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## Symmetric polynomials in the free metabelian associative algebra of rank 2

Dedicated to the 70<sup>th</sup> anniversary of Vesselin Drensky

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**Abstract:** Let  $F$  be the free metabelian associative algebra generated by  $x$  and  $y$  over a field of characteristic zero. We call a polynomial  $f \in F$  symmetric, if  $f(x, y) = f(y, x)$ . The set of all symmetric polynomials coincides with the algebra  $F^{S_2}$  of invariants of the symmetric group  $S_2$ . In this paper, we give the full description of the algebra  $F^{S_2}$ .

**Key words:** Metabelian, symmetric polynomial

### 1. Introduction

Let  $K[X_n]$  be the algebra of polynomials in  $n$  commuting variables over a field  $K$  of characteristic zero, where  $X_n = \{x_1, \dots, x_n\}$ . It is well known that the algebra

$$K[X_n]^{S_n} = \{p \in K[X_n] \mid p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots, x_{\pi(n)}), \forall \pi \in S_n\}$$

of symmetric polynomials is generated by elementary symmetric polynomials  $\sigma_1, \dots, \sigma_n$ , where

$$\sigma_1 = x_1 + \dots + x_n, \sigma_2 = x_1x_2 + \dots + x_1x_n + \dots + x_{n-1}x_n, \dots, \sigma_n = x_1 \cdots x_n.$$

One may consider noncommutative or nonassociative analogues of the above result. As a pioneer, Wolf [12] in 1936 handled the problem for the algebra  $K\langle X_n \rangle^{S_n}$ , where  $K\langle X_n \rangle$  is the free associative algebra. One may also see the work of Bergeron et al. [3] on the invariants and coinvariants of the symmetric groups in noncommuting variables. For a survey on symmetric polynomials in noncommutative variables, we suggest the paper by Boumova et al. [4]. In a recent work [1], the algebra of symmetric polynomials of the free algebra of rank three in the variety of Grassmann algebras was described. One may also see the work [8] on the symmetric polynomials of the algebra generated by two  $2 \times 2$  generic traceless matrices and of its Lie subalgebra. When considering the nonassociative case, the recent papers [6, 9], [10], and [7] consider symmetric polynomials of free metabelian Lie algebras, free metabelian Leibniz algebras, and free metabelian Poisson algebras, respectively.

In the present paper, we describe the algebra  $F^{S_2}$  of symmetric polynomials in the free metabelian associative algebra  $F$  of rank two, and provide a finite generating set for  $F^{S_2}$ .

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**2. Preliminaries**

Let  $A$  be the free associative algebra of rank two over a field  $K$  of characteristic zero. Then, the algebra  $F = A/(A')^2$  is the free metabelian associative algebra of rank two, where  $A' = A[A, A]A$  stands for the commutator ideal of  $A$  generated by all elements of the form  $[a, b] = ab - ba$ , when  $a, b \in A$ . The algebra  $F$  satisfies the metabelian identity  $[a, b][c, d] = 0$  (see [11]). Assume that  $F$  is freely generated by  $x$  and  $y$ . Let all commutators be left normed:  $[a, b, c] = [[a, b], c]$ . Then it is well known that (see [2, 5]) the commutator ideal  $F'$  of  $F$  is of a basis consisting of elements of the form

$$x^m y^n [x, y, \underbrace{x, \dots, x}_k, \underbrace{y, \dots, y}_l], \quad m, n, k, l \geq 0.$$

However, for the needs of the paper we use another basis of the algebra  $F$  as follows.

$$\underbrace{x^m y^n}_{\text{basis of } F/F'}, \quad \underbrace{x^m y^n [x, y] x^k y^l}_{\text{basis of } F'}, \quad m, n, k, l \geq 0.$$

The metabelian identity implies that  $xyu = yxu$  and  $uxy = uyx$  for every element  $u \in F'$ . This yields the following construction. We consider the action of the commutative polynomial algebra  $K[x_1, y_1, x_2, y_2]$  on  $F'$  defined as

$$(x_1^a y_1^b x_2^c y_2^d)u = x^a y^b u x^c y^d, \quad u \in F'.$$

Hence, the vector space  $F'$  is the free left  $K[x_1, y_1, x_2, y_2]$ -module generated by  $[x, y]$  via this action.

Recall that every element of the set

$$F^{S_2} = \{f(x, y) \in F \mid f(x, y) = f(y, x)\}$$

is called a symmetric polynomial of the free associative algebra  $F$ . Note that  $F^{S_2}$  coincides with the algebra of invariants of the symmetric group  $S_2$ . In the next section, we give a generating set for the algebra  $F^{S_2}$ .

**3. Main results**

The next lemma describes the forms of symmetric polynomials in the left  $K[x_1, y_1, x_2, y_2]$ -module  $F'$ .

**Lemma 3.1** *Let  $p \in K[x_1, y_1, x_2, y_2]$ . Then the followings are equivalent.*

- (1)  $p(x_1, y_1, x_2, y_2)[x, y] \in (F')^{S_2}$ .
- (2)  $p(x_1, y_1, x_2, y_2) = -p(y_1, x_1, y_2, x_2)$ .
- (3)  $p(x_1, y_1, x_2, y_2) = (x_1 - y_1)p_1(x_1, y_1, x_2, y_2) + (x_2 - y_2)p_2(x_1, y_1, x_2, y_2)$ , for some  $p_1, p_2 \in K[x_1, y_1, x_2, y_2]^{S_2}$ .

**Proof** (1)  $\Rightarrow$  (2) Let  $\tau_{12} \in S_2$  be the transposition exchanging  $x$  and  $y$ . Then,

$$\begin{aligned} p(x_1, y_1, x_2, y_2)[x, y] &= \tau_{12} \left( p(x_1, y_1, x_2, y_2)[x, y] \right) \\ &= p(y_1, x_1, y_2, x_2)[y, x] \\ &= -p(y_1, x_1, y_2, x_2)[x, y]. \end{aligned}$$

Hence,  $(p(x_1, y_1, x_2, y_2) + p(y_1, x_1, y_2, x_2))[x, y] = 0$ . Therefore,  $p(x_1, y_1, x_2, y_2) + p(y_1, x_1, y_2, x_2) = 0$  in the free left  $K[x_1, y_1, x_2, y_2]$ -module  $F'$  generated by the single element  $[x, y]$ .

(2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (2) are clear.

(2)  $\Rightarrow$  (3) We may assume that  $p(x_1, y_1, x_2, y_2) = (x_1 - y_1)p_1(x_1, y_1, x_2, y_2) + q(y_1, x_2, y_2)$ , where  $q$  does not depend on  $x_1$ . Then,

$$\begin{aligned} p(y_1, x_1, y_2, x_2) &= (y_1 - x_1)p_1(y_1, x_1, y_2, x_2) + q(x_1, y_2, x_2) \\ &= -p(x_1, y_1, x_2, y_2) \\ &= (y_1 - x_1)p_1(x_1, y_1, x_2, y_2) - q(y_1, x_2, y_2). \end{aligned}$$

Substituting  $x_1 = y_1$ , we get that  $q(y_1, y_2, x_2) = -q(y_1, x_2, y_2)$ . Hence, one may express that

$$q(y_1, x_2, y_2) = (x_2 - y_2)q_1(y_1, x_2, y_2) + q_2(y_1, y_2),$$

where  $q_2$  does not depend on  $x_1, x_2$ . Then

$$\begin{aligned} q(y_1, y_2, x_2) &= (y_2 - x_2)q_1(y_1, y_2, x_2) + q_2(y_1, x_2) \\ &= -q(y_1, x_2, y_2) \\ &= (y_2 - x_2)q_1(y_1, x_2, y_2) - q_2(y_1, y_2). \end{aligned}$$

Now  $x_2 = y_2$  yields that  $2q_2(y_1, y_2) = 0$ , and hence  $q_2 = 0$ . □

**Remark 3.2** Note that the algebra  $K[x_1, y_1, x_2, y_2]^{S_2}$  is generated by  $x_1 + y_1$ ,  $x_2 + y_2$ ,  $x_1y_1$ ,  $x_2y_2$  and  $x_1y_2 + x_2y_1$  (see [6]). In addition, the following holds.

$$(x_1y_2 + x_2y_1)^2 + A(x_1y_2 + x_2y_1) + B = 0,$$

where  $A = -(x_1 + y_1)(x_2 + y_2)$  and  $B = x_1y_1((x_2 + y_2)^2 - 2x_2y_2) + x_2y_2((x_1 + y_1)^2 - 2x_1y_1)$ . Hence,  $A$  and  $B$  depend on  $x_1 + y_1$ ,  $x_2 + y_2$ ,  $x_1y_1$ ,  $x_2y_2$ . Therefore, every polynomial  $p \in K[x_1, y_1, x_2, y_2]^{S_2}$  is of the form

$$p = q(x_1 + y_1, x_2 + y_2, x_1y_1, x_2y_2) + r(x_1 + y_1, x_2 + y_2, x_1y_1, x_2y_2)(x_1y_2 + x_2y_1)$$

for some  $q, r \in K[x_1 + y_1, x_2 + y_2, x_1y_1, x_2y_2]$ .

In the next theorem, we give generators of the algebra  $F^{S_2}$  of symmetric polynomials.

**Theorem 3.3** Let  $\text{char}K \neq 2$ . Then  $F^{S_2}$  is generated by

$$\begin{aligned} x + y, \quad xy + yx, \quad u_1 = x[x, y] - y[x, y], \quad u_2 = [x, y]x - [x, y]y, \\ u_3 = xu_1y + yu_1x, \quad u_4 = xu_2y + yu_2x. \end{aligned}$$

**Proof** Initially, it follows from  $\text{char}K \neq 2$  that  $x + y$  and  $xy + yx$  generate  $K[x, y]^{S_2} \cong (F/F')^{S_2}$ . They act on  $F'$  as the polynomials

$$x_1 + y_1, \quad x_2 + y_2, \quad 2x_1y_1, \quad 2x_2y_2 \in K[x_1, y_1, x_2, y_2].$$

If we show that  $(F')^{S_2}$  is generated by  $u_1, u_2, u_3, u_4$  as a  $K[x_1 + y_1, x_2 + y_2, x_1y_1, x_2y_2]$ -module, then the proof will be completed. We prove the theorem in two steps.

Step 1 Let  $f = f_1 + f_2 \in F^{S_2}$ , where  $f_1 \in K[x, y]^{S_2}$  and  $f_2 \in (F')^{S_2}$ . Then  $f_1 = p(x + y, 2xy)$  for some  $p \in K[v_1, v_2]$ . Thus,  $f_1 - p(x + y, 2xy) \equiv 0 \pmod{(F')^{S_2}}$ . This implies that  $f_1 - p(x + y, 2xy) \in (F')^{S_2}$ , and for some  $q \in K[x_1, y_1, x_2, y_2]$  we have

$$f - p(x + y, 2xy) = q(x_1, y_1, x_2, y_2)[x, y],$$

i.e. can be presented.

Step 2 Now let  $p(x_1, y_1, x_2, y_2)[x, y] \in (F')^{S_2}$ . Then

$$p(x_1, y_1, x_2, y_2) = (x_1 - y_1)p_1(x_1, y_1, x_2, y_2) + (x_2 - y_2)p_2(x_1, y_1, x_2, y_2),$$

for some  $p_1, p_2 \in K[x_1, y_1, x_2, y_2]^{S_2}$  by Lemma 3.1. Then, we have that

$$p_i = q_i(x_1 + y_1, x_2 + y_2, x_1y_1, x_2y_2) + r_i(x_1 + y_1, x_2 + y_2, x_1y_1, x_2y_2)(x_1y_2 + x_2y_1), \quad i = 1, 2,$$

for some explicitly given  $q_1, q_2, r_1, r_2$  depending on  $x_1 + y_1, x_2 + y_2, x_1y_1, x_2y_2$  by Remark 3.2. This implies that

$$\begin{aligned} p(x_1, y_1, x_2, y_2)[x, y] &= \left( (x_1 - y_1)(q_1 + r_1(x_1y_2 + x_2y_1)) + (x_2 - y_2)(q_2 + r_2(x_1y_2 + x_2y_1)) \right) [x, y] \\ &= q_1 \underbrace{(x[x, y] - y[x, y])}_{u_1} + r_1(xu_1y + yu_1x) + q_2 \underbrace{([x, y]x - [x, y]y)}_{u_2} + r_2(xu_2y + yu_2x). \end{aligned}$$

The action of  $q_i$  and  $r_i$  is a linear combination of composition of multiplications from both sides by  $x + y$  and  $\frac{xy+yx}{2}$ . □

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