

1-1-2021

Zero-divisor graphs of partial transformation semigroups

KEMAL TOKER

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>




Part of the [Mathematics Commons](#)

Recommended Citation

TOKER, KEMAL (2021) "Zero-divisor graphs of partial transformation semigroups," *Turkish Journal of Mathematics*: Vol. 45: No. 5, Article 34. <https://doi.org/10.3906/mat-2012-94>
Available at: <https://journals.tubitak.gov.tr/math/vol45/iss5/34>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Zero-divisor graphs of partial transformation semigroups

Kemal TOKER* 

Department of Mathematics, Faculty of Science and Literature, Harran University, Şanlıurfa, Turkey

Received: 28.12.2020

Accepted/Published Online: 01.09.2021

Final Version: 16.09.2021

Abstract: Let \mathcal{P}_n be the partial transformation semigroup on $X_n = \{1, 2, \dots, n\}$. In this paper, we find the left zero-divisors, right zero-divisors and two sided zero-divisors of \mathcal{P}_n , and their numbers. For $n \geq 3$, we define an undirected graph $\Gamma(\mathcal{P}_n)$ associated with \mathcal{P}_n whose vertices are the two sided zero-divisors of \mathcal{P}_n excluding the zero element θ of \mathcal{P}_n with distinct two vertices α and β joined by an edge in case $\alpha\beta = \theta = \beta\alpha$. First, we prove that $\Gamma(\mathcal{P}_n)$ is a connected graph, and find the diameter, girth, domination number and the degrees of the all vertices of $\Gamma(\mathcal{P}_n)$. Furthermore, we give lower bounds for clique number and chromatic number of $\Gamma(\mathcal{P}_n)$.

Key words: Partial transformation semigroup, zero-divisor graph, clique number, chromatic number

1. Introduction

In 1988, the zero-divisor graphs on commutative rings were defined by Beck [2]. However, the zero element is a vertex in the zero-divisor graph within Beck's definition, later the standard definition of zero-divisor graphs on commutative rings was given by Anderson and Livingston [1]. Let R be a commutative ring and let $Z(R)$ be the set of the zero-divisors of R . The zero-divisor graph of R is defined by an undirected graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, where distinct vertices x and y of $\Gamma(R)$ are adjacent if and only if $xy = 0$. Demeyer et al. have considered this definition for commutative semigroups and they defined and found some basic properties of the zero-divisor graph of a commutative semigroup with zero [5, 6]. There are some papers about zero-divisor graphs on some classes of commutative semigroups [4, 11]. For noncommutative rings, a directed zero-divisor graph and some undirected zero-divisor graphs were defined by Redmond [9]. Suppose that R is a ring, let $Z_R(T)$ be the set of all two sided zero-divisor elements of R . Then Redmond defined an undirected zero-divisor graph $\Gamma(R)$ with vertices $Z_R(T) \setminus \{0\}$, where distinct vertices x and y are adjacent with a single edge if and only if $xy = 0 = yx$. If R is a noncommutative ring, then $\Gamma(R)$ does not need to be connected and if R is a commutative ring then $\Gamma(R)$ coincide with the standard zero-divisor graph of R . Furthermore, these definitions can be considered for noncommutative semigroups with zero. Recently, some properties of zero-divisor graphs of Catalan monoids have been researched in [12].

Suppose that $X_n = \{1, 2, \dots, n\}$ is a finite set. Let $\mathcal{P}_n, \mathcal{T}_n$ and \mathcal{S}_n be the partial transformation semigroup, the full transformation semigroup and the symmetric group on X_n , respectively. Let $\alpha \in \mathcal{P}_n$, the domain of α denoted by $\text{dom}(\alpha)$ and image of α denoted by $\text{im}(\alpha)$, moreover, codomain of α is the complement of $\text{dom}(\alpha)$ and it is denoted by $\text{codom}(\alpha)$.

*Correspondence: ktoker@harran.edu.tr

2010 AMS Mathematics Subject Classification: 20M20, 97K30.

Let $\theta \in \mathcal{P}_n$ such that $\text{dom}(\theta) = \emptyset$ and let α be any element of \mathcal{P}_n then it is clear that $\theta\alpha = \theta = \alpha\theta$, so θ is the zero element of \mathcal{P}_n . Furthermore, \mathcal{P}_n is a noncommutative semigroup for $n \geq 2$.

For $n \geq 2$, let $\mathcal{P}_n^* = \mathcal{P}_n \setminus \{\theta\}$. Then we define the following sets

$$\begin{aligned} L = L(\mathcal{P}_n) &= \{\alpha \in \mathcal{P}_n \mid \alpha\beta = \theta \text{ for some } \beta \in \mathcal{P}_n^*\}, \\ R = R(\mathcal{P}_n) &= \{\alpha \in \mathcal{P}_n \mid \gamma\alpha = \theta \text{ for some } \gamma \in \mathcal{P}_n^*\} \text{ and} \\ T = T(\mathcal{P}_n) &= \{\alpha \in \mathcal{P}_n \mid \alpha\beta = \theta = \gamma\alpha \text{ for some } \beta, \gamma \in \mathcal{P}_n^*\} = L \cap R \end{aligned}$$

which are called the set of left zero-divisors, right zero-divisors and two sided zero-divisors of \mathcal{P}_n , respectively. It is known that $|\mathcal{P}_n| = (n + 1)^n$, $|\mathcal{T}_n| = n^n$ and $|\mathcal{S}_n| = n!$ for $n \in \mathbb{Z}^+$. In this paper we find the left zero-divisors, right zero-divisors and zero-divisors of \mathcal{P}_n , and then their numbers.

For a semigroup S with (zero) 0 if $T(S) \setminus \{0\} \neq \emptyset$ where $T(S) = \{z \in S \mid zx = 0 = yz \text{ for } x, y \in S \setminus \{0\}\}$, then we similarly define the (undirected) zero-divisor graph $\Gamma(S)$ associated with S whose the set of vertices is $T(S) \setminus \{0\}$ with distinct two vertices joined by an edge in case $xy = 0 = yx$ for some $x, y \in T(S) \setminus \{0\}$. Notice that $\theta \in T(\mathcal{P}_n)$ for all $n \geq 1$. In this paper, we prove that $\Gamma(\mathcal{P}_n)$ is a connected graph and find the diameter, girth, the degrees of all vertices, lower bounds for clique number and chromatic number of $\Gamma(\mathcal{P}_n)$ for $n \geq 3$.

For semigroup terminology see [7, 8] and for graph theoretical terminology see [10].

2. Zero-divisors of \mathcal{P}_n

In this section, we find the left zero-divisors, right zero-divisors and two-sided zero-divisors of \mathcal{P}_n and their numbers.

Since, for any $\alpha, \beta \in \mathcal{P}_n$, it is well-known that $\text{dom}(\alpha\beta) = [\text{im}(\alpha) \cap \text{dom}(\beta)]\alpha^{-1}$ and $\text{im}(\alpha\beta) = [\text{im}(\alpha) \cap \text{dom}(\beta)]\beta$ (see Proposition 1.4.3 in [8]), we can write the following immediate result.

Lemma 2.1 *If $\alpha, \beta \in \mathcal{P}_n$, then $\alpha\beta = \theta \iff \text{im}(\alpha) \subseteq \text{codom}(\beta)$. In particular, $\alpha^2 = \theta \iff \text{im}(\alpha) \subseteq \text{codom}(\alpha)$.*

Lemma 2.2 *For $n \geq 1$, let L be the set of left zero-divisors of \mathcal{P}_n . Then we have*

$$L = \mathcal{P}_n \setminus \mathcal{S}_n \quad \text{and} \quad |L| = (n + 1)^n - n!.$$

Proof For any $\alpha \in \mathcal{P}_n \setminus \mathcal{S}_n$, notice that $X_n \setminus \text{im}(\alpha) \neq \emptyset$. If we take any $\beta \in \mathcal{P}_n$ with $\text{dom}(\beta) = X_n \setminus \text{im}(\alpha)$, then it follows from Lemma 2.1 that α is a left zero divisor.

Since, for every $\alpha \in \mathcal{S}_n$, $|\text{im}(\alpha\beta)| = |\text{im}(\beta)|$ for all $\beta \in \mathcal{P}_n$, it follows that $\alpha\beta = \theta$ if and only if $\beta = \theta$, the zero element of \mathcal{P}_n . Thus, the permutations on X_n are not left zero divisors. Therefore, $|L| = |\mathcal{P}_n| - |\mathcal{S}_n| = (n + 1)^n - n!$, as required. \square

Since, for any $\alpha, \beta \in \mathcal{P}_n$, $\text{im}(\beta\alpha) = [\text{im}(\beta) \cap \text{dom}(\alpha)]\alpha$, we also have immediate lemma.

Lemma 2.3 *For $\alpha, \beta \in \mathcal{P}_n$, $\beta\alpha = \theta$ if and only if $\text{im}(\beta) \cap \text{dom}(\alpha) = \emptyset$.*

Lemma 2.4 *For $n \geq 1$, let R be the set of right zero-divisors of \mathcal{P}_n . Then we have*

$$R = \mathcal{P}_n \setminus \mathcal{T}_n \quad \text{and} \quad |R| = (n + 1)^n - n^n.$$

Proof For any $\alpha \in \mathcal{P}_n \setminus \mathcal{T}_n$, notice that $X_n \setminus \text{dom}(\alpha) \neq \emptyset$. If we take any $\beta \in \mathcal{P}_n$ with $\text{im}(\beta) = X_n \setminus \text{dom}(\alpha)$, then it follows from Lemma 2.3 that α is a right zero divisor.

Since, for every $\alpha \in \mathcal{T}_n$, $\text{dom}(\beta\alpha) = \text{dom}(\beta)$ for all $\beta \in \mathcal{P}_n$, it follows that $\beta\alpha = \theta$ if and only if $\beta = \theta$. Thus, the full transformations on X_n are not right zero divisors. Therefore, $|R| = |\mathcal{P}_n| - |\mathcal{T}_n| = (n+1)^n - n^n$, as required. \square

Since $\mathcal{S}_n \subseteq \mathcal{T}_n \subseteq \mathcal{P}_n$, it follows that $R \subseteq L$. Thus we have the following corollary.

Corollary 2.5 For $n \geq 1$, let T be the set of zero-divisors of \mathcal{P}_n . Then we have

$$T = R = \mathcal{P}_n \setminus \mathcal{T}_n \quad \text{and} \quad |T| = (n+1)^n - n^n.$$

3. Zero-divisor graph of \mathcal{P}_n

Let $G = (V(G), E(G))$ be an undirected graph, $V(G)$ denotes vertex set of G and $E(G)$ denotes the edge set of G . If G does not have any loops and multiple edges, then G is called a simple graph. We consider simple graphs for the following definitions. Two vertices u and v of G are said to be connected if there is a path from u to v . If u and v connected for every $u, v \in V(G)$, then G is called a connected graph. Let $u, v \in V(G)$, the length of the shortest path between u and v is denoted by $d_G(u, v)$.

The diameter of G is denoted by $\text{diam}(G)$ and defined by

$$\text{diam}(G) = \max\{d_G(u, v) \mid u, v \in V(G)\}.$$

Let $v \in V(G)$ then the degree of a vertex is denoted by $\text{deg}_G(v)$ and it is the number of adjacent vertices to v in G . $\Delta(G)$ shows that the maximum degree and $\delta(G)$ shows that the minimum degree among all the degrees in G .

Let $\emptyset \neq D \subseteq V(G)$, for each vertex of G if the vertex in D or the vertex is adjacent to any vertex in D then D is called a dominating set for G . The domination number of G is

$$\min\{|D| \mid D \text{ is a dominating set of } G\}$$

and this number is denoted by $\gamma(G)$. The length of the shortest cycle in G is called girth of G and it is denoted by $\text{gr}(G)$, if G does not contain any cycles then its girth is defined to be infinity.

Let $\emptyset \neq C \subseteq V(G)$, if u and v are adjacent vertices for all $u, v \in C$ in G , then C is called a clique. The number of all the vertices in any maximal clique of G is called clique number of G and it is denoted by $\omega(G)$. The chromatic number of G is defined by the number of the minimum number of colours required to colour of the all vertices G with the rule no two adjacent vertices have the same colour, and it is denoted by $\chi(G)$.

Let G be a graph with n vertices, if every vertex is adjacent to each other vertices, then G is called a complete graph and it is denoted by K_n .

Let $V' \subseteq V(G)$. The (vertex) induced subgraph $G' = (V', E')$ is a subgraph of G and its vertex set is V' , furthermore, its edge set consists of all of the edges in $E(G)$ that have both endpoints in V' .

In this section, we prove that $\Gamma(\mathcal{P}_n)$ is a connected graph and find the diameter, girth, domination number, the vertex degrees, and give lower bounds for clique number and chromatic number of $\Gamma(\mathcal{P}_n)$ for $n \geq 3$. In this paper, we use Γ instead of $\Gamma(\mathcal{P}_n)$ for convenience.

Let $T^* = T \setminus \{\theta\}$, then we have $V(\Gamma) = T^*$ from definitions. It follows from Lemmas 2.1 and 2.3 α is adjacent to β if and only if $\text{im}(\alpha) \subseteq \text{codom}(\beta)$ and $\text{im}(\beta) \subseteq \text{codom}(\alpha)$.

Lemma 3.1 Γ is a connected graph for $n \geq 3$.

Proof For $n \geq 3$, let $\gamma_i \in \mathcal{P}_n$ such that $\text{dom}(\gamma_i) = \{i\}$ and $i\gamma_i = i$ for $1 \leq i \leq n$, then it is clear that $\gamma_i \in V(\Gamma)$ for $1 \leq i \leq n$. Let G be an induced subgraph of Γ induced by the vertex set $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$. Then it is clear that $G = K_n$. For $\alpha \in V(\Gamma) \setminus \{\gamma_1, \gamma_2, \dots, \gamma_n\}$, there exists at least one $i \in \text{codom}(\alpha)$.

Case 1: If $i \notin \text{im}(\alpha)$, then $\alpha\gamma_i = \gamma_i\alpha = \theta$.

Case 2: If $i \in \text{im}(\alpha)$, then $i\alpha^{-1} \neq \emptyset$. So, in this case, we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r & A_{r+1} \\ i & a_2 & \dots & a_r & - \end{pmatrix}$$

where $\text{im}(\alpha) = \{i = a_1, a_2, \dots, a_r\}$, $A_k = a_k\alpha^{-1}$ for $1 \leq k \leq r$ and $A_{r+1} = \text{codom}(\alpha)$. If $r \geq 2$, then we take

$$\beta = \begin{pmatrix} X_n \setminus \{i, a_2, \dots, a_r\} & \{i, a_2, \dots, a_r\} \\ i & - \end{pmatrix},$$

then we have $\beta \in V(\Gamma)$ and $\alpha\beta = \beta\alpha = \theta$. Moreover, $\beta\gamma_{a_2} = \gamma_{a_2}\beta = \theta$. So, in this case, we have a path in Γ such that $\alpha - \beta - \gamma_{a_2}$ where $\gamma_{a_2} \in V(G)$. In other case, we can write

$$\alpha = \begin{pmatrix} A_1 & A_2 \\ i & - \end{pmatrix},$$

so, $i \in A_2$ since $i \in \text{codom}(\alpha)$. There exists $j \in X_n$ such that $i \neq j$, moreover, $X_n \setminus \{i, j\} \neq \emptyset$ since $n \geq 3$. Let

$$\beta = \begin{pmatrix} X_n \setminus \{i, j\} & \{i, j\} \\ i & - \end{pmatrix}.$$

Then we have $\beta \in V(\Gamma)$ and $\alpha\beta = \beta\alpha = \theta$. Moreover, $\beta\gamma_j = \gamma_j\beta = \theta$. So, in this case, we have a path in Γ such that $\alpha - \beta - \gamma_j$ where $\gamma_j \in V(G)$. It follows that Γ is a connected graph. \square

Lemma 3.2 $\text{diam}(\Gamma) = 4$ for $n \geq 3$.

Proof Let $n \geq 3$ and $\alpha, \beta \in V(\Gamma)$. Then we have

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r & A_{r+1} \\ a_1 & a_2 & \dots & a_r & - \end{pmatrix}, \beta = \begin{pmatrix} B_1 & B_2 & \dots & B_k & B_{k+1} \\ b_1 & b_2 & \dots & b_k & - \end{pmatrix}$$

where $a_i, b_j \in X_n$ for $1 \leq i \leq r$ and $1 \leq j \leq k$, $1 \leq r \leq n - 1$ and $1 \leq k \leq n - 1$, $\{A_1, A_2, \dots, A_{r+1}\}$ is a partition of X_n and $\{B_1, B_2, \dots, B_{k+1}\}$ is a partition of X_n . Firstly we will show that $d_\Gamma(\alpha, \beta) \leq 4$. Let $Y = \{a_1, a_2, \dots, a_r\} \cup \{b_1, b_2, \dots, b_k\}$.

Case 1: If $Y \neq X_n$ and $A_{r+1} \cap B_{k+1} \neq \emptyset$. Then there exists $z, t \in X_n$ such that $z \in X_n \setminus Y$ and $t \in A_{r+1} \cap B_{k+1}$. Let

$$\gamma = \begin{pmatrix} z & X_n \setminus \{z\} \\ t & - \end{pmatrix}.$$

It is clear that $\gamma \in V(\Gamma)$ and $\alpha - \gamma - \beta$ in Γ .

Case 2: If $Y = X_n$ and $A_{r+1} \cap B_{k+1} \neq \emptyset$. There exists $x, y \in X_n$ such that $x \in \{b_1, b_2, \dots, b_k\} \setminus \{a_1, a_2, \dots, a_r\}$ and $y \in \{a_1, a_2, \dots, a_r\} \setminus \{b_1, b_2, \dots, b_k\}$ since $1 \leq r \leq n-1$, $1 \leq k \leq n-1$ and $Y = X_n$. So we have $x \neq y$ and there exists $z \in X_n$ such that $z \in X_n \setminus \{x, y\}$ since $n \geq 3$. Let $t \in A_{r+1} \cap B_{k+1}$. Let

$$\lambda_1 = \begin{pmatrix} x & X_n \setminus \{x\} \\ t & - \end{pmatrix}, \lambda_2 = \begin{pmatrix} y & X_n \setminus \{y\} \\ t & - \end{pmatrix}, \lambda_3 = \begin{pmatrix} X_n \setminus \{t\} & t \\ z & - \end{pmatrix}.$$

Then it is clear that $\lambda_1, \lambda_2, \lambda_3 \in V(\Gamma)$. If $t \notin \{x, y\}$, then $\alpha - \lambda_1 - \lambda_2 - \beta \in \Gamma$. If $t \in \{x, y\}$, then $\alpha - \lambda_1 - \lambda_3 - \lambda_2 - \beta \in \Gamma$.

Case 3: If $Y \neq X_n$ and $A_{r+1} \cap B_{k+1} = \emptyset$. There exists $z \in X_n$ such that $z \in X_n \setminus Y$. Let $x \in A_{r+1}$, $y \in B_{k+1}$ and $t \in X_n \setminus \{z\}$. Let

$$\lambda_1 = \begin{pmatrix} z & X_n \setminus \{z\} \\ x & - \end{pmatrix}, \lambda_2 = \begin{pmatrix} z & X_n \setminus \{z\} \\ y & - \end{pmatrix}, \lambda_3 = \begin{pmatrix} X_n \setminus \{x, y\} & \{x, y\} \\ t & - \end{pmatrix}.$$

Then it is clear that $\lambda_1, \lambda_2, \lambda_3 \in V(\Gamma)$. If $z \notin \{x, y\}$, then $\alpha - \lambda_1 - \lambda_2 - \beta \in \Gamma$. If $z \in \{x, y\}$, then $\alpha - \lambda_1 - \lambda_3 - \lambda_2 - \beta \in \Gamma$.

Case 4: If $Y = X_n$ and $A_{r+1} \cap B_{k+1} = \emptyset$. There exists $x, y \in X_n$ such that $x \in \{b_1, b_2, \dots, b_k\} \setminus \{a_1, a_2, \dots, a_r\}$ and $y \in \{a_1, a_2, \dots, a_r\} \setminus \{b_1, b_2, \dots, b_k\}$ since $1 \leq r \leq n-1$, $1 \leq k \leq n-1$ and $Y = X_n$. So we have $x \neq y$ and there exists $z \in X_n$ such that $z \in X_n \setminus \{x, y\}$ since $n \geq 3$. Let $a \in A_{r+1}$ and $b \in B_{k+1}$. Then $a \neq b$ since $A_{r+1} \cap B_{k+1} = \emptyset$. Let

$$\lambda_1 = \begin{pmatrix} x & X_n \setminus \{x\} \\ a & - \end{pmatrix}, \lambda_2 = \begin{pmatrix} y & X_n \setminus \{y\} \\ b & - \end{pmatrix}.$$

Then it is clear that $\lambda_1, \lambda_2 \in V(\Gamma)$. If $x \neq b$ and $y \neq a$, then $\alpha - \lambda_1 - \lambda_2 - \beta \in \Gamma$. If $x = b$, then we take

$$\lambda_3 = \begin{pmatrix} X_n \setminus \{x, a\} & \{x, a\} \\ z & - \end{pmatrix}.$$

Then it is clear that $\lambda_3 \in V(\Gamma)$ and $\alpha - \lambda_1 - \lambda_3 - \lambda_2 - \beta \in \Gamma$. If $y = a$, then we take

$$\lambda_4 = \begin{pmatrix} X_n \setminus \{y, b\} & \{y, b\} \\ z & - \end{pmatrix}.$$

Then it is clear that $\lambda_4 \in V(\Gamma)$ and $\alpha - \lambda_1 - \lambda_4 - \lambda_2 - \beta \in \Gamma$.

So we have concluded that $d_\Gamma(\alpha, \beta) \leq 4$ for all cases. Let

$$\alpha_1 = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-1 & - \end{pmatrix}, \beta_1 = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 3 & \dots & n & - \end{pmatrix}.$$

α_1 and β_1 are not adjacent vertices in Γ . Moreover, α_1 has only one adjacent vertex in Γ which is $\mu_1 = \begin{pmatrix} n & X_n \setminus \{n\} \\ n & - \end{pmatrix}$, similarly β_1 has only one adjacent vertex in Γ which is $\mu_2 = \begin{pmatrix} 1 & X_n \setminus \{1\} \\ n & - \end{pmatrix}$. So we have $\mu_1 \neq \mu_2$, moreover, μ_1 and μ_2 are not adjacent vertices in Γ , thus $d_\Gamma(\alpha_1, \beta_1) = 4$. It follows that $\text{diam}(\Gamma) = 4$ for $n \geq 3$. \square

Theorem 3.3 $gr(\Gamma) = 3$ for $n \geq 3$.

Proof First of all, since Γ is a simple graph $gr(\Gamma) \geq 3$. For $n \geq 3$, let $\gamma_i \in \mathcal{P}_n$ such that $\text{dom}(\gamma_i) = \{i\}$ and $i\gamma_i = i$ for $1 \leq i \leq n$, then it is clear that $\gamma_i \in V(\Gamma)$ for $1 \leq i \leq n$. Then we have a cycle such that $\gamma_1 - \gamma_2 - \gamma_3 - \gamma_1$, it follows that $gr(\Gamma) = 3$ for $n \geq 3$. \square

Theorem 3.4 For $n \geq 3$, let $\alpha \in V(\Gamma)$. If $r = |\text{im}(\alpha)|$ and $k = |\text{codom}(\alpha)|$, then

$$\text{deg}_\Gamma(\alpha) = \begin{cases} (k+1)^{(n-r)} - 1 & \text{if } \text{im}(\alpha) \not\subseteq \text{codom}(\alpha) \\ (k+1)^{(n-r)} - 2 & \text{if } \text{im}(\alpha) \subseteq \text{codom}(\alpha). \end{cases}$$

Proof For $n \geq 3$, let $\alpha \in V(\Gamma)$, $r = |\text{im}(\alpha)|$ and $k = |\text{codom}(\alpha)|$. For any $\beta \in \mathcal{P}_n$ suppose that $\alpha\beta = \theta = \beta\alpha$. It follows from Lemma 2.1 that $\text{im}(\beta) \subseteq \text{codom}(\alpha)$ and $\text{im}(\alpha) \subseteq \text{codom}(\beta)$. If $\text{im}(\alpha) \not\subseteq \text{codom}(\alpha)$, then we have $\alpha^2 \neq \theta$, so $\beta \neq \alpha$. Moreover, we have if $x \in \text{im}(\alpha)$, then $x \in \text{codom}(\beta)$ and if $x \in X_n \setminus \text{im}(\alpha)$, then $x\beta \in \text{codom}(\alpha)$ or $x \in \text{codom}(\beta)$. So we have $k+1$ different choice for $\forall x \in X_n \setminus \text{im}(\alpha)$ since $k = |\text{codom}(\alpha)|$. However, if $x \in \text{codom}(\beta)$ for $\forall x \in X_n \setminus \text{im}(\alpha)$, then $\beta = \theta \notin V(\Gamma)$. It follows that, $\text{deg}_\Gamma(\alpha) = (k+1)^{(n-r)} - 1$ since $|X_n \setminus \text{im}(\alpha)| = n-r$. If $\text{im}(\alpha) \subseteq \text{codom}(\alpha)$, then we have $\alpha^2 = \theta$, so if we take $\beta = \alpha$, then $\alpha\beta = \theta = \beta\alpha$. So from the definition of vertex degree we have $\text{deg}_\Gamma(\alpha) = ((k+1)^{(n-r)} - 1) - 1 = (k+1)^{(n-r)} - 2$. \square

For $n \geq 3$, if we consider $\alpha \in V(\Gamma)$ such that $|\text{im}(\alpha)| = n-1$ and $|\text{codom}(\alpha)| = 1$, then we have $\text{im}(\alpha) \not\subseteq \text{codom}(\alpha)$ and so $\text{deg}_\Gamma(\alpha) = 1$ from Theorem 3.4. If we consider $\beta \in V(\Gamma)$ such that $|\text{im}(\beta)| = r$ and $|\text{codom}(\beta)| = k$, then it is clear that $1 \leq r \leq n-1$ and $1 \leq k \leq n-1$. For the maximum degree, we take $k = n-1$ and $r = 1$. Moreover, let $x\beta = x$ for $x \in \text{dom}(\beta)$ then $\text{im}(\beta) \not\subseteq \text{codom}(\beta)$, so $\text{deg}_\Gamma(\beta) = n^{(n-1)} - 1$. Thus we have the following immediate corollary.

Corollary 3.5 $\Delta(\Gamma) = n^{(n-1)} - 1$ and $\delta(\Gamma) = 1$ for $n \geq 3$.

Theorem 3.6 $\gamma(\Gamma) = n^2$ for $n \geq 3$.

Proof For $n \geq 3$, let $\alpha_{ij} \in \mathcal{P}_n$ such that $\text{dom}(\alpha_{ij}) = \{i\}$ and $i\alpha_{ij} = j$ for $1 \leq i \leq n$, $1 \leq j \leq n$. We have $\alpha_{ij} \in V(\Gamma)$. Let

$$\mathcal{A} = \{\alpha_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}$$

then it is clear that $|\mathcal{A}| = n^2$. If $i = j$ then $\text{deg}_\Gamma(\alpha_{ij}) = n^{n-1} - 1$ and if $i \neq j$ then $\text{deg}_\Gamma(\alpha_{ij}) = n^{n-1} - 2$. Let $X_n \setminus \{i\} = \{i_1, i_2, \dots, i_{n-1}\}$ with $i_1 < i_2 < \dots < i_{n-1}$ and $X_n \setminus \{j\} = \{j_1, j_2, \dots, j_{n-1}\}$ with $j_1 < j_2 < \dots < j_{n-1}$. For $1 \leq i \leq n$, $1 \leq j \leq n$, let $\beta_{ij} \in \mathcal{P}_n$ such that $\text{codom}(\beta_{ij}) = \{j\}$ and $j_k\beta_{ij} = i_k$ for $1 \leq k \leq n-1$. Then we have $\beta_{ij} \in V(\Gamma)$ and $\text{deg}_\Gamma(\beta_{ij}) = 1$. Let

$$\mathcal{B} = \{\beta_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}.$$

Then we have $|\mathcal{B}| = n^2$, moreover, \mathcal{A} and \mathcal{B} are disjoint sets. In Γ , β_{ij} has only one adjacent vertex which is α_{ij} for $1 \leq i \leq n$, $1 \leq j \leq n$. It follows that $\gamma(\Gamma) \geq n^2$ from the pigeonhole principle. Let $\alpha \in V(\Gamma) \setminus \mathcal{A}$. Then there exist $i, j \in X_n$ such that $i \in X_n \setminus \text{im}(\alpha)$ and $j \in \text{codom}(\alpha)$. We have α and α_{ij} are adjacent vertices

in Γ . Thus \mathcal{A} is a dominating set for Γ . We have $\gamma(\Gamma) \geq n^2$, \mathcal{A} is a dominating set for Γ and $|\mathcal{A}| = n^2$. So $\gamma(\Gamma) = n^2$ for $n \geq 3$. \square

We have $\omega(\Gamma) \geq n$ from the proof of Lemma 3.1. In the following theorem, we give better lower bound for clique number of Γ for $n \geq 3$.

Theorem 3.7 *If $n \geq 3$, then $\omega(\Gamma) \geq (r + 1)^{n-r} - 1$ for $1 \leq r \leq n - 1$.*

Proof Let $n \geq 3$ and $A = \{x_1, x_2, \dots, x_r\} \subseteq X_n$ for $1 \leq r \leq n - 1$. Let

$$B = \{\alpha \in \mathcal{P}_n \mid A \subseteq \text{codom}(\alpha) \text{ and } \emptyset \neq \text{im}(\alpha) \subseteq A\}.$$

It is clear that $B \neq \emptyset$ and if $\alpha \in B$, then $\alpha \in V(\Gamma)$. Let $\alpha_1, \alpha_2 \in B$ and $\alpha_1 \neq \alpha_2$, then we have $\text{im}(\alpha_1) \subseteq A \subseteq \text{codom}(\alpha_2)$ and $\text{im}(\alpha_2) \subseteq A \subseteq \text{codom}(\alpha_1)$, so α_1 and α_2 adjacent vertices in Γ from Lemma 2.1. Let G be an induced subgraph of Γ induced by the vertex set B , then we have G is a complete graph. Moreover $|B| = (r + 1)^{n-r} - 1$. Thus for $n \geq 3$, $\omega(\Gamma) \geq (r + 1)^{n-r} - 1$ for $1 \leq r \leq n - 1$. \square

For any graph G , it is known that $\chi(G) \geq \omega(G)$ (see Corollary 6.2 in [3]), so we have the following corollary.

Corollary 3.8 *If $n \geq 3$, then $\chi(\Gamma) \geq (r + 1)^{n-r} - 1$ for $1 \leq r \leq n - 1$.*

Example 3.9 *Let $\Gamma = \Gamma(\mathcal{P}_5)$, then Γ is a connected graph, $|V(\Gamma)| = 4650$, $\text{diam}(\Gamma) = 4$, $\text{gr}(\Gamma) = 3$, $\Delta(\Gamma) = 624$, $\delta(\Gamma) = 1$, $\gamma(\Gamma) = 25$, $\omega(\Gamma) \geq 26$ and $\chi(\Gamma) \geq 26$. Moreover, if $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & - & 2 \end{pmatrix}$, then $\alpha \in V(\Gamma)$ and $\text{deg}_\Gamma(\alpha) = 63$.*

Acknowledgment

We would like to thank anonymous referees for the comments and suggestions, which significantly contributed to improving the quality of the publication.

References

- [1] Anderson DF, Livingston PS. The zero-divisor graph of a commutative ring. *Journal of Algebra* 1999; 217 (2): 434-447.
- [2] Beck I. Coloring of commutative rings. *Journal of Algebra* 1988; 116 (1): 208-226.
- [3] Chartrand G, Zhang P. *Chromatic Graph Theory*. Boca Raton, FL, USA: CRC Press, 2009.
- [4] Das KC, Akgüneş N, Çevik AS. On a graph of monogenic semigroup. *Journal of Inequalities and Applications* 2013; 44: 1-13.
- [5] DeMeyer F, DeMeyer L. Zero divisor graphs of semigroups. *Journal of Algebra* 2005; 283 (1): 190-198.
- [6] DeMeyer F, McKenzie T, Schneider K. The zero-divisor graph of a commutative semigroup. *Semigroup Forum* 2002; 65 (2): 206-214.
- [7] Ganyushkin O, Mazorchuk V. *Classical Finite Transformation Semigroups*. London, UK: Springer-Verlag, 2009.
- [8] Howie JM. *Fundamentals of Semigroup Theory*. New York, NY, USA: Oxford University Press, 1995.

- [9] Redmond SP. The zero-divisor graph of a non-commutative ring. *International Journal of Commutative Rings* 2002; 1 (4): 203-211.
- [10] Thulasiraman K, Arumugam S, Brandstädt A, Nishizeki T. *Handbook of Graph Theory, Combinatorial Optimization, and Algorithms*. Boca Raton, CRC Press, 2015.
- [11] Toker K. On the zero-divisor graphs of finite free semilattices. *Turkish Journal of Mathematics* 2016; 40 (4): 824-831.
- [12] Toker K. Zero-divisor graphs of Catalan monoid. *Hacettepe Journal of Mathematics and Statistics* 2021; 50 (2): 387-396.