 & \cdots & k \\ \pi(i_1) & \pi(i_2) & & \pi(i_k) \end{pmatrix} \tag{2.9}$$

and

$$\pi(\underset{\rightarrow}{i}) = \begin{pmatrix} k+1 & k+2 & \cdots & 2n \\ \pi(j_1) & \pi(j_2) & & \pi(j_{2n-k}) \end{pmatrix}. \tag{2.10}$$

For an integer $1 \le k \le 2n$ define a $\mathbb{Z}_2$-graded polynomial of degree $2n$ as follows:

$$\begin{aligned} f_k(X,Y) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2n} \mathrm{sgn}(\tau(\overrightarrow{i})) \Bigg(\sum_{\pi \in \Pi(\overrightarrow{i})} \mathrm{sgn}(\pi(\underset{\rightarrow}{i})) x_{\pi(1)} \cdots x_{\pi(i_1-1)} y_{\pi(i_1)} x_{\pi(i_1+1)} \cdots \\ \cdots x_{\pi(i_2-1)} y_{\pi(i_2)} x_{\pi(i_2+1)} \cdots x_{\pi(i_k-1)} y_{\pi(i_k)} x_{\pi(i_k+1)} \cdots x_{\pi(2n)} \Bigg), \end{aligned} \tag{2.11}$$

where

$$X = \{x_{k+1}, x_{k+2}, \ldots, x_{2n}\} \text{ and } Y = \{y_1, y_2, \ldots, y_k\}$$

are the sets of even and odd indeterminates (variables).

Theorem 2.1 *If* $1 \le k \le 2n$, *then* $f_k(X,Y) = 0$ *is a* $\mathbb{Z}_2$*-graded polynomial identity of the* $\mathbb{Z}_2$*-graded full matrix algebra* $\mathrm{M}_n(E) = \mathrm{M}_n(E_0) \oplus \mathrm{M}_n(E_1)$.

Proof (First proof of 2.1). First notice that for a permutation $\pi \in \Pi(\overrightarrow{i})$ we have $\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i}) = \pi \circ \tau(\overrightarrow{i})$, where

$\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i}) \in \mathrm{Sym}\{1, 2, \ldots, k, k+1, \ldots, 2n\}$ is the "disjoint union" of $\pi(\overrightarrow{i})$ and $\pi(\underrightarrow{i})$. Clearly, the number of even cycles of $\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i})$ is the sum of the numbers of the even cycles in $\pi(\overrightarrow{i})$ and in $\pi(\underrightarrow{i})$.

It follows that

$$\mathrm{sgn}(\pi(\overrightarrow{i}))\mathrm{sgn}(\pi(\underrightarrow{i})) = \mathrm{sgn}(\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i})) = \mathrm{sgn}(\pi)\mathrm{sgn}(\tau(\overrightarrow{i})), \tag{2.12}$$

whence $\mathrm{sgn}(\pi)\mathrm{sgn}(\pi(\overrightarrow{i})) = \mathrm{sgn}(\tau(\overrightarrow{i}))\mathrm{sgn}(\pi(\underrightarrow{i}))$ can be derived. In order to show that $f_k(X, Y) = 0$ is a $\mathbb{Z}_2$-graded polynomial identity on $\mathrm{M}_n(E) = \mathrm{M}_n(E_0) \oplus \mathrm{M}_n(E_1)$ take the substitutions

$$x_{k+1} = A_{k+1}, x_{k+2} = A_{k+2}, \ldots, x_{2n} = A_{2n}$$

and

$$y_1 = B_1, y_2 = B_2, \ldots, y_k = B_k,$$

where $A_{k+1}, A_{k+2}, \ldots, A_{2n} \in \mathrm{M}_n(E_0)$ and $B_1, B_2, \ldots, B_k \in \mathrm{M}_n(E_1)$ and consider the "companion" matrices

$$w_1 B_1, w_2 B_2, \ldots, w_k B_k \in \mathrm{M}_n(F_0)$$

($w_1, w_2, \ldots, w_k$ are generators in G) over the even part F_0 of the extended Grassmann algebra $F = K\langle V \cup W \rangle$. In view of $\mathrm{M}_n(E_0) \subseteq \mathrm{M}_n(F_0)$, the application of the Amitsur-Levitzki theorem on $\mathrm{M}_n(F_0)$ yields that

$$S_{2n}(w_1 B_1, \ldots, w_k B_k, A_{k+1}, A_{k+2}, \ldots, A_{2n}) = 0. \tag{2.13}$$

Any summand in

$$S_{2n}(w_1 B_1, \ldots, w_k B_k, A_{k+1}, A_{k+2}, \ldots, A_{2n})$$

is a signed product of the terms $w_1 B_1, \ldots, w_k B_k, A_{k+1}, A_{k+2}, \ldots, A_{2n}$ in a certain order and appears as

$$\begin{aligned}
&\mathrm{sgn}(\pi) A_{\pi(1)} \cdots A_{\pi(i_1-1)} w_{\pi(i_1)} B_{\pi(i_1)} A_{\pi(i_1+1)} \cdots A_{\pi(i_2-1)} w_{\pi(i_2)} B_{\pi(i_2)} A_{\pi(i_2+1)} \cdots \\
&\qquad \cdots A_{\pi(i_k-1)} w_{\pi(i_k)} B_{\pi(i_k)} A_{\pi(i_k+1)} \cdots A_{\pi(2n)} = \\
&\mathrm{sgn}(\pi)(-1)^{1+2+\cdots+(k-1)} w_{\pi(i_1)} w_{\pi(i_2)} \cdots w_{\pi(i_k)} A_{\pi(1)} \cdots A_{\pi(i_1-1)} B_{\pi(i_1)} A_{\pi(i_1+1)} \cdots \\
&\qquad \cdots A_{\pi(i_2-1)} B_{\pi(i_2)} A_{\pi(i_2+1)} \cdots A_{\pi(i_k-1)} B_{\pi(i_k)} A_{\pi(i_k+1)} \cdots A_{\pi(2n)} = \\
&\mathrm{sgn}(\pi)(-1)^{1+2+\cdots+(k-1)} \mathrm{sgn}(\pi(\overrightarrow{i})) w_1 w_2 \cdots w_k A_{\pi(1)} \cdots A_{\pi(i_1-1)} B_{\pi(i_1)} A_{\pi(i_1+1)} \cdots \\
&\qquad \cdots A_{\pi(i_2-1)} B_{\pi(i_2)} A_{\pi(i_2+1)} \cdots A_{\pi(i_k-1)} B_{\pi(i_k)} A_{\pi(i_k+1)} \cdots A_{\pi(2n)},
\end{aligned} \tag{2.14}$$

where $1 \le i_1 < i_2 < \cdots < i_k \le 2n$ and $\pi \in \Pi(\overrightarrow{i})$ are uniquely determined. In the above calculations we used

$$A_t w_r = w_r A_t, \ B_s w_r = -w_r B_s, \ 1 \le r, s \le k < t \le 2n,$$

and

$$w_{\pi(i_1)} w_{\pi(i_2)} \cdots w_{\pi(i_k)} = \mathrm{sgn}(\pi(\overrightarrow{i})) w_1 w_2 \cdots w_k.$$

Thus, we can write that

$$
\begin{aligned}
&S_{2n}(w_1B_1,\dots,w_kB_k,A_{k+1},A_{k+2},\dots,A_{2n})= \\
&\sum_{1\le i_1<i_2<\cdots<i_k\le 2n}\Bigg(\sum_{\pi\in\Pi(\overrightarrow{i})}\operatorname{sgn}(\pi)A_{\pi(1)}\cdots A_{\pi(i_1-1)}w_{\pi(i_1)}B_{\pi(i_1)}A_{\pi(i_1+1)}\cdots \\
&\cdots A_{\pi(i_2-1)}w_{\pi(i_2)}B_{\pi(i_2)}A_{\pi(i_2+1)}\cdots A_{\pi(i_k-1)}w_{\pi(i_k)}B_{\pi(i_k)}A_{\pi(i_k+1)}\cdots A_{\pi(2n)}\Bigg)= \\
&(-1)^{1+2+\cdots+(k-1)}w_1w_2\cdots w_k\sum_{1\le i_1<i_2<\cdots<i_k\le 2n}\operatorname{sgn}(\tau(\overrightarrow{i}))\cdot \qquad (2.15)\\
&\Bigg(\sum_{\pi\in\Pi(\overrightarrow{i})}\operatorname{sgn}(\pi(\underrightarrow{i}))A_{\pi(1)}\cdots A_{\pi(i_1-1)}B_{\pi(i_1)}A_{\pi(i_1+1)}\cdots \\
&A_{\pi(i_2-1)}B_{\pi(i_2)}A_{\pi(i_2+1)}\cdots A_{\pi(i_k-1)}B_{\pi(i_k)}A_{\pi(i_k+1)}\cdots A_{\pi(2n)}\Bigg)= \\
&(-1)^{1+2+\cdots+(k-1)}w_1w_2\cdots w_kf_k(A_{k+1},A_{k+2},\dots,A_{2n},B_1,\dots,B_k),
\end{aligned}
$$

whence $f_k(A_{k+1},A_{k+2},\dots,A_{2n},B_1,\dots,B_k)=0$ follows. □

Remark 2.2 *The case $k=2n$ in the above Theorem 2.1 gives Rosset's key observation that*

$$f_{2n}(Y)=\sum_{\pi\in\mathrm{Sym}\{1,2,\dots,2n\}}y_{\pi(1)}\cdots y_{\pi(2n)}=0 \tag{2.16}$$

(the multilinearization of $y^{2n}=0$) is a polynomial identity of the odd component $\mathrm{M}_n(E_1)$. The case $k=1$ has already appeared in the proof of Theorem 2.4 of [4].

A $\mathbb{Z}_2$-graded polynomial $h(x_1,x_2,\dots,x_m,y_1,y_2,\dots,y_k)$ which is linear in each odd variable y_i can be written as

$$h(x_1,x_2,\dots,x_m,y_1,y_2,\dots,y_k)=\sum_u\sum_{\sigma\in\mathrm{Sym}\{1,2,\dots,k\}}a_{\sigma,u}u_1y_{\sigma(1)}u_2y_{\sigma(2)}\cdots u_ky_{\sigma(k)}u_{k+1}, \tag{2.17}$$

where $a_{\sigma,u}\in K$ and the u_i's are words, possibly empty, in the even variables x_j, $1\le j\le m$. The $*$-transform of h is defined as

$$h^*(x_1,x_2,\dots,x_m,y_1,y_2,\dots,y_k)=\sum_u\sum_{\sigma\in\mathrm{Sym}\{1,2,\dots,k\}}\operatorname{sgn}(\sigma)a_{\sigma,u}u_1y_{\sigma(1)}u_2y_{\sigma(2)}\cdots u_ky_{\sigma(k)}u_{k+1}. \tag{2.18}$$

Lemma 19.4.10 (in [1]) asserts that $h=0$ is a $\mathbb{Z}_2$-graded identity of the $\mathbb{Z}_2$-graded K-algebra $R=R_0\oplus R_1$ if and only if $h^*=0$ is a $\mathbb{Z}_2$-graded identity of the Grassmann envelope $G(R)=(R_0\otimes E_0)\oplus(R_1\otimes E_1)=(R\otimes E)_0$ (the even part of $R\otimes E$).

Proof (Second proof of 2.1). Take $R=\mathrm{M}_n(K\oplus cK)$ with $R_0=\mathrm{M}_n(K)$ and $R_1=c\mathrm{M}_n(K)$, where $K\oplus cK\cong K[c]/(c^2-1)$ is the commutative group algebra of the two element group $\{1,c\}$ with $c^2=1$. Clearly

$\mathrm{M}_n(E)$ can be naturally identified with the Grassmann envelope $G(R)$. Since the Amitsur-Levitzki theorem trivially ensures that

$$f(x_{k+1}, x_{k+2}, \ldots, x_{2n}, y_1, y_2, \ldots, y_k) = S_{2n}(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_{2n}) = 0 \tag{2.19}$$

is a $\mathbb{Z}_2$-graded identity of $R = R_0 \oplus R_1$, the application of the above Lemma 19.4.10 gives that $\mathrm{M}_n(E)$ satisfies the $\mathbb{Z}_2$-graded identity $f^*(x_{k+1}, x_{k+2}, \ldots, x_{2n}, y_1, y_2, \ldots, y_k) = 0$. In view of

$$\begin{aligned} f &= S_{2n}(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_{2n}) = \\ &\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2n} \Bigg(\sum_{\pi \in \Pi(\overrightarrow{i})} \mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(i_1-1)} y_{\pi(\overrightarrow{i})(1)} x_{\pi(i_1+1)} \cdots \\ &\cdots x_{\pi(i_2-1)} y_{\pi(\overrightarrow{i})(2)} x_{\pi(i_2+1)} \cdots x_{\pi(i_k-1)} y_{\pi(\overrightarrow{i})(k)} x_{\pi(i_k+1)} \cdots x_{\pi(2n)} \Bigg), \end{aligned} \tag{2.20}$$

we obtain that

$$\begin{aligned} f^* = &\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2n} \Bigg(\sum_{\pi \in \Pi(\overrightarrow{i})} \mathrm{sgn}(\pi(\overrightarrow{i}))\mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(i_1-1)} y_{\pi(\overrightarrow{i})(1)} x_{\pi(i_1+1)} \cdots \\ &\cdots x_{\pi(i_2-1)} y_{\pi(\overrightarrow{i})(2)} x_{\pi(i_2+1)} \cdots x_{\pi(i_k-1)} y_{\pi(\overrightarrow{i})(k)} x_{\pi(i_k+1)} \cdots x_{\pi(2n)} \Bigg). \end{aligned} \tag{2.21}$$

Now $f^* = f_k(X, Y)$ is a consequence of $\mathrm{sgn}(\pi(\overrightarrow{i}))\mathrm{sgn}(\pi) = \mathrm{sgn}(\tau(\overrightarrow{i}))\mathrm{sgn}(\pi(\underset{\rightarrow}{i}))$. □

Acknowledgment

The authors would like to sincerely thank the several insightful comments and suggestions of the referees.

The second named author was partially supported by the National Research, Development and Innovation Office of Hungary (NKFIH) K138828.

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