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Symmetric polynomials in free associative algebras

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Abstract: By a result of Margarete Wolf in 1936, we know that the algebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ of symmetric polynomials in noncommuting variables is not finitely generated. In 1984, Koryukin proved that if we equip the homogeneous component of degree n with the additional action of $\mathrm{Sym}(n)$ by permuting the positions of the variables, then the algebra of invariants $K\langle X_d\rangle^G$ of every reductive group G is finitely generated. First, we make a short comparison between classical invariant theory of finite groups and its noncommutative counterpart. Then, we expose briefly the results of Wolf. Finally, we present the main result of our paper, which is, over a field of characteristic 0 or of characteristic $p > d$, the algebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ with the action of Koryukin is generated by the elementary symmetric polynomials.

Key words: Free associative algebra, noncommutative invariant theory, symmetric polynomials, finite generation

1. Introduction

Following Rota [39, 40] "Invariant theory is the great romantic story of mathematics." Its origin can be found in the work of Lagrange in the 1770s and of Gauss (in his *"Disquititiones Arithmeticae"* in 1801) who studied the representation of integers by quadratic binary forms and used the discriminant to distinguish nonequivalent forms. But the real invariant theory began in the 1840s with the works by George Boole in England and by Otto Hesse in Germany. The early years of invariant theory continued in the work of a pleiad of distinguished mathematicians, among which are Cayley, Sylvester, Clebsch, Gordan, and Hilbert. "Seldom in history has an international community of scholars felt so united by a common scientific ideal for so long a stretch of time" ([39, 40]). See, respectively, [22] and [45] for the contributions of Hesse and Boole and [12] for those of Cayley. See also [15, 39, 40] for the history of invariant theory.

The purpose of our paper is threefold. First, we compare three cornerstone results of invariant theory of finite groups and their noncommutative counterparts. Then, we summarize and translate in the modern language the pioneering results by Margarete Wolf [44] on the symmetric polynomials in the free associative algebra $K\langle X_d\rangle = K\langle x_1,\dots,x_d\rangle$. Finally, we present our main result. Following Koryukin [31], we equip the homogeneous component of degree n of $K\langle X_d\rangle$ with the additional action of the symmetric group $\mathrm{Sym}(n)$ of

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degree n by permuting the positions of the variables. We show that if the ground field K is of characteristic 0 or of characteristic $p > d$, then the subalgebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ of symmetric polynomials in $K\langle X_d\rangle$ with the additional action of Koryukin is generated by the elementary symmetric polynomials.

2. Commutative and noncommutative invariant theory

Traditionally in classical invariant theory the considerations are over the complex field $\mathbb{C}$ although many of the results hold over any field K of characteristic 0. The general linear group $\mathrm{GL}_d(\mathbb{C})$ acts on the d-dimensional vector space V_d with basis $\{v_1,\dots,v_d\}$ and $\mathbb{C}[X_d]=\mathbb{C}[x_1,\dots,x_d]$ is the algebra of polynomial functions, where

$$x_i : V_d \to \mathbb{C}, \quad i=1,\dots,d,$$

is defined by

$$x_i(\xi_1v_1+\cdots+\xi_dv_d)=\xi_i, \quad \xi_1,\dots,\xi_d\in\mathbb{C}.$$

The action of $\mathrm{GL}_d(\mathbb{C})$ on V_d induces an action on $\mathbb{C}[X_d]$ by

$$g(f): v\to f\big(g^{-1}(v)\big), \quad g\in \mathrm{GL}_d(\mathbb{C}),\, f(X_d)\in\mathbb{C}[X_d],\, v\in V_d.$$

If G is a subgroup of $\mathrm{GL}_d(\mathbb{C})$, then the algebra of G-invariants is

$$\mathbb{C}[X_d]^G=\big\{f\in\mathbb{C}[X_d]\mid g(f)=f \text{ for all } g\in G\big\}.$$

For our purposes, it is more convenient to assume that $\mathrm{GL}_d(\mathbb{C})$ acts canonically on the vector space KX_d with basis X_d and to extend its action diagonally on $\mathbb{C}[X_d]$ by

$$g\big(f(x_1,\dots,x_d)\big)=f\big(g(x_1),\dots,g(x_d)\big), \quad g\in\mathrm{GL}_d(\mathbb{C}), f\in\mathbb{C}[X_d].$$

The action of $\mathrm{GL}_d(\mathbb{C})$ on the polynomial functions $\mathbb{C}[X_d]$ in the former case is the same as the diagonal action of its opposite group $\mathrm{GL}_d^{\mathrm{op}}(\mathbb{C})$ induced by the canonical action of $\mathrm{GL}_d^{\mathrm{op}}(\mathbb{C})$ on the vector space KX_d in the latter case. Both actions of $\mathrm{GL}_d(\mathbb{C})$ on $\mathbb{C}[X_d]$ give the same algebras of invariants because the mapping $g\to g^{-1}$ defines an isomorphism of $\mathrm{GL}_d(\mathbb{C})$ and $\mathrm{GL}_d^{\mathrm{op}}(\mathbb{C})$.

Every mathematics student knows at least one theorem from invariant theory – the Fundamental theorem of symmetric polynomials:

Every symmetric polynomial can be expressed in a unique way as a polynomial of the elementary symmetric polynomials.

Translated in the language of invariant theory, K is an arbitrary field of any characteristic and the symmetric group $\mathrm{Sym}(d)$ of degree d acts on the vector space KX_d by

$$\sigma(x_i)=x_{\sigma(i)}, \quad \sigma\in\mathrm{Sym}(d),\, i=1,\dots,d.$$

Theorem 2.1 (Fundamental theorem of symmetric polynomials) (i) *The algebra of symmetric polynomials $K[X_d]^{\mathrm{Sym}(d)}$ is generated by the elementary symmetric polynomials*

$$e_1=x_1+\cdots+x_d=\sum_{i=1}^{d}x_i,$$

$$e_2 = x_1x_2 + x_1x_3 + \cdots + x_{d-1}x_d = \sum_{i<j} x_ix_j,$$

$$\ldots$$

$$e_d = x_1 \cdots x_d;$$

(ii) *If* $f \in K[X_d]^{\mathrm{Sym}(d)}$*, then there exists a unique polynomial* $p \in K[y_1, \ldots, y_d]$ *such that* $f = p(e_1, \ldots, e_d)$*. In other words, the elementary symmetric polynomials are algebraically independent.*

Even more well known are the Vieta formulas:
If the algebraic equation

$$f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_{d-1}x + a_d = 0$$

has roots $\alpha_1, \ldots, \alpha_d$ *(in some extension of the ground field* K*), then*

$$a_i = (-1)^i a_0 e_i(\alpha_1, \ldots, \alpha_d), \quad i = 1, \ldots, d.$$

But not so many people know the history of these theorems. Details can be found in [19] and [8].

If not explicitly stated, in the sequel, we shall work over an arbitrary field K. From the very beginning of invariant theory, one of the main problems has been the description of the algebra $K[X_d]^G$ in terms of generators and defining relations. In particular, the following problem was the main motivation for the 14-th problem of Hilbert [25] in his famous lecture *"Mathematische Probleme"* given at the International Congress of Mathematicians held in Paris in 1900.

Problem 2.2 *Is the algebra* $K[X_d]^G$ *finitely generated for all subgroups* G *of* $\mathrm{GL}_d(K)$*?*

For finite groups, G the answer into affirmative was given by Emmy Noether [35] in 1916 when the ground field is of characteristic 0 and in 1926 for fields of any characteristic [36]. Although not stated in this generality, the (nonconstructive) proof for the finite representability of the algebra $K[X_d]^G$ for reductive groups G in characteristic 0 is contained in the work of Hilbert [24] in 1890–1893. In the general case, Nagata [34] in the 1950s gave a counterexample to Problem 2.2.

The invariant theory of finite groups acting on free associative algebras is quite different from the invariant theory of finite groups acting on $K[X_d]$. Below, we state three of the cornerstones of the theory in the commutative case and compare them with the corresponding results in the noncommutative case.

Theorem 2.3 (Endlichkeitssatz of Emmy Noether [35]) *Let* $\mathrm{char}(K) = 0$ *and let* G *be a finite subgroup of* $\mathrm{GL}_d(K)$*. Then, the algebra of invariants* $K[X_d]^G$ *is finitely generated. It has a system of generators* $f_1(X_d), \ldots, f_m(X_d)$ *where every* $f_i(X_d)$ *is a homogeneous polynomial of degree bounded by the order* $|G|$ *of the group* G*.*

Hence, the mapping $K[Y_m] \to K[X_d]^G$ defined by $y_i \to f_i(X_d)$ defines an isomorphism $K[X_d]^G \cong K[Y_m]/I$ for some ideal I of $K[Y_d]$. Together with the Basissatz of Hilbert [23] that *every ideal of* $K[Y_m]$ *is finitely generated,* $K[X]^G$ *is finitely presented, i.e., it can be defined by a finite system of relations, for any finite group* G*.*

Theorem 2.4 (Chevalley–Shephard–Todd [11, 41]) *For G finite and in characteristic 0 the algebra of invariants $K[X_d]^G$ is isomorphic to a polynomial algebra, i.e., $K[X_d]^G$ has a system of algebraically independent generators and $K[X_d]^G \cong K[Y_d]$ if and only if $G < \mathrm{GL}_d(K)$ is generated by pseudo-reflections (matrices of finite multiplicative order with all eigenvalues except one equal to 1).*

The third cornerstone theorem answers the question "how many invariants are there?". The algebra $K[X_d]^G$ is graded for any group G and

$$K[X_d]^G = K \oplus \left(K[X_d]^G\right)^{(1)} \oplus \left(K[X_d]^G\right)^{(2)} \oplus \cdots ,$$

where $\left(K[X_d]^G\right)^{(n)}$ is the vector space of the homogeneous invariants of degree n. The formal power series

$$H\big(K[X_d]^G, t\big) = \sum_{n\geq 0} \dim \left(K[X_d]^G\right)^{(n)} \cdot t^n$$

is called the Hilbert (or Poincaré) series of $K[X_d]^G$. Since $K[X_d]^G$ is finitely generated for finite groups G, the Hilbert–Serre theorem for *the rationality of the Hilbert series of finitely generated commutative algebras* gives that

$$H\big(K[X_d]^G, t\big) = p(t) \prod_{i=1}^{m} \frac{1}{1-t^{a_i}}, \quad p(t) \in \mathbb{Z}[t].$$

The explicit form of $H\big(K[X_d]^G, t\big)$ is given in 1897 by the Molien formula [33].

Theorem 2.5 *Let $\mathrm{char}\,(K) = 0$. For a finite group G*

$$H\big(K[X_d]^G, t\big) = \frac{1}{|G|} \sum_{g\in G} \frac{1}{\det\big(1-gt\big)}.$$

Going to noncommutative generalizations of invariant theory, the first problem is to find a candidate to replace the polynomial algebra $K[X_d]$ with a noncommutative algebra, which shares many of the properties of $K[X_d]$. The most natural candidate is the free unitary associative algebra $K\langle X_d\rangle$ (or the algebra of polynomials in d noncommuting variables). This algebra has the same universal property as $K[X_d]$:

- *If R is a unitary commutative algebra, then every mapping $X_d \to R$ can be extended in a unique way to a homomorphism $K[X_d] \to R$.*
- *If R is a unitary associative algebra, then every mapping $X_d \to R$ can be extended in a unique way to a homomorphism $K\langle X_d\rangle \to R$.*

In our paper, we consider invariant theory of groups acting on $K\langle X_d\rangle$ only. We assume that $\mathrm{GL}_d(K)$ acts canonically on the vector space KX_d and extend this action diagonally on $K\langle X_d\rangle$:

$$g\big(f(x_1,\dots,x_d)\big) = f\big(g(x_1),\dots,g(x_d)\big), \quad g \in \mathrm{GL}_d(K),\ f \in K\langle X_d\rangle.$$

For a subgroup G of $\mathrm{GL}_d(K)$ the algebra of G-invariants $K\langle X_d\rangle^G$ consists of all polynomials in $K\langle X_d\rangle$, which are fixed under the action of G.

As in the case of polynomial algebras, the first results of invariant theory are for the algebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ of the symmetric polynomials in $K\langle X_d\rangle$. They are in the paper [44] of Margarete Wolf in 1936. We shall discuss them in detail in the next section.

Going back to invariant theory of $K\langle X_d\rangle$, $d \geq 2$, the following three theorems show the differences and the similarity with Theorems 2.3, 2.4 and 2.5 from invariant theory of $K[X_d]$.

The first theorem is obtained independently by Dicks and Formanek [14] and Kharchenko [29] for finite groups and by Koryukin [31] in the general case. It shows that $K[X_d]^G$ and $K\langle X_d\rangle^G$ behave in a completely different way concerning the finite generation.

Theorem 2.6 [31] *Let G be an arbitrary subgroup of $\mathrm{GL}_d(K)$ over a field K of any characteristic and let KY_m be the minimal subspace of KX_d such that $K\langle X_d\rangle^G \subseteq K\langle Y_m\rangle$. Then $K\langle X_d\rangle^G$ is finitely generated if and only if G acts on KY_m by scalar multiplication.*

Corollary 2.7 [14, 29] *If G is a finite subgroup of $\mathrm{GL}_d(K)$, then $K\langle X_d\rangle^G$ is finitely generated if and only if G is a finite cyclic group consisting of scalar matrices.*

Corollary 2.8 [31] *If G acts irreducibly on KX_d, i.e. KX_d does not have nontrivial subspaces W such that $G(W) = W$, then $K\langle X_d\rangle^G$ is either trivial or not finitely generated.*

It has turned out that the analogue of the Chevalley–Shephard–Todd theorem also sounds differently for $K\langle X_d\rangle$.

Theorem 2.9 (i) (Lane [32] and Kharchenko [28]) *The algebra $K\langle X_d\rangle^G$ is free for any subgroup G of $\mathrm{GL}_d(K)$ and for any field K.*

(ii) (Kharchenko [28]) *When G is finite, there is a Galois correspondence between the free subalgebras of $K\langle X_d\rangle$ containing $K\langle X_d\rangle^G$ and the subgroups of G: The subalgebra F of $K\langle X_d\rangle$ with $K\langle X_d\rangle^G \subseteq F$ is free if and only if $F = K\langle X_d\rangle^H$ for a subgroup H of G.*

Concerning the Molien formula, there is a complete analogue for $K\langle X_d\rangle$, but the determinants are replaced by the traces of the matrices.

Theorem 2.10 (Dicks and Formanek [14]) *If $G \subset \mathrm{GL}_d(K)$ is a finite group and* $\mathrm{char}\,(K) = 0$, *then*

$$H\big(K\langle X_d\rangle^G, t\big) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{1-\mathrm{tr}(g)t}.$$

We conclude this section with a result of Koryukin [31], which was the motivation for the original result in our paper. Let $\big(K\langle X_d\rangle\big)^{(n)}$ be the vector space of the homogeneous elements of degree n in $K\langle X_d\rangle$. The symmetric group $\mathrm{Sym}(n)$ acts from the right on $\big(K\langle X_d\rangle\big)^{(n)}$ by the rule

$$(x_{i_1}\cdots x_{i_n})\circ\sigma = x_{i_{\sigma^{-1}(1)}}\cdots x_{i_{\sigma^{-1}(n)}}, \quad \sigma\in\mathrm{Sym}(n).$$

We call this action the S-action. We denote by $(K\langle X_d\rangle,\circ)$ the algebra $K\langle X_d\rangle$ with the additional action of $\mathrm{Sym}(n)$ on $\big(K\langle X_d\rangle\big)^{(n)}$, $n = 0,1,2,\ldots$. If F is a graded subalgebra of $K\langle X_d\rangle$ and $F^{(n)}\circ\mathrm{Sym}(n) = F^{(n)}$, then F inherits the S-action. We denote it by $(F,\circ)$ and call $(F,\circ)$ an S-algebra. We say that $(F,\circ)$ is finitely generated as an S-algebra if there exists a finite subset U of F consisting of homogeneous polynomials such that $(F,\circ)$ is the minimal S-subalgebra of $(K\langle X_d\rangle,\circ)$ containing U. Since the left action of $\mathrm{GL}_d(K)$ on $\left(K\langle X_d\rangle\right)^{(n)}$ commutes with the right action of $\mathrm{Sym}(n)$, if G is an arbitrary subgroup of $\mathrm{GL}_d(K)$, then $\left(K\langle X_d\rangle^G,\circ\right)$ is an S-algebra.

Theorem 2.11 ([31]) *Let the field K be arbitrary and let G be a reductive subgroup of $\mathrm{GL}_d(K)$ (i.e., all rational representations of G are completely reducible). Then, the S-algebra $\left(K\langle X_d\rangle^G,\circ\right)$ is finitely generated.*

This theorem immediately inspires the following problem.

Problem 2.12 *Let $\mathrm{char}\,(K)=0$ and let G be a finite subgroup of $\mathrm{GL}_d(K)$.*

(i) *Consider a minimal homogeneous system of generators of the S-algebra $\left(K\langle X_d\rangle^G,\circ\right)$. Is there a bound of the degree of the generators in terms of the order $|G|$ of G and the rank d of $K\langle X_d\rangle$?*

(ii) *Find a finite system of generators of $\left(K\langle X_d\rangle^G,\circ\right)$ for concrete groups G.*

(iii) *If the commutative algebra $K[X_d]^G$ is generated by a homogeneous system $\{f_1,\dots,f_m\}$, can this system be lifted to a system of generators of $\left(K\langle X_d\rangle^G,\circ\right)$?*

We shall answer the cases (ii) and (iii) of Problem 2.12 when G is the symmetric group of degree d.

In this section, we considered invariant theory of groups acting on $K\langle X_d\rangle$ only. There are also other algebras which share the same universal properties as $K[X_d]$ and $K\langle X_d\rangle$, e.g., free Lie algebras or relatively free algebras in varieties of associative or Lie algebras. We refer to the survey articles by Formanek [18] and by one of the authors [16] to get some idea about invariant theory for such algebras.

3. The results of Margarete Wolf

In this section, we summarize and translate in the modern language some of the results of Margarete Wolf in [44].

The free associative algebra $K\langle X_d\rangle = K\langle x_1,\dots,x_d\rangle$ has a basis consisting of the set $\langle X_d\rangle$ of all monomials $x_{i_1}\cdots x_{i_n}$ in the noncommutative variables X_d. We consider the deglex order in $\langle X_d\rangle$, ordering the monomials $u\in\langle X_d\rangle$ first by degree and then lexicographically assuming that $x_1>\cdots>x_d$. We denote the leading monomial of $f\in K\langle X_d\rangle$, $f\neq 0$, by $\overline{f}$. Since the deglex order is admissible, the leading monomial $\overline{f_1f_2}$ of the product of two nonzero polynomials f_1 and f_2 in $K\langle X_d\rangle$ is equal to the product of their leading monomials $\overline{f_1}$ and $\overline{f_2}$.

The symmetric group $\mathrm{Sym}(d)$ acts on the set of monomials $\langle X_d\rangle$ and splits it in orbits. If $u\in\langle X_d\rangle$, then we denote by $\sum u$ the sum of all monomials in the orbit generated by u. If we choose one monomial u from each orbit, then $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ has a basis consisting of all such $\sum u$.

Theorem 3.1 [44] (i) *The algebra of symmetric polynomials $K\langle X_d\rangle^{\mathrm{Sym}(d)}$, $d\geq 2$, is a free associative algebra over any field K.*

(ii) *It has a homogeneous system of free generators $\{f_j\mid j\in J\}$ such that for any $n\geq 1$, there is at least one generator of degree n.*

(iii) *The number of homogeneous polynomials of degree n is the same in every homogeneous free generating system.*

(iv) *If $f \in K\langle X_d\rangle^{\mathrm{Sym}(d)}$ has the presentation*

$$f = \sum_{j=(j_1,\dots,j_m)} \alpha_j f_{j_1}\cdots f_{j_m}, \quad \alpha_j \in K,$$

then the coefficients α_j are linear combinations with integer coefficients of the coefficients of $f(X_d)$.

Proof (i) The leading monomial $u = x_{i_1}\cdots x_{i_n}$ of the symmetric polynomial $\sum v$, $v \in \langle X_d\rangle$, $\deg(v) = n > 0$, has the following properties:

(i) $i_1 = 1$;

(ii) If u has the form $u = x_{i_1}\cdots x_{i_k}x_{i_{k+1}}w$, where $v = x_{i_1}\cdots x_{i_k} \in \langle X_d\rangle$ depends essentially on all $x_1,\dots,x_p$ and if $i_{k+1} \neq 1,\dots,p$, then $i_{k+1} = p+1$.

(iii) Every monomial $u \in \langle X_d\rangle$ satisfying (i) and (ii) is the leading monomial of $\sum u$.

We apply induction on the leading monomials in the basis of the vector space $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ constructed above. The basis of the induction is $\sum x_1$ and we add it as the first element to the generating set of the algebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ which we shall construct.

If the leading monomial of $f \in K\langle X_d\rangle^{\mathrm{Sym}(d)}$ is of the form $\overline{f} = vx_1$, $\deg(v) > 0$, then v is the leading monomial of $\sum v$ and $\overline{f} = \left(\overline{\sum v}\right)\left(\overline{\sum x_1}\right)$. Hence, the leading monomial $\overline{h}$ of the symmetric polynomial $h = f - \sum v \sum x_1$ is smaller than $\overline{f}$ and by the inductive assumption h can be expressed as a polynomial of the already constructed polynomials in the generating set of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$.

Similarly, if the leading monomial of $f \in K\langle X_d\rangle^{\mathrm{Sym}(d)}$ is of the form $\overline{f} = v_1x_1x_2v_2$, $\deg(v_1) > 0$, then v_1 is the leading monomial of $\sum v_1$ and $x_1x_2v_2$ is the leading monomial of $\sum x_1x_2v_2$. Hence, $\overline{f} = \left(\overline{\sum v_1}\right)\left(\overline{\sum x_1x_2v_2}\right)$ and again we apply the inductive assumption for $h = f - \sum v_1 \sum x_1x_2v_2$.

It is easy to see that the leading monomial u of a symmetric polynomial f, $\deg(u) > 1$, cannot be presented as a product of two leading monomials of symmetric polynomials of lower degree if u is neither of the form $u = vx_1$ nor of the form $u = v_1x_1x_2v_2$, $\deg(v_1) > 0$. Then, we add the symmetric polynomial $\sum u$ to the generating system of $K\langle X_d\rangle^{S_d}$. The polynomials of the constructed generating system of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ are free generators of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ because the leading monomial of every $f \in K\langle X_d\rangle^{\mathrm{Sym}(d)}$ can be presented in a unique way as a product of the leading monomials of the system. It follows from the proof that the symmetric polynomials $\sum u$, $u \in \langle X_d\rangle$, are presented as linear combinations with integer coefficients of the constructed free generating system, and this proves also (iv).

(ii) For the proof, it is sufficient to see that the symmetric polynomials

$$u_1 = \sum x_1 \text{ and } u_n = \sum x_1x_2^{n-1},\ n = 2,3,\dots,$$

participate in the free generating system of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ constructed in (i).

(iii) The statement is true for any free graded subalgebra F of $K\langle X_d\rangle$. Let Y be a homogeneous free generating set of F and let g_n be the number of polynomials of degree n in Y. Then it is well known that the Hilbert series of F is

$$H(F,t)=\frac{1}{1-g(t)},\ \text{where } g(t)=\sum_{n\geq 1}g_nt^n$$

is the generating function of the sequence $g_1,g_2,\ldots$. Since the Hilbert series of F does not depend on the choice of the system of free generators, the same holds for the generating function $g(t)$, and this completes the proof of (iii). $\square$

Example 3.2 (i) *The leading monomials u of the symmetric polynomials $\sum v$, $v\in\langle X_d\rangle$, of degree $n\leq 4$ are the following:*

$$\begin{array}{llllll}
n=1: & x_1; & & & & \\
n=2: & x_1x_1, & x_1x_2; & & & \\
n=3: & x_1x_1x_1, & x_1x_1x_2, & x_1x_2x_1, & x_1x_2x_2, & x_1x_2x_3; \\
n=4: & x_1x_1x_1x_1, & x_1x_1x_1x_2, & x_1x_1x_2x_1, & x_1x_1x_2x_2, & x_1x_1x_2x_3, \\
 & x_1x_2x_1x_1, & x_1x_2x_1x_2, & x_1x_2x_1x_3, & x_1x_2x_2x_1, & x_1x_2x_2x_2, \\
 & x_1x_2x_2x_3, & x_1x_2x_3x_1, & x_1x_2x_3x_2, & x_1x_2x_3x_3, & x_1x_2x_3x_4.
\end{array}$$

(ii) *The leading monomials of the system of free generators of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ constructed in Theorem 3.1 for degrees $n\leq 4$ and $d\geq 4$ are*

$$\begin{array}{lllllll}
n=1: & x_1; & & & & & \\
n=2: & x_1x_2; & & & & & \\
n=3: & x_1x_2x_2, & x_1x_2x_3; & & & & \\
n=4: & x_1x_2x_1x_3, & x_1x_2x_2x_2, & x_1x_2x_2x_3, & x_1x_2x_3x_2, & x_1x_2x_3x_3, & x_1x_2x_3x_4.
\end{array}$$

The paper [44] contains also a detailed description of the free generating set of the algebra $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ of symmetric polynomials in two variables.

Theorem 3.3 [44] *In every homogeneous free generating set of $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ there is precisely one element of degree n for each $n\geq 1$.*

Proof **First proof.** It follows immediately from the proof of Theorem 3.1 that $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ is freely generated by the symmetric polynomials

$$\sum x_1x_2^{n-1}=x_1x_2^{n-1}+x_2x_1^{n-1},\quad n\geq 1.$$

Second proof. We divide the monomials of degree $n\geq 1$ in $\langle X_2\rangle$ in two groups. The fist group $x_1\langle X_2\rangle$ consists of the monomials starting with x_1 and similarly for the second group $x_2\langle X_2\rangle$. Then the vector space $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ has a basis

$$\{1\}\cup\{u(x_1,x_2)+u(x_2,x_1)\mid u(x_1,x_2)\in x_1\langle X_2\rangle\}$$

and the number of homogeneous elements of degree n in this basis is 2^{n-1}. (The above counting arguments were used also in the original proof of the theorem in [44].) Hence the Hilbert series of $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ is

$$H\big(K\langle X_2\rangle^{\mathrm{Sym}(2)},t\big)=1+\sum_{n\geq 1}2^{n-1}t^n=1+\frac{t}{1-2t}=\frac{1-t}{1-2t}=\frac{1}{1-g(t)},$$

where $g(t)$ is the generating function counting the elements of degree $n=1,2,\dots$ in the homogeneous free generating system of $K\langle X_2\rangle^{\mathrm{Sym}(2)}$. Hence

$$g(t)=\frac{t}{1-t}=t+t^2+t^3+\cdots,$$

i.e. there is exactly one generator of each degree n.

In characteristic 0 the Hilbert series of $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ can be computed also using the formula of Dicks and Formanek in Theorem 2.10 because the matrices of the elements of Sym(2) with respect to the basis X_2 are

$$\mathrm{id}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad (12)=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$H\big(K\langle X_2\rangle^{\mathrm{Sym}(2)},t\big)=\frac{1}{2}\left(\frac{1}{1-\mathrm{tr(id)}t}+\frac{1}{1-\mathrm{tr}((12))t}\right)=\frac{1}{2}\left(\frac{1}{1-2t}+\frac{1}{1-0t}\right)=\frac{1-t}{1-2t}.$$

Third proof. Let $\mathrm{char}\,(K)\neq 2$. We change linearly the free generators of $K\langle X_2\rangle$ by

$$y_1=\frac{1}{2}(x_1+x_2),y_2=\frac{1}{2}(x_1-x_2).$$

Then $\sigma=(12)\in\mathrm{Sym}(2)$ acts on X_2 and $Y_2=\{y_1,y_2\}$ by

$$x_1\to x_2,x_2\to x_1\text{ and }y_1\to y_1,y_2\to -y_2.$$

Hence $K\langle Y_2\rangle^{\mathrm{Sym}(2)}=K\langle X_2\rangle^{\mathrm{Sym}(2)}$ is spanned by the monomials $u\in\langle Y_2\rangle$ which are of even degree in y_2. Such monomials are written as

$$u=y_1^{m_0}(y_2y_1^{n_1}y_2)y_1^{m_1}\cdots y_1^{m_{k-1}}(y_2y_1^{n_k}y_2)y_1^{m_k}$$

and $K\langle Y_2\rangle^{\mathrm{Sym}(2)}$ is freely generated by

$$y_1,\quad y_2y_1^ny_2,\quad n\geq 0.$$

The original proof of Margarete Wolf. The proof goes by induction. Let $n\geq 1$. We assume that all symmetric polynomials of degree $\leq n-1$ can be expressed as polynomials in a set of symmetric polynomials $f_1,\dots,f_{n-1}$, $\deg(f_k)=k$, $k=1,\dots,n-1$. As in the second proof of the theorem given above, $\dim((K\langle X_2\rangle^{\mathrm{Sym}(2)})^{(n)})=2^{n-1}$ for $n\geq 1$. Let $f_{k_1}f_{k_2}\cdots f_{k_p}=\sum\alpha_jx_{j_1}\cdots x_{j_n}$ be a product of degree n. There is a 1-1 correspondence between such products and the $(p-1)$-tuples $(k_1+1,k_1+k_2+1,\dots,k_1+\cdots+k_{p-1}+1)$. The $(p-1)$-tuple indicates the positions in the monomials $x_{j_1}\cdots x_{j_n}$ where the monomials in $f_2,\dots,f_p$ start, respectively. For example, the product

$$f_2f_4f_1=\left(\sum\alpha x_{a_1}x_{a_2}\right)\left(\sum\beta x_{b_1}x_{b_2}x_{b_3}x_{b_4}\right)\left(\sum\gamma x_c\right)$$

corresponds to $(3,7)$. There are $\binom{n-1}{p-1}$ possibilities to choose $f_{k_1}\cdots f_{k_p}$ but one of them corresponds to the case $p=1$ and has to be excluded. Hence all possibilities are

$$\sum_{p=2}^{n}\binom{n-1}{p-1}=2^{n-1}-1.$$

The products $f_{k_1}\cdots f_{k_p}$ of degree n are linearly independent and span a vector subspace of codimension 1 of $(K\langle X_2\rangle^{(n)})^{S_2}$. Hence, we need one more symmetric polynomial f_n of degree n to express all homogeneous symmetric polynomials of degree n. $\square$

Symmetric functions in commuting variables have been studied from different points of view. The same has happened in the noncommutative case. In [44] Margarete Wolf studied the algebraic properties of $K\langle X_d\rangle^{S_d}$. The next result in this direction appeared more than 30 years later in [7] where Bergman and Cohn generalized the main result in [44]. There is an enormous literature devoted to different aspects of the theory, see for example [1–6, 9, 13, 17, 20, 26, 27, 30, 37, 38, 42, 43].

Biographical data for Margarete Wolf can be found in the Companion website of [21].*

4. The S-algebra of symmetric polynomials in noncommutative variables

This section contains our new result on the generation of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ as an S-algebra.

As a vector space the homogeneous component $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}\right)^{(n)}$ of degree n has a basis consisting of symmetric polynomials of the form $\sum v$ where $v\in(\langle X_d\rangle)^{(n)}$. We may choose v to be such that

$$\deg_{x_1}(v)\geq\cdots\geq\deg_{x_d}(v)$$

and attach to it the partition of n

$$\lambda=(\lambda_1,\ldots,\lambda_d)=(\deg_{x_1}(v),\ldots,\deg_{x_d}(v)).$$

There is a permutation $\sigma\in\mathrm{Sym}(n)$ such that $\sum v=p_\lambda\circ\sigma$, where

$$p_\lambda=\sum x_1^{\lambda_1}\cdots x_d^{\lambda_d}.$$

In particular,

$$p_{(n)}=x_1^n+\cdots+x_d^n,\quad n=1,2,\ldots,$$

are the power sums and

$$p_{(1^n)}=\sum_{\sigma\in\mathrm{Sym}(d)}x_{\sigma(1)}\cdots x_{\sigma(n)},\quad n\leq d,$$

are the noncommutative analogues of the elementary symmetric polynomials.

Lemma 4.1 *Over any field K of arbitrary characteristic the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)},\circ)$ is generated by the power sums $p_{(m)}$, $m=1,2\ldots$.*

Proof It is sufficient to show that for any partition $\lambda=(\lambda_1,\ldots,\lambda_k)$ of $n>0$, $\lambda_1\geq\cdots\geq\lambda_k>0$ the polynomial p_λ can be expressed as a linear combination of products $p_{(m_1)}\cdots p_{(m_j)}\circ\sigma$, $m_1+\cdots+m_j=n$, $\sigma\in\mathrm{Sym}(n)$. We apply induction on the number k of parts of the partition λ. If $k=1$, then $\lambda=(n)$ and $p_\lambda=p_{(n)}$. If $k=2$, then $\lambda=(\lambda_1,\lambda_2)$,

$$p_{(\lambda_1,\lambda_2)}=p_{(\lambda_1)}p_{(\lambda_2)}-p_{(\lambda_1+\lambda_2)}.$$

*http://www.ams.org/publications/authors/books/postpub/hmath-34-PioneeringWomen.pdf

In the general case we use that the difference $f_\lambda = p_{(\lambda_1,\dots,\lambda_k)} - p_{(\lambda_1)}p_{(\lambda_2)}\cdots p_{(\lambda_k)}$ is a sum of monomials $x_{i_1}^{n_1}\cdots x_{i_l}^{n_l}\circ\sigma$, $l<k$. Hence, f_λ is a linear combination of polynomials $p_\mu\circ\sigma$ where $\mu=(\mu_1,\dots,\mu_j)$ is a partition of n in less than k parts. By the inductive assumption, f_λ belongs to the S-algebra generated by $p_{(m)}$, $m=1,2\dots,n$, and hence the same holds for p_λ. $\square$

Remark 4.2 *If we project the polynomials $p_{(1^n)}\in K\langle X_d\rangle$ to the polynomial algebra $K[X_d]$, we shall not obtain the elementary symmetric functions $e_n\in K[X_d]$ but integer multiples of them. This explains why Lemma 4.1 works in the noncommutative case for $K\langle X_d\rangle$ and does not work in the commutative case for $K[X_d]$.*

We shall need a noncommutative analogue of the Newton formulas

$$ke_k=\sum_{i=1}^{k}(-1)^{i-1}e_{k-i}p_i$$

which relate the elementary symmetric polynomials and the power sums in $K[X_d]$. In order to state our version of the Newton formulas for $k\le d$, we denote by Sh_i, $i=0,1,\dots,k$, the set of all “shuffles” $\sigma\in\mathrm{Sym}(k)$ with the property that σ^{-1} preserves the orders both of $1,\dots,k-i$ and of $k-i+1,\dots,k$. For $k>d$ the set Sh_i, $i=0,1,\dots,d$, consists of all permutations $\sigma\in\mathrm{Sym}(k)$, which fix $d+1,\dots,k$ and σ^{-1} preserve the orders both of $1,\dots,d-i$ and of $d-i+1,\dots,d$.

Lemma 4.3 *In $K\langle X_d\rangle$*

$$k!p_{(k)}+(-1)^k kp_{(1^k)}+\sum_{i=1}^{k-1}(-1)^{k-i}i!\left(p_{(1^{k-i})}p_{(i)}\circ\sum_{\sigma\in Sh_i}\sigma\right)=0,\quad k\le d,$$

$$d!p_{(k)}+(-1)^d dp_{(1^d)}p_{(k-d)}+\sum_{i=1}^{d-1}(-1)^{d-i}i!\left(p_{(1^{d-i})}p_{(k-d+i)}\circ\sum_{\sigma\in Sh_i}\sigma\right)=0,\quad k>d.$$

Proof We mimic the proof in the classical case for polynomial algebras, see e.g. the page for Newton identities in Wikipedia[†]. First we handle the case $k=d$. We start with the polynomial

$$f(z,X_d)=\sum_{\sigma\in\mathrm{Sym}(d)}(z-x_{\sigma(1)})(z-x_{\sigma(2)})\cdots(z-x_{\sigma(d)})\in K\langle X_d,z\rangle$$

and expand it in the form

$$f(z,X_d)=\sum_{i=0}^{d}f_i(z,X_d),$$

where $f_i(z,X_d)$ is homogeneous of degree i in z. In the notation of S-algebras f_i has the form

$$f_i=(-1)^{d-i}i!p_{(1^{d-i})}z^i\circ\sum_{\sigma\in Sh_i}\sigma.$$

[†]https://en.wikipedia.org/wiki/Newton%27s_identities

The easiest way to see that the coefficients of f_i are correct is to evaluate f_i in the polynomial algebra $K[X_d]$. Then, $p_{(1^i)}$ becomes $i!e_i$, and the S-action by permuting the positions of the variables in the monomials of degree d is trivial. There are $\binom{d}{i} = \frac{d!}{i!(d-i)!}$ shuffles, so we obtain the usual Vieta expansion of the product $(z-x_1)\cdots(z-x_d) \in K[X_d,z]$ multiplied by $d!$.

For example, for $d=3$

$$f(z,X_3) = 3!z^3 - 2!\sum_{i=1}^{3}(x_iz^2 + zx_iz + z^2x_i) + 1!\sum_{\substack{i,j=1\\ i\neq j}}^{3}(x_ix_jz + x_izx_j + zx_ix_j) - \sum_{\sigma\in\mathrm{Sym}(3)} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}.$$

Since $f(x_j,X_d)=0$ for $j=1,\dots,d$, we obtain that

$$0 = \sum_{j=1}^{d} f(x_j,X_d) = d!p_{(d)} + (-1)^d dp_{(1^d)} + \sum_{i=1}^{d-1}(-1)^{d-i}i!\left(p_{(1^{d-i})}p_{(i)}\circ \sum_{\sigma\in Sh_i}\sigma\right)$$

which gives the proof for $k=d$. (Again, going from $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ to $K[X_d]^{\mathrm{Sym}(d)}$ this is the usual Newton identity multiplied by $d!$.)

Now let $k>d$. As in the case $k=d$ we start with the polynomial $f(z,X_d)$ but replace it by

$$f(z,X_d)z^{k-d} = \left(\sum_{\sigma\in\mathrm{Sym}(d)}(z-x_{\sigma(1)})(z-x_{\sigma(2)})\cdots(z-x_{\sigma(d)})\right)z^{k-d}.$$

Repeating the arguments for $k=d$, we obtain

$$0 = \sum_{j=1}^{d} f(x_j,X_d) = d!p_{(k)} + (-1)^d p_{(1^d)}p_{(k-d)} + \sum_{i=1}^{d-1}(-1)^{d-i}i!\left(p_{(1^{d-i})}p_{(k-d+i)}\circ \sum_{\sigma\in Sh_i}\sigma\right).$$

Finally, let $k<d$. We consider the expression

$$h(x_1,\dots,x_d) = k!p_{(k)} + (-1)^k kp_{(1^k)} + \sum_{i=1}^{k-1}(-1)^{k-i}i!\left(p_{(1^{k-i})}p_{(i)}\circ \sum_{\sigma\in Sh_i}\sigma\right).$$

If we replace $x_{k+1},\dots,x_d$ by 0, then we obtain that $h(x_1,\dots,x_k,0,\dots,0)=0$ in $K\langle X_k\rangle^{\mathrm{Sym}(k)}$. The symmetric polynomial $h(x_1,\dots,x_d)$ is of degree k. Every symmetric polynomial of degree k is completely determined by its component which depends on k variables only. Hence $h(x_1,\dots,x_d)=0$ and this completes also the proof for $k<d$. □

We give examples for small $d=3,4,5$.

$$\begin{aligned}
6p_{(3)} &= 3p_{(1,1,1)} - p_{(1,1)}p_{(1)}\circ\big(\mathrm{id} + (321) + (23)\big) + 2p_{(1)}p_{(2)}\circ\big(\mathrm{id} + (12) + (123)\big)\\
24p_{(4)} &= -4p_{(1,1,1,1)} + p_{(1,1,1)}p_{(1)}\circ\big(\mathrm{id} + (34) + (432) + (4321)\big)\\
&\quad - 2p_{(1,1)}p_{(2)}\circ\big(\mathrm{id} + (23) + (321) + (13)(24) + (234) + (2134)\big)+\\
&\quad + 6p_{(1)}p_{(3)}\circ\big(\mathrm{id} + (12) + (123) + (1234)\big)
\end{aligned}$$

$$\begin{aligned}120p_{(5)} = 5p_{(1^5)} &- p_{(1^4)}p_{(1)} \circ \big(\mathrm{id} + (45) + (543) + (5432) + (54321)\big)\\ &+ 2p_{(1^3)}p_{(2)} \circ \big(\mathrm{id} + (34) + (432) + (4321) + (345) + (3245) + (32145) + (24)(35) + (142)(35)\big)\\ &- 6p_{(1^2)}p_{(3)} \circ \big(\mathrm{id} + (23) + (234) + (2345) + (132) + (1342) + (13452) + (13)(24) + (13)(245) + (35241)\big)\\ &+ 24p_{(1)}p_{(4)} \circ \big(\mathrm{id} + (12) + (123) + (1234) + (12345)\big)\end{aligned}$$

Problem 4.4 *By the Newton formulas, the elementary symmetric polynomial e_k is expressed in terms of elementary symmetric polynomials of lower degree and power sums. Chamberlin and Rafizadeh [10] found an analogue of the Newton formulas where the monomial symmetric polynomials*

$$m_\lambda = \sum x_1^{\lambda_1} \cdots x_d^{\lambda_d} \in K[X_d], \quad \lambda \vdash k,$$

are expressed in terms of monomial symmetric polynomials of lower degree and power sums. It would be interesting to find similar formulas for the symmetric polynomials $p_\lambda \in (K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$.

Now, we go to the main new result of our paper.

Theorem 4.5 *Let $\mathrm{char}\,(K) = 0$ or $\mathrm{char}\,(K) = p > d$. Then the algebra $\big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\big)$ of the symmetric polynomials in d variables is generated as an S-algebra by the elementary symmetric polynomials $p_{(1^i)}$, $i = 1, \ldots, d$.*

Proof By Lemma 4.1 the S-algebra $\big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\big)$ is generated by the power sums $p_{(m)}$, $m = 1, 2 \ldots$. By Lemma 4.3 the power sums $p_{(n)}$ multiplied by a suitable $k!$, $k \leq d$, belong to the S-algebra generated by $p_{(m)}$, $m < n$ and $p_{(1^i)}$, $i = 1, \ldots, d$. Since $k!$ is invertible in the ground field K and the first power sum $p_{(1)}$ coincides with the first elementary symmetric polynomial, the proof follows immediately by induction on n. □

Remark 4.6 *We have also several direct proofs of Theorem 4.5 for small d. For $\mathrm{char}\,(K) \neq 2$ we present two proofs for $d = 2$.*

First proof. *As we have mentioned in the first proof of Theorem 3.3 it follows immediately from the proof of Theorem 3.1 that $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ is freely generated by the symmetric polynomials*

$$u_n = \sum x_1 x_2^{n-1} = x_1 x_2^{n-1} + x_2 x_1^{n-1}, \quad n \geq 1,$$

which for $n = 1$ and $n = 2$ are equal to $p_{(1)}$ and $p_{(1^2)}$, respectively. Further we apply induction on $n \geq 3$. By the inductive assumption $f = p_{(1^2)}p_{(n-2)}$ belongs to the S-algebra generated by $p_{(1)}$ and $p_{(1^2)}$. Then

$$f = (x_1x_2 + x_2x_1)(x_1^{n-2} + x_2^{n-2}) = (x_1x_2^{n-1} + x_2x_1^{n-1}) + (x_2x_1x_2^{n-2} + x_1x_2x_1^{n-2}),$$

$$f \circ (13) = (x_2^2x_1x_2^{n-3} + x_1^2x_2x_1^{n-3}) + (x_2x_1x_2^{n-2} + x_1x_2x_1^{n-2}),$$

$$f \circ (23) = (x_1x_2 + x_2x_1)(x_1^{n-2} + x_2^{n-2}) = (x_1x_2^{n-1} + x_2x_1^{n-1}) + (x_2^2x_1x_2^{n-3} + x_1^2x_2x_1^{n-3}),$$

$$u_n = \frac{1}{2}(f - f \circ (13) + f \circ (23)).$$

Second proof. *In the third proof of Theorem 3.3 we have changed the free generating system of* $K\langle X_2\rangle$ *to*

$$y_1 = \frac{1}{2}(x_1 + x_2), y_2 = \frac{1}{2}(x_1 - x_2)$$

and have shown that $K\langle Y_2\rangle^{\mathrm{Sym}(2)}$ *is freely generated by*

$$y_1 \textit{ and } v_n = y_2 y_1^{n-2} y_2, \quad n \geq 2.$$

Since

$$v_n \circ (2n) = y_2^2 y_1^{n-2} = v_2 y_1^{n-2}$$

we immediately obtain that the S*-algebra* $\left(K\langle Y_2\rangle^{\mathrm{Sym}(2)}, \circ\right)$ *is generated by* y_1 *and* v_2*. This completes the proof because*

$$p_{(1)} = y_1 \textit{ and } p_{(1^2)} = \frac{1}{2}(y_1^2 - v_2).$$

Conjecture 4.7 *Let* $\mathrm{char}\,(K) = p \leq d$*. Then the* S*-algebra* $\left(K[X_d]^{\mathrm{Sym}(d)}, \circ\right)$ *of symmetric polynomials in* d *variables is not finitely generated.*

5. Conclusion

We start our paper with a brief historical picture on the origins of classical invariant theory. Our first goal then is to present several important theorems in invariant theory of finite groups acting on polynomial algebras and to state their counterparts when finite groups act on free associative algebras. In the end of this part of the paper we give a result of Koryukin in 1984 who equipped the free associative algebra with an additional action of symmetric groups (which we call an S-action) and showed that the algebras of noncommutative invariants of reductive groups are S-finitely generated.

The second goal of the paper is to present from a modern point of view the results of Margarete Wolf in 1936 about symmetric polynomials in noncommuting variables. These results are the first steps in noncommutative invariant theory, and we think that they deserve to be made more popular.

Finally, we present our main new result which states that the algebra of symmetric polynomials in noncommuting variables is S-generated by the elementary symmetric polynomials under a natural restriction on the characteristic of the ground field.

We give several proofs of some of the results in the paper to show that the problems in consideration can be attacked from different points of view.

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