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Conformal generic submersions

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Abstract: Akyol and Şahin (2017) introduced the notion of conformal semiinvariant submersions from almost Hermitian manifolds. The present paper deals with the study of conformal generic submersions from almost Hermitian manifolds which extends semiinvariant Riemannian submersions, generic Riemannian submersions and conformal semiinvariant submersions in a natural way. We mention some examples of such maps and obtain characterizations and investigate some properties, including the integrability of distributions, the geometry of foliations and totally geodesic foliations. Moreover, we obtain some conditions for such submersions to be totally geodesic and harmonic, respectively.

Key words: Kähler manifold, Riemannian submersion, generic Riemannian submersion, conformal submersion, conformal generic submersion, vertical distribution

1. Introduction

Let $\tilde{M}$ be an almost Hermitian manifold with almost complex structure J and M a Riemannian manifold isometrically immersed in $\tilde{M}$. We note that submanifolds of a Kähler manifold are determined by the behavior of tangent bundle of the submanifold under the action of the almost complex structure of the ambient manifold. A submanifold M is called holomorphic(complex) if $J(T_qM) \subset T_qM$, for every $q \in M$, where T_qM denotes the tangent space to M at the point q. M is called totally real if $J(T_qM) \subset T_q^{\perp}M$, for every $q \in M$, where $T_q^{\perp}M$ denotes the normal space to M at the point q. As a generalization of holomorphic and totally real submanifolds, $CR-$submanifolds were introduced by Bejancu [10]. A $CR-$submanifold M of an almost Hermitian manifold $\tilde{M}$ with an almost complex structure J admits two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ defined on M such that $\mathcal{D}$ is invariant under J and $\mathcal{D}^{\perp}$ is totally real [10]. There is yet another generalization of $CR-$submanifolds known as generic submanifolds [12]. These submanifolds are defined by relaxing the condition on the complementary distribution of holomorphic distribution. Let M be a real submanifold of an almost Hermitian manifold $\tilde{M}$, and let $\mathcal{D}_q = T_qM \cap JT_qM$ be the maximal holomorphic subspace of T_qM. If $\mathcal{D} : q \longrightarrow \mathcal{D}_q$ defines a smooth holomorphic distribution on M, then M is called a generic submanifold of $\tilde{M}$. The complementary distribution $\mathcal{D}'$ of $\mathcal{D}$ is called purely real distribution on M. A generic submanifold is a $CR-$submanifold if the purely real distribution on M is totally real. A purely real distribution $\mathcal{D}'$ on a generic submanifold M is called proper if it is not totally real. A generic submanifold is called proper if its purely real distribution is proper. Generic submanifolds have been studied widely by many authors and the theory of such submanifolds is still an active research area, see [14–16, 24, 25, 39] for recent papers on this topic.

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The notion of Riemannian submersions between Riemannian manifolds was studied by O'Neill [27] and Gray [21]. Later on, such submersions have been studied widely in differential geometry. Riemannian submersions between Riemannian manifolds equipped with an additional structure of almost complex type were firstly studied by Watson [42]. Watson defined an almost Hermitian submersion between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases. We note that almost Hermitian submersions have been extended to the almost contact manifolds [13], locally conformal Kähler manifolds [26], quaternionic Kähler manifolds [23], paraquaternionic manifolds [11], [40] and statistical manifolds [41].

Recently, Şahin [35] introduced the notion of semiinvariant Riemannian submersions as a generalization of antiinvariant Riemannian submersions [34] from almost Hermitian manifolds onto Riemannian manifolds. Later such submersions and their extensions are studied [4, 5, 28–30, 37, 38].

As a generalization of semiinvariant submersions, Ali and Fatima [1] introduced the notion of generic Riemannian submersions in the sense of Chen (see also [2]). For the notion of generic Riemannian submersions, there are also two notions which are given by Yano and Kon [44], Ronsse [32] in literature. (See also [33]).

On the other hand, a related topic of growing interest deals with the study of the so-called horizontally conformal submersions: these maps, which provide a natural generalization of Riemannian submersion, were introduced independently Fuglede [18] and Ishihara [22]. As a generalization of holomorphic submersions, the notion of conformal holomorphic submersions were defined by Gudmundsson and Wood [20]. In 2017, Akyol and Şahin [7] defined a conformal semiinvariant submersion from an almost Hermitian manifolds onto a Riemannian manifold (for conformal submersions see also [3, 6, 8]). In the present paper, we introduce the notion of conformal generic submersions in the sense of Chen as a generalization of semiinvariant submersions, generic Riemannian submersions and conformal semiinvariant submersions and investigate the geometry of the total space, the base space and the fibres for the existence of such submersions.

The present article is organized as follows: In Section 2, we give some background about conformal submersions and the second fundamental maps. In Section 3, we define and study conformal generic submersions from almost Hermitian manifolds onto Riemannian manifolds, give examples and investigate the geometry of leaves of the horizontal distribution and the vertical distribution. In this section, we also show that there are certain product structures on the total space of a conformal generic submersion. In the last section of this paper, we find necessary and sufficient conditions for a conformal generic submersion to be totally geodesic and harmonic, respectively.

2. Preliminaries

The manifolds, maps, vector fields etc. considered in this paper are assumed to be smooth, i.e. differentiable of class C^∞.

2.1. Conformal submersions

Let $\psi : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds, and let $p \in M_1$. Then ψ is called horizontally weakly conformal or semiconformal at p [9] if either (i) $d\psi_p = 0$, or (ii) $d\psi_p$ is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$g_2(d\psi_p\xi, d\psi_p\eta) = \Lambda(p)g_1(\xi, \eta) \ (\xi, \eta \in \mathcal{H}_p),$$

where $\mathcal{H}_p$ is horizontal space in total space. We call the point p as a critical point if it satisfies the type (i), and we shall call the point p a regular point if it satisfied the type (ii). At a critical point, $d\psi_p$ has rank 0; at a regular point, $d\psi_p$ has rank n and ψ is submersion. Further, the positive number $\Lambda(p)$ is called the square dilation (of ψ at p). The map ψ is called horizontally weakly conformal or semiconformal (on M_1) if it is horizontally weakly conformal at every point of M_1 and if it has no critical point, then we call it a horizontally conformal submersion.

A vector field $\xi_1 \in \Gamma(TM_1)$ is called a basic vector field if $\xi_1 \in \Gamma((\ker d\psi)^{\perp})$ and $\psi-$related with a vector field $\bar{\xi_1} \in \Gamma(TM_2)$ which means that $(d\psi_p \xi_{1p}) = \bar{\xi_1}_{\psi(p)} \in \Gamma(TM_2)$ for any $p \in M_1$.

Define $\mathcal{T}$ and $\mathcal{A}$, which are O'Neill's tensors, as follows:

$$\mathcal{A}_{E_1} E_2 = \mathcal{V} \overset{M}{\nabla}_{\mathcal{H}E_1} \mathcal{H}E_2 + \mathcal{H} \overset{M}{\nabla}_{\mathcal{H}E_1} \mathcal{V}E_2 \tag{2.1}$$

$$\mathcal{T}_{E_1} E_2 = \mathcal{H} \overset{M}{\nabla}_{\mathcal{V}E_1} \mathcal{V}E_2 + \mathcal{V} \overset{M}{\nabla}_{\mathcal{V}E_1} \mathcal{H}E_2 \tag{2.2}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [19]). On the other hand, from (2.1) and (2.2), we have

$$\overset{M}{\nabla}_V W = \mathcal{T}_V W + \hat{\nabla}_V W \tag{2.3}$$

$$\overset{M}{\nabla}_V \xi = \mathcal{H} \overset{M}{\nabla}_V \xi + \mathcal{T}_V \xi \tag{2.4}$$

$$\overset{M}{\nabla}_\xi V = \mathcal{A}_\xi V + \mathcal{V} \overset{M}{\nabla}_\xi V \tag{2.5}$$

$$\overset{M}{\nabla}_\xi \eta = \mathcal{H} \overset{M}{\nabla}_\xi \eta + \mathcal{A}_\xi \eta \tag{2.6}$$

for $\xi, \eta \in \Gamma((\ker d\psi)^{\perp})$ and $V, W \in \Gamma(\ker d\psi)$, where $\overset{M}{\nabla}$ is the Levi–Civita connection on M_1 and $\hat{\nabla}_V W = \mathcal{V} \overset{M}{\nabla}_V W$. If ξ is basic, then $\mathcal{H} \overset{M}{\nabla}_V \xi = \mathcal{A}_\xi V$.

It is easily seen that for $q \in M_1$, $\xi \in \mathcal{H}_q$ and $V \in \mathcal{V}_q$ the linear operators $\mathcal{T}_V, \mathcal{A}_\xi : T_q M_1 \to T_q M_1$ are skew-symmetric, that is

$$-g_1(\mathcal{T}_V E_1, E_2) = g_1(E_1, \mathcal{T}_V E_2) \text{ and } -g_1(\mathcal{A}_\xi E_1, E_2) = g_1(E_1, \mathcal{A}_\xi E_2)$$

for all $E_1, E_2 \in T_q M_1$. We also see that the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}\mid_{\ker d\psi \times \ker d\psi}$ is exactly the second fundamental form of the fibres of ψ. Since $\mathcal{T}_V$ is skew-symmetric we get: ψ has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

Let (M_1, g_1) and (M_1, g_2) be Riemannian manifolds and suppose that $\psi : M_1 \to M_2$ is a smooth map between them. Then the differential $d\psi$ of ψ can be viewed a section of the bundle $Hom(TM_1, \psi^{-1}TM_2) \to M_1$, where $\psi^{-1}TM_2$ is the pullback bundle which has fibres $(\psi^{-1}TM_2)_p = T_{\psi(p)}M_2$, $p \in M_1$. $Hom(TM_1, \psi^{-1}TM_2)$ has a connection ∇ induced from the Levi–Civita connection ∇^{M_1} and the pullback connection. Then the second fundamental form of ψ is given by

$$(\nabla d\psi)(\xi, \eta) = \nabla^{\psi}_\xi d\psi(\eta) - d\psi(\overset{M}{\nabla}_\xi \eta) \tag{2.7}$$

for $\xi, \eta \in \Gamma(TM_1)$, where ∇^{ψ} is the pullback connection. It is known that the second fundamental form is symmetric.

Lemma 2.1 *[43] Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\psi : M \longrightarrow N$ is a smooth map between them. Then we have*

$$\nabla^{\psi}_{\xi} \mathrm{d}\psi(\eta) - \nabla^{\psi}_{\eta} \mathrm{d}\psi(\xi) - \mathrm{d}\psi([\xi, \eta]) = 0 \tag{2.8}$$

for $\xi, \eta \in \Gamma(TM)$.

Finally, we have the following from [9]:

Lemma 2.2 *(Second fundamental form of an HC submersion) Suppose that $\psi : M_1 \to M_2$ is a horizontally conformal submersion. Then, for any horizontal vector fields ξ, η and vertical vector fields V, W, we have*

(i) $(\nabla \mathrm{d}\psi)(\xi, \eta) = \xi(\ln \lambda)\mathrm{d}\psi\eta + \eta(\ln \lambda)\mathrm{d}\psi\xi - g(\xi, \eta)\mathrm{d}\psi(\nabla \ln \lambda)$;

(ii) $(\nabla \mathrm{d}\psi)(V, W) = -\mathrm{d}\psi(\mathcal{T}_V W)$;

(iii) $(\nabla \mathrm{d}\psi)(\xi, V) = -\mathrm{d}\psi(\nabla^1_{\xi} V) = -\mathrm{d}\psi(\mathcal{A}_{\xi} V)$.

3. Conformal generic submersions from almost Hermitian manifolds

In this section, we define conformal generic submersions from an almost Hermitian manifold onto a Riemannian manifold, give lots of examples and investigate the geometry of leaves of distributions and show that there are certain product structures on the total space of a conformal generic submersion.

Let (M_1, g_1, J) be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g such that [45]:

$$(i)\ J^2 = -I, \qquad (ii)\ g(Z_1, Z_2) = g(JZ_1, JZ_2), \tag{3.1}$$

for all vector fields Z_1, Z_2 on M_1, where I is the identity map. An almost Hermitian manifold M_1 is called Kähler manifold if the almost complex structure J satisfies

$$(\nabla_{Z_1} J)Z_2 = 0,\ \forall Z_1, Z_2 \in \Gamma(TM_1), \tag{3.2}$$

where ∇ denotes the Levi–Civita connection on M_1.

First of all, we recall the definition of generic Riemannian submersions as follows:

Definition 3.1 *[1] Let N_1 be a complex m-dimensional almost Hermitian manifold with Hermitian metric h_1 and almost complex structure J_1 and N_2 be a Riemannian manifold with Riemannian metric h_2. A Riemannian submersion $\psi : N_1 \longrightarrow N_2$ is called generic Riemannian submersion if there is a distribution $\bar{\mathcal{D}}_1 \subseteq \ker d\psi$ such that*

$$ker d\psi = \bar{\mathcal{D}}_1 \oplus \bar{\mathcal{D}}_2 \quad J(\bar{\mathcal{D}}_1) = \bar{\mathcal{D}}_1,$$

where $\bar{\mathcal{D}}_2$ is orthogonal complementary to $\bar{\mathcal{D}}_1$ in $(ker d\psi)$, and is purely real distribution on the fibres of the submersion ψ.

Now, we will give our definition as follows:

Let φ be a conformal submersion from an almost Hermitian manifold (M, g, J) to a Riemannian manifold (B, h). Define

$$\mathcal{D}_q = (\mathrm{kerd}\varphi \cap J(\mathrm{kerd}\varphi)), \quad q \in M$$

the complex subspace of the vertical subspace $\mathcal{V}_q$.

Definition 3.2 *Let $\varphi : (M, g, J) \longrightarrow (B, h)$ be a horizontally conformal submersion, where (M, g, J) is an almost Hermitian manifold and (B, h) is a Riemannian manifold with Riemannian metric h. If the dimension $\mathcal{D}_q$ is constant along M and it defines a differential distribution on M then we say that φ is conformal generic submersion. A conformal generic submersion is purely real (respectively, complex) if $\mathcal{D}_q = \{0\}$ (respectively, $\mathcal{D}_q = \mathrm{kerd}\varphi_q$). For a conformal generic submersion, the orthogonal complementary distribution $\mathcal{D}^{'}$, called purely real distribution, satisfies*

$$\mathrm{kerd}\varphi = \mathcal{D} \oplus \mathcal{D}^{'}, \tag{3.3}$$

and

$$\mathcal{D} \cap \mathcal{D}^{'} = \{0\}. \tag{3.4}$$

Remark 3.3 *It is known that the distribution* $\mathrm{kerd}\varphi$ *is integrable. Hence, above definition implies that the integral manifold (fiber) $\varphi^{-1}(q)$, $q \in B$, of kerdφ is a generic submanifold of M. For generic submanifolds, see:[12].*

First of all, we give lots of examples for conformal generic submersions from almost Hermitian manifolds to Riemannian manifolds.

Example 3.4 *Every semiinvariant Riemannian submersion [35] φ from an almost Hermitian manifold to a Riemannian manifold is a conformal generic submersion with $\lambda = 1$ and $\mathcal{D}^{'}$ is a totally real distribution.*

Example 3.5 *Every conformal semiinvariant submersion [7] φ from an almost Hermitian manifold to a Riemannian manifold is a conformal generic submersion such that $\mathcal{D}^{'}$ is a totally real distribution.*

Example 3.6 *Every generic Riemannian submersion [1] φ from an almost Hermitian manifold to a Riemannian manifold is a conformal generic submersion with $\lambda = 1$.*

Remark 3.7 *We would like to point out that since conformal semiinvariant submersions include conformal holomorphic submersions and conformal antiinvariant submersions, such conformal submersions are also examples of conformal generic submersions. We say that a conformal generic submersion is proper if $\lambda \neq 1$ and $\mathcal{D}^{'}$ is neither complex nor purely real.*

In the following $\mathbb{R}^{2m}$ denotes the Euclidean $2m$-space with the standard metric. Define the compatible almost complex structure J on $\mathbb{R}^8$ by

$$J\partial_1 = \frac{1}{\sqrt{2}}(-\partial_3 - \partial_2), J\partial_2 = \frac{1}{\sqrt{2}}(-\partial_4 + \partial_1), J\partial_3 = \frac{1}{\sqrt{2}}(\partial_1 + \partial_4), J\partial_4 = \frac{1}{\sqrt{2}}(\partial_2 - \partial_3),$$

$$J\partial_5 = \frac{1}{\sqrt{2}}(-\partial_7 - \partial_6), J\partial_6 = \frac{1}{\sqrt{2}}(-\partial_8 + \partial_5), J\partial_7 = \frac{1}{\sqrt{2}}(\partial_5 + \partial_8), J\partial_8 = \frac{1}{\sqrt{2}}(\partial_6 - \partial_7),$$

where $\partial_k = \frac{\partial}{\partial u_k}$, $k = 1, ..., 8$ and $(u_1, ..., u_8)$ natural coordinates of $\mathbb{R}^8$.

Example 3.8 *Let* $\varphi : (\mathbb{R}^8, g) \longrightarrow (\mathbb{R}^2, h)$ *be a submersion defined by*

$$\varphi(u_1, u_2, ..., u_8) = (t_1, t_2),$$

where

$$t_1 = e^{u_1} \sin u_3 \quad and \quad t_2 = e^{u_1} \cos u_3.$$

Then, the Jacobian matrix of φ *is:*

$$d\varphi = \begin{bmatrix} e^{u_1} \sin u_3 & 0 & e^{u_1} \cos u_3 & 0 & 0 & 0 & 0 & 0 \\ e^{u_1} \cos u_3 & 0 & -e^{u_1} \sin u_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since, the rank of this matrix is equaled to 2, the map φ *is a submersion. A straight computation yields*

$$ker d\varphi = span\{T_1 = \partial_2,\ T_2 = \partial_4,\ T_3 = \partial_5,\ T_4 = \partial_6,\ T_5 = \partial_7,\ T_6 = \partial_8\}$$

and

$$(ker d\varphi)^{\perp} = span\{H_1 = e^{u_1} \sin u_3 \partial_1 + e^{u_1} \cos u_3 \partial_3, H_2 = e^{u_1} \cos u_3 \partial_1 - e^{u_1} \sin u_3 \partial_3\}.$$

Hence, we get

$$JT_3 = -\frac{1}{\sqrt{2}}(T_4 + T_5),\ JT_4 = \frac{1}{\sqrt{2}}(T_3 - T_6),$$

$$JT_5 = \frac{1}{\sqrt{2}}(T_3 + T_6),\ JT_6 = \frac{1}{\sqrt{2}}(T_4 - T_5)$$

and

$$JT_1 = \frac{1}{\sqrt{2}} T_2 - \frac{e^{-u_1} \sin u_3}{\sqrt{2}} H_1 - \frac{e^{-u_1} \cos u_3}{\sqrt{2}} H_2,$$

$$JT_2 = -\frac{1}{\sqrt{2}} T_1 + \frac{e^{-u_1} \cos u_3}{\sqrt{2}} H_1 - \frac{e^{-u_1} \sin u_3}{\sqrt{2}} H_2,$$

where J *is the complex structure of* $\mathbb{R}^8$*. It follows that* $\mathcal{D} = span\{T_3, T_4, T_5, T_6\}$ *and* $\mathcal{D}' = span\{T_1, T_2\}$*. Also by direct computations yields*

$$d\varphi(H_1) = (e^{u_1})^2 \partial v_1 \quad and \quad d\varphi(H_2) = (e^{u_1})^2 \partial v_2.$$

Hence, it is easy to see that

$$g_{\mathbb{R}^2}(d\varphi(H_i), d\varphi(H_i)) = (e^{u_1})^2 g_{\mathbb{R}^8}(H_i, H_i), \quad i = 1, 2.$$

Thus φ *is a conformal generic submersion with* $\lambda = e^{u_1}$*.*

Example 3.9 *Let* $\psi : (\mathbb{R}^8, g_1) \longrightarrow (\mathbb{R}^2, g_2)$ *be a submersion defined by*

$$\psi(v_1, v_2, ..., v_8) = \pi^{17}(\frac{-v_1 + v_3}{\sqrt{2}}, \frac{-v_1 - v_3}{\sqrt{2}}).$$

Then ψ *is a conformal generic submersion with* $\lambda = \pi^{17}$*.*

Remark 3.10 *Throughout this paper, we assume that all horizontal vector fields are basic vector fields.*

Let φ be a conformal generic submersion from a Kähler manifold (M,g,J) onto a Riemannian manifold (B,h). Then for $Z\in\Gamma(\ker d\varphi)$, we write

$$JZ=\psi Z+\omega Z \tag{3.5}$$

where $\psi Z\in\Gamma(\ker d\varphi)$ and $\omega Z\in\Gamma((\ker d\varphi)^{\perp})$. We denote the orthogonal complement of $\omega\mathcal{D}^{'}$ in $(\ker d\varphi)^{\perp}$ by μ. Then we have

$$(\ker d\varphi)^{\perp}=\omega\mathcal{D}^{'}\oplus\mu \tag{3.6}$$

and that μ is invariant under J. Also for $\xi\in\Gamma((\ker d\varphi)^{\perp})$, we write

$$J\xi=\mathcal{B}\xi+\mathcal{C}\xi \tag{3.7}$$

where $\mathcal{B}\xi\in\Gamma(\mathcal{D}^{'})$ and $\mathcal{C}\xi\in\Gamma(\mu)$. From (3.5), (3.6) and (3.7), we have the following result.

Proposition 3.11 *Let φ be a conformal generic submersion from a Kähler manifold (M,g,J) onto a Riemannian manifold (B,h). Then we have*

$$(i)\ \psi\mathcal{D}=\mathcal{D},\quad (ii)\ \omega\mathcal{D}=0,\quad (iii)\ \psi\mathcal{D}^{'}\subset\mathcal{D}^{'},\quad (iv)\ \mathcal{B}((\ker d\varphi)^{\perp})=\mathcal{D}^{'},$$
$$(a)\ \psi^2+\mathcal{B}\omega=-id,\ (b)\ \mathcal{C}^2+\omega\mathcal{B}=-id,\ (c)\ \omega\psi+\mathcal{C}\omega=0,\ (d)\ \mathcal{B}\mathcal{C}=0.$$

Next, we easily have the following lemma:

Lemma 3.12 *Let (M,g,J) be a Kähler manifold and (B,h) a Riemannian manifold. Let $\varphi:(M,g,J)\to(B,h)$ be a conformal generic submersion. Then we have*

(i)

$$\mathcal{C}\mathcal{H}\overset{M}{\nabla}_{\xi}\eta+\omega\mathcal{A}_{\xi}\eta=\mathcal{A}_{\xi}\mathcal{B}\eta+\mathcal{H}\overset{M}{\nabla}_{\xi}\mathcal{C}\eta$$
$$\mathcal{B}\mathcal{H}\overset{M}{\nabla}_{\xi}\eta+\psi\mathcal{A}_{\xi}\eta=\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta+\mathcal{A}_{\xi}\mathcal{C}\eta,$$

(ii)

$$\mathcal{C}\mathcal{T}_ZW+\omega\hat{\nabla}_ZW=\mathcal{T}_U\psi W+\mathcal{A}_{\omega W}Z$$
$$\mathcal{B}\mathcal{T}_ZW+\psi\hat{\nabla}_ZW=\hat{\nabla}_Z\psi W+\mathcal{T}_Z\omega W,$$

(iii)

$$\mathcal{C}\mathcal{A}_{\xi}Z+\omega\mathcal{V}\overset{M}{\nabla}_{\xi}Z=\mathcal{A}_{\xi}\psi Z+\mathcal{H}\overset{M}{\nabla}_{\xi}\omega Z$$
$$\mathcal{B}\mathcal{A}_{\xi}Z+\psi\mathcal{V}\overset{M}{\nabla}_{\xi}Z=\mathcal{V}\overset{M}{\nabla}_{\xi}\psi Z+\mathcal{A}_{\xi}\omega Z,$$

for $\xi,\eta\in\Gamma((\ker d\varphi)^{\perp})$ and $Z,W\in\Gamma(\ker d\varphi)$.

3.1. The geometry of $\varphi : (M, g, J) \to (B, h)$

Lemma 3.13 *Let φ be a conformal generic submersion from a Kähler manifold (M, g, J) onto a Riemannian manifold (B, h). Then the distribution $\mathcal{D}$ is integrable if and only if the following is satisfied*

$$\lambda^{-2}\{h((\nabla \mathrm{d}\varphi)(U, JV) - (\nabla \mathrm{d}\varphi)(V, JU), \mathrm{d}\varphi(\omega Z)\} = g(\varphi(\hat{\nabla}_V JU - \hat{\nabla}_U JV), Z) \tag{3.8}$$

for $U, V \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^{'})$.

Proof The distribution $\mathcal{D}$ is integrable if and only if

$$g([U, V], Z) = 0, \quad \text{and} \quad g([U, V], \xi) = 0$$

for any $U, V \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^{'})$ and $\xi \in \Gamma((\mathrm{ker}\mathrm{d}\varphi)^{\perp})$. Since $\mathrm{ker}\mathrm{d}\varphi$ is integrable $g([U, V], \xi) = 0$. Therefore, $\mathcal{D}$ is integrable if and only if $g([U, V], Z) = 0$. By Eqs. (3.1)(i), (3.2), (2.3), and (3.5) we have

$$g([U, V], Z) = g(\mathcal{H}\overset{M}{\nabla}_U JV, \omega Z) + g(\hat{\nabla}_U JV, \varphi Z) - g(\mathcal{H}\overset{M}{\nabla}_V JU, \omega Z) - g(\hat{\nabla}_V JU, \varphi Z).$$

By using the property of φ, Eq. (3.5) and Lemma 2.2 yield

$$\begin{aligned} g([U, V], Z) &= \lambda^{-2} h\big(- (\nabla \mathrm{d}\varphi)(U, JV) + \nabla^{\varphi}_U \mathrm{d}\varphi(JV), \mathrm{d}\varphi(\omega Z)\big) - \lambda^{-2} h\big(- (\nabla \mathrm{d}\varphi)(V, JU) \\ &\quad + \nabla^{\varphi}_V \mathrm{d}\varphi(JU), \mathrm{d}\varphi(\omega Z)\big) + g(\varphi(\hat{\nabla}_V JU - \hat{\nabla}_U JV), Z) \\ &= \lambda^{-2}\{h((\nabla \mathrm{d}\varphi)(V, JU) - (\nabla \mathrm{d}\varphi)(U, JV), \mathrm{d}\varphi(\omega Z))\} + g(\varphi(\hat{\nabla}_V JU - \hat{\nabla}_U JV), Z) \end{aligned}$$

which gives Eq. (3.8).

In a similar way, we get: □

Lemma 3.14 *Let φ be a conformal generic submersion from a Kähler manifold (M, g, J) onto a Riemannian manifold (B, h). Then the distribution $\mathcal{D}^{'}$ is integrable if and only if*

$$\hat{\nabla}_V \psi U - \hat{\nabla}_U \psi V + \mathcal{T}_V \omega U - \mathcal{T}_U \omega V \in \Gamma(\mathcal{D}^{'}) \tag{3.9}$$

for $U, V \in \Gamma(\mathcal{D}^{'})$.

We now investigate the geometry of leaves of $\mathcal{D}$ and $\mathcal{D}^{'}$.

Lemma 3.15 *Let φ be a conformal generic submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (B, h). Then $\mathcal{D}$ defines a totally geodesic foliation on M if and only if*

(a) $\lambda^{-2} h((\nabla \mathrm{d}\varphi)(X_1, JY_1), \mathrm{d}\varphi(\omega X_2)) = g(\hat{\nabla}_{X_1} JY_1, \psi X_2)$

(b) $\lambda^{-2} h((\nabla \mathrm{d}\varphi)(X_1, JY_1), \mathrm{d}\varphi(\mathcal{C}\xi)) = g(\hat{\nabla}_{X_1} \psi \mathcal{B}\xi + \mathcal{T}_{X_1} \omega \mathcal{B}\xi, Y_1)$

for $X_1, Y_1 \in \Gamma(\mathcal{D}), X_2 \in \Gamma(\mathcal{D}^{'})$ and $\xi \in \Gamma((\mathrm{ker}\mathrm{d}\varphi)^{\perp})$.

Proof The distribution $\mathcal{D}$ defines a totally geodesic foliation on M if and only if

$$g(\overset{M}{\nabla}_{X_1} Y_1, X_2) = 0 \text{ and } g(\overset{M}{\nabla}_{X_1} Y_1, \xi) = 0$$

for any $X_1, Y_1 \in \Gamma(\mathcal{D})$, $X_2 \in \Gamma(\mathcal{D}')$ and $\xi \in \Gamma((\ker d\varphi)^\perp)$. By virtue of Eqs. (3.1)(i) and (3.1)(ii), we get

$$g(\overset{M}{\nabla}_{X_1} Y_1, X_2) = g(\hat{\nabla}_{X_1} JY_1, \psi X_2) + g(\mathcal{H}\overset{M}{\nabla}_{X_1} JY_1, \omega X_2).$$

Since φ is a conformal generic submersion, by Eq. (2.7) yields

$$g(\overset{M}{\nabla}_{X_1} Y_1, X_2) = g(\hat{\nabla}_{X_1} JY_1, \psi X_2) - \lambda^{-2} h((\nabla d\varphi)(X_1, JY_1), d\varphi(\omega X_2)). \tag{3.10}$$

On the other hand, by Eqs. (3.1)(i), (3.1)(ii), (2.3), and (3.7) yields

$$g(\overset{M}{\nabla}_{X_1} Y_1, \xi) = g(Y_1, \overset{M}{\nabla}_{X_1} J\mathcal{B}\xi) + g(\mathcal{H}\overset{M}{\nabla}_{X_1} JY_1, \mathcal{C}\xi).$$

By Eqs. (2.4), (2.7), and (3.5) we get

$$\begin{aligned} g(\overset{M}{\nabla}_{X_1} Y_1, \xi) &= g(Y_1, \hat{\nabla}_{X_1} \psi\mathcal{B}\xi) + g(Y_1, \mathcal{T}_{X_1} \omega\mathcal{B}\xi) \\ &\quad - \lambda^{-2} h((\nabla d\varphi)(X_1, JY_1), d\varphi(\mathcal{C}\xi)). \end{aligned} \tag{3.11}$$

Hence proof follows from Eqs. (3.10) and (3.11). □

In a similar way, we have the following result.

Lemma 3.16 *Let φ be a conformal generic submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (B, h). Then $\mathcal{D}'$ defines a totally geodesic foliation on M if and only if*

(a) $\lambda^{-2} h((\nabla d\varphi)(X_2, JX_1), d\varphi(\omega Y_2)) = g(\hat{\nabla}_{X_2} JX_1, \psi Y_2)$,

(b) $\lambda^{-2} h\big((\nabla d\varphi)(X_2, Y_2), d\varphi(JC\xi)\big) = g(\mathcal{T}_{X_2} \mathcal{B}\xi, \omega Y_2) - g(\hat{\nabla}_{X_2} \psi Y_2, \mathcal{B}\xi)$

for $X_1, Y_1 \in \Gamma(\mathcal{D}), X_2, Y_2 \in \Gamma(\mathcal{D}')$ and $\xi \in \Gamma((\ker d\varphi)^\perp)$.

From Lemma 3.15 and Lemma 3.16, we have the following result.

Lemma 3.17 *Let $\varphi : (M, g, J) \longrightarrow (B, h)$ be a conformal generic submersion from a Kähler manifold (M, g, J) onto a Riemannian manifold (B, h). Then the fibers of φ are locally product manifolds of the form $M_{\mathcal{D}} \times M_{\mathcal{D}'}$ if and only if*

(i) $\lambda^{-2} h((\nabla d\varphi)(X_1, JY_1), d\varphi(\omega X_2)) = g(\hat{\nabla}_{X_1} JY_1, \psi X_2)$

(ii) $\lambda^{-2} h((\nabla d\varphi)(X_1, JY_1), d\varphi(\mathcal{C}\xi)) = g(\hat{\nabla}_{X_1} \psi\mathcal{B}\xi + \mathcal{T}_{X_1} \omega\mathcal{B}\xi, Y_1)$

(iii) $\lambda^{-2} h((\nabla d\varphi)(X_2, JX_1), d\varphi(\omega Y_2)) = g(\hat{\nabla}_{X_2} JX_1, \psi Y_2)$,

(iv) $\lambda^{-2} h\big((\nabla d\varphi)(X_2, Y_2), d\varphi(JC\xi)\big) = g(\mathcal{T}_{X_2} \mathcal{B}\xi, \omega Y_2) - g(\hat{\nabla}_{X_2} \psi Y_2, \mathcal{B}\xi)$

for any $X_1, Y_1 \in \Gamma(\mathcal{D}), X_2, Y_2 \in \Gamma(\mathcal{D}^{'})$ *and* $\xi \in \Gamma((\mathrm{kerd}\varphi)^{\perp})$.

Since the distribution $\mathrm{kerd}\varphi$ is integrable, we only study the integrability of the distribution $(\mathrm{kerd}\varphi)^{\perp}$ and then we discuss the geometry of leaves of $\mathrm{kerd}\varphi$ and $(\mathrm{kerd}\varphi)^{\perp}$.

Theorem 3.18 *Let* φ *be a conformal generic submersion from a Kähler manifold* (M, g, J) *to a Riemannian manifold* (B, h). *Then the distribution* $(\mathrm{kerd}\varphi)^{\perp}$ *is integrale if and only if*

$$\mathcal{V}(\overset{M}{\nabla}_{\xi}\mathcal{B}\eta - \overset{M}{\nabla}_{\eta}\mathcal{B}\xi) + \mathcal{A}_{\xi}\mathcal{C}\eta - \mathcal{A}_{\eta}\mathcal{C}\xi \in \Gamma(\mathcal{D}^{'})$$

and

$$\begin{aligned}&\lambda^{-2}h\big((\nabla \mathrm{d}\varphi)(\xi, \mathcal{B}\eta) - (\nabla \mathrm{d}\varphi)(\eta, \mathcal{B}\xi) - \nabla^{\varphi}_{\xi}\mathrm{d}\varphi(\mathcal{C}\eta) + \nabla^{\varphi}_{\eta}\mathrm{d}\varphi(\mathcal{C}\xi), \mathrm{d}\varphi(\omega W)\big)\\&= g\big(\eta(\ln\lambda)\mathcal{C}\xi - \xi(\ln\lambda)\mathcal{C}\eta - \mathcal{C}\eta(\ln\lambda)\xi + \mathcal{C}\xi(\ln\lambda)\eta + 2g(\xi, \mathcal{C}\eta)\nabla\ln\lambda, \omega W\big)\\&+ g(-\varphi(\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta - \mathcal{V}\overset{M}{\nabla}_{\eta}\mathcal{B}\xi + \mathcal{A}_{\xi}\mathcal{C}\eta - \mathcal{A}_{\eta}\mathcal{C}\xi), W)\end{aligned}$$

for $\xi, \eta \in \Gamma(\mathrm{kerd}\varphi)^{\perp})$, $V \in \Gamma(\mathcal{D})$ *and* $W \in \Gamma(\mathcal{D}^{'})$.

Proof By virtue of (3.1)(i) and (3.1)(ii), we get

$$g([\xi, \eta], JV) = g(-J[\xi, \eta], V) = -g(J\overset{M}{\nabla}_{\xi}J\eta, JV) + g(J\overset{M}{\nabla}_{\eta}J\xi, JV)$$

for $\xi, \eta \in \Gamma((\mathrm{kerd}\varphi)^{\perp})$ and $V \in \Gamma(\mathcal{D})$. Then by using (3.7), (3.5), (2.5), and (2.6) yields

$$g([\xi, \eta], JV) = -g(\psi(\mathcal{V}(\overset{M}{\nabla}_{\xi}\mathcal{B}\eta - \overset{M}{\nabla}_{\eta}\mathcal{B}\xi) + \mathcal{A}_{\xi}\mathcal{C}\eta - \mathcal{A}_{\eta}\mathcal{C}\xi), JV)$$

So that

$$g([\xi, \eta], JV) = 0 \Longleftrightarrow \mathcal{V}(\overset{M}{\nabla}_{\xi}\mathcal{B}\eta - \overset{M}{\nabla}_{\eta}\mathcal{B}\xi) + \mathcal{A}_{\xi}\mathcal{C}\eta - \mathcal{A}_{\eta}\mathcal{C}\xi \in \Gamma(\mathcal{D}^{'}). \tag{3.12}$$

Also using (2.5), (2.6), and (3.7) we get

$$\begin{aligned}g([\xi, \eta], W) &= g(\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta - \mathcal{V}\overset{M}{\nabla}_{\eta}\mathcal{B}\xi + \mathcal{A}_{\xi}\mathcal{C}\eta - \mathcal{A}_{\eta}\mathcal{C}\xi, \psi W) + g(\mathcal{H}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta, \omega W)\\&- g(\mathcal{H}\overset{M}{\nabla}_{\eta}\mathcal{B}\xi, \omega W) + g_1(\mathcal{H}\overset{M}{\nabla}_{\xi}\mathcal{C}\eta, \omega W) - g(\mathcal{H}\overset{M}{\nabla}_{\eta}\mathcal{C}\xi, \omega W).\end{aligned}$$

Taking into account (2.7) and Lemma 2.2, we get

$$\begin{aligned}g([\xi,\eta], W) &= g(\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta - \mathcal{V}\overset{M}{\nabla}_{\eta}\mathcal{B}\xi + \mathcal{A}_{\xi}\mathcal{C}\eta - \mathcal{A}_{\eta}\mathcal{C}\xi, \psi W)\\&-\lambda^{-2}h((\nabla \mathrm{d}\varphi)(\xi, \mathcal{B}\eta), \mathrm{d}\varphi(\omega W)) + \lambda^{-2}h((\nabla \mathrm{d}\varphi(\eta, \mathcal{B}\xi), \mathrm{d}\varphi(\omega W))\\&+\lambda^{-2}h\{-\xi(\ln\lambda)\mathrm{d}\varphi(\mathcal{C}\eta) - \mathcal{C}\eta(\ln\lambda)\mathrm{d}\varphi(\xi) + g(\xi, \mathcal{C}\eta)\mathrm{d}\varphi(\nabla\ln\lambda) + \nabla^{\varphi}_{\xi}\mathrm{d}\varphi(\mathcal{C}\eta), \mathrm{d}\varphi(\omega W)\}\\&- \lambda^{-2}h\{-\eta(\ln\lambda)\mathrm{d}\varphi(\mathcal{C}\xi) - \mathcal{C}\xi(\ln\lambda)\mathrm{d}\varphi(\eta) + g(\eta, \mathcal{C}\xi)\mathrm{d}\varphi(\nabla\ln\lambda) + \nabla^{\varphi}_{\eta}\mathrm{d}\varphi(\mathcal{C}\xi), \mathrm{d}\varphi(\omega W)\}\end{aligned}$$

by virtue of (2.5) and (2.7)

$$\begin{aligned}g([\xi,\eta],W) = {} & g(\mathcal{V}\overset{M}{\nabla}_\xi \mathcal{B}\eta - \mathcal{V}\overset{M}{\nabla}_\eta \mathcal{B}\xi + \mathcal{A}_\xi\mathcal{C}\eta - \mathcal{A}_\eta\mathcal{C}\xi, \psi W)\\
& - \lambda^{-2}h\big((\nabla \mathrm{d}\varphi)(\xi,\mathcal{B}\eta) - (\nabla \mathrm{d}\varphi)(\eta,\mathcal{B}\xi), \mathrm{d}\varphi(\omega W)\big)\\
& - \lambda^{-2}g(\nabla \ln\lambda,\xi)h(\mathrm{d}\varphi(\mathcal{C}\eta),\mathrm{d}\varphi(\omega W)) - \lambda^{-2}g(\nabla\ln\lambda,\mathcal{C}\eta)h(\mathrm{d}\varphi(\xi),\mathrm{d}\varphi(\omega W))\\
& + \lambda^{-2}g(\xi,\mathcal{C}\eta)h(\mathrm{d}\varphi(\nabla\ln\lambda),\mathrm{d}\varphi(\omega W)) + \lambda^{-2}h(\nabla^\varphi_\xi \mathrm{d}\varphi(\mathcal{C}\eta),\mathrm{d}\varphi(\omega W))\\
& + \lambda^{-2}g(\nabla \ln\lambda,\eta)h(\mathrm{d}\varphi(\mathcal{C}\xi),\mathrm{d}\varphi(\omega W)) + \lambda^{-2}g(\nabla\ln\lambda,\mathcal{C}\xi)h(\mathrm{d}\varphi(\eta),\mathrm{d}\varphi(\omega W))\\
& - \lambda^{-2}g(\eta,\mathcal{C}\xi)h(\mathrm{d}\varphi(\nabla\ln\lambda),\mathrm{d}\varphi(\omega W)) - \lambda^{-2}h(\nabla^\varphi_\eta \mathrm{d}\varphi(\mathcal{C}\xi),\mathrm{d}\varphi(\omega W)).\end{aligned}$$

A straight computation yields

$$\begin{aligned}g([\xi,\eta],W) = {} & g\big(\eta(\ln\lambda)\mathcal{C}\xi - \xi(\ln\lambda)\mathcal{C}\eta - \mathcal{C}\eta(\ln\lambda)\xi + \mathcal{C}\xi(\ln\lambda)\eta + 2g(\xi,\mathcal{C}\eta)\nabla\ln\lambda, \omega W\big)\\
& + g(-\varphi(\mathcal{V}\overset{M}{\nabla}_\xi \mathcal{B}\eta - \mathcal{V}\overset{M}{\nabla}_\eta \mathcal{B}\xi + \mathcal{A}_\xi\mathcal{C}\eta - \mathcal{A}_\eta\mathcal{C}\xi), W)\\
& - \lambda^{-2}h\big((\nabla \mathrm{d}\varphi)(\xi,\mathcal{B}\eta) - (\nabla \mathrm{d}\varphi)(\eta,\mathcal{B}\xi) - \nabla^\varphi_\xi \mathrm{d}\varphi(\mathcal{C}\eta) + \nabla^\varphi_\eta \mathrm{d}\varphi(\mathcal{C}\xi), \mathrm{d}\varphi(\omega W)\big)\end{aligned}$$

so that

$$\begin{aligned}g([\xi,\eta],W) = 0 \Longleftrightarrow {} & \lambda^{-2}h\big((\nabla \mathrm{d}\varphi)(\xi,\mathcal{B}\eta) - (\nabla \mathrm{d}\varphi)(\eta,\mathcal{B}\xi)\\
& - \nabla^\varphi_\xi \mathrm{d}\varphi(\mathcal{C}\eta) + \nabla^\varphi_\eta \mathrm{d}\varphi(\mathcal{C}\xi), \mathrm{d}\varphi(\omega W)\big)\\
& = g\big(\eta(\ln\lambda)\mathcal{C}\xi - \xi(\ln\lambda)\mathcal{C}\eta - \mathcal{C}\eta(\ln\lambda)\xi + \mathcal{C}\xi(\ln\lambda)\eta + 2g(\xi,\mathcal{C}\eta)\nabla\ln\lambda, \omega W\big)\\
& + g(-\psi(\mathcal{V}\overset{M}{\nabla}_\xi \mathcal{B}\eta - \mathcal{V}\overset{M}{\nabla}_\eta \mathcal{B}\xi + \mathcal{A}_\xi\mathcal{C}\eta - \mathcal{A}_\eta\mathcal{C}\xi), W).\end{aligned} \tag{3.13}$$

The proof follows from (3.12) and (3.13). □

From Theorem 3.18, we deduce.

Theorem 3.19 *Let φ be a conformal generic submersion from a Kähler manifold (M,g,J) to a Riemannian manifold (B,h) with integrable distribution $(\ker \mathrm{d}\varphi)^\perp$. If φ is a horizontally homothetic map then we have*

$$\begin{aligned}& \lambda^{-2}h\big((\nabla \mathrm{d}\varphi)(\xi,\mathcal{B}\eta) - (\nabla \mathrm{d}\varphi)(\eta,\mathcal{B}\xi) - \nabla^\varphi_\xi \mathrm{d}\varphi(\mathcal{C}\eta) + \nabla^\varphi_\eta \mathrm{d}\varphi(\mathcal{C}\xi), \mathrm{d}\varphi(\omega W)\big)\\
& = g(-\psi(\mathcal{V}\overset{M}{\nabla}_\xi \mathcal{B}\eta - \mathcal{V}\overset{M}{\nabla}_\eta \mathcal{B}\xi + \mathcal{A}_\xi\mathcal{C}\eta - \mathcal{A}_\eta\mathcal{C}\xi), W)\end{aligned} \tag{3.14}$$

for $\xi,\eta \in \Gamma((\ker \mathrm{d}\varphi)^\perp)$ and $W \in \Gamma(\mathcal{D}')$.

For the geometry of leaves of the horizontal distribution, we have the following theorem.

Theorem 3.20 *Let φ be a conformal generic submersion from a Kähler manifold (M,g,J) to a Riemannian manifold (B,h). Then the horizontal distribution defines a totally geodesic foliation on M if and only if*

$$\lambda^{-2}h((\nabla \mathrm{d}\varphi)(\xi,JV_1),\mathrm{d}\varphi(\eta)) = g(\eta,\mathcal{V}\overset{M}{\nabla}_\xi JV_1)$$

and

$$\lambda^{-2}h(\nabla^{\varphi}_{\xi}\mathrm{d}\varphi(\omega V_2),\mathrm{d}\varphi(\mathcal{C}\eta)) = -g(\psi(\mathcal{A}_{\xi}\mathcal{C}\eta+\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta),V_2)$$
$$+\, g(\mathcal{A}_{\xi}\mathcal{B}\eta-\xi(\ln\lambda)\mathcal{C}\eta-\mathcal{C}\eta(\ln\lambda)\xi+g(\xi,\mathcal{C}\eta)\nabla\ln\lambda,\omega V_2)$$

for $\xi,\eta\in\Gamma((\ker\mathrm{d}\varphi)^{\perp}),\quad V_1\in\Gamma(\mathcal{D})$ *and* $V_2\in\Gamma(\mathcal{D}')$.

Proof Given $\xi,\eta\in\Gamma((\ker\mathrm{d}\varphi)^{\perp})$ and $JV_1\in\Gamma(\mathcal{D})$, by virtue of (3.1)(ii), (2.5), (2.6), (3.5) and (3.7) we obtain

$$g(\overset{M}{\nabla}_{\xi}\eta,JV_1) = -g(\eta,\mathcal{H}\overset{M}{\nabla}_{\xi}JV_1+\mathcal{V}\overset{M}{\nabla}_{\xi}JV_1)$$
$$= \lambda^{-2}h((\nabla\mathrm{d}\varphi)(\xi,JV_1),\mathrm{d}\varphi(\eta)) - g(\eta,\mathcal{V}\overset{M}{\nabla}_{\xi}JV_1)$$

so that

$$g(\overset{M}{\nabla}_{\xi}\eta,JV_1)=0 \Longleftrightarrow \lambda^{-2}h((\nabla\mathrm{d}\varphi)(\xi,JV_1),\mathrm{d}\varphi(\eta)) = g(\eta,\mathcal{V}\overset{M}{\nabla}_{\xi}JV_1). \tag{3.15}$$

Given $V_2\in\Gamma(\mathcal{D}')$, by using (3.1)(ii), (2.5), (2.6), (3.5) and (3.7) we get

$$g(\overset{M}{\nabla}_{\xi}\eta,V_2) = -g(\psi(\mathcal{A}_{\xi}\mathcal{C}\eta+\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta),V_2) - g(\mathcal{B}\eta,\overset{M}{\nabla}_{\xi}\omega V_2) + g(\overset{M}{\nabla}_{\xi}\mathcal{C}\eta,\omega V_2)$$
$$= -g(\psi(\mathcal{A}_{\xi}\mathcal{C}\eta+\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta),V_2) - g(\mathcal{B}\eta,\mathcal{A}_{\xi}\omega V_2)$$
$$-\,\lambda^{-2}g(\nabla\ln\lambda,\xi)h(\mathrm{d}\varphi(\omega V_2),\mathrm{d}\varphi(\mathcal{C}\eta)) - \lambda^{-2}g(\nabla\ln\lambda,\mathcal{C}\eta)h(\mathrm{d}\varphi(\xi),\mathrm{d}\varphi(\omega V_2))$$
$$+\, g(\xi,\mathcal{C}\eta)\lambda^{-2}h(\mathrm{d}\varphi(\nabla\ln\lambda),\mathrm{d}\varphi(\omega V_2)) + \lambda^{-2}h(\nabla^{\varphi}_{\xi}\mathrm{d}\varphi(\mathcal{C}\eta),\mathrm{d}\varphi(\omega V_2))$$

so that

$$g(\overset{M}{\nabla}_{\xi}\eta,V_2) = g(\mathcal{A}_{\xi}\mathcal{B}\eta-\xi(\ln\lambda)\mathcal{C}\eta-\mathcal{C}\eta(\ln\lambda)\xi+g(\xi,\mathcal{C}\eta)\nabla\ln\lambda,\omega V_2) \tag{3.16}$$
$$-\, g(\psi(\mathcal{A}_{\xi}\mathcal{C}\eta+\mathcal{V}\overset{M_1}{\nabla}_{\xi}\mathcal{B}\eta),V_2) - \lambda^{-2}h(\nabla^{\varphi}_{\xi}\varphi(\omega V_2),\mathrm{d}\varphi(\mathcal{C}\eta)).$$

The proof follows (3.15) and (3.16). □

From Theorem 3.20, we immediately deduce.

Theorem 3.21 *Let* φ *be a conformal generic submersion from a Kähler manifold* (M,g,J) *to a Riemannian manifold* (B,h) *with a totally geodesic foliation* $(\ker\mathrm{d}\varphi)^{\perp}$. *If* φ *is a horizontally homothetic map. then we have*

$$\lambda^{-2}h(\nabla^{\varphi}_{\xi}\mathrm{d}\varphi(\omega V_2),\mathrm{d}\varphi(\mathcal{C}\eta)) = g(\mathcal{A}_{\xi}\mathcal{B}\eta,\omega V_2) - g(\psi(\mathcal{A}_{\xi}\mathcal{C}\eta+\mathcal{V}\overset{M}{\nabla}_{\xi}\mathcal{B}\eta),V_2) \tag{3.17}$$

for any $\xi,\eta\in\Gamma((\ker\mathrm{d}\varphi)^{\perp})$ *and* $V_2\in\Gamma(\mathcal{D}')$.

Proof Since $(ker\varphi_*)^{\perp}$ defines a totally geodesic foliation on M_1, from (3.16) we have

$$g(\overset{M}{\nabla}_{\xi}\eta,V_2) = g(\mathcal{A}_{\xi}\mathcal{B}\eta-\xi(\ln\lambda)\mathcal{C}\eta-\mathcal{C}\eta(\ln\lambda)\xi+g(\xi,\mathcal{C}\eta)\nabla\ln\lambda,\omega V_2)$$
$$-\, g(\psi(\mathcal{A}_{\xi}\mathcal{C}\eta+\mathcal{V}\overset{M_1}{\nabla}_{\xi}\mathcal{B}\eta),V_2) - \lambda^{-2}h(\nabla^{\varphi}_{\xi}\varphi(\omega V_2),\mathrm{d}\varphi(\mathcal{C}\eta))$$

for any $\xi,\eta\in\Gamma((\ker\mathrm{d}\varphi)^{\perp})$ and $V_2\in\Gamma(\ker\mathrm{d}\varphi)$. Now, one can easily see that if λ is a constant on $(\ker\mathrm{d}\varphi)^{\perp}$, we obtain (3.17.) □

In the sequel we are going to investigate the geometry of leaves of the distribution $\ker\mathrm{d}\varphi$.

Theorem 3.22 *Let φ be a conformal generic submersion from a Kähler manifold (M,g,J) to a Riemannian manifold (B,h). Then the vertical distribution defines a totally geodesic foliation on M if and only if*

$$\lambda^{-2}h((\nabla \mathrm{d}\varphi)(U,\omega V),\mathrm{d}\varphi(\mathcal{C}\xi)) = g(\hat{\nabla}_U\psi V,\mathcal{B}\xi) - g(\omega V,\mathcal{T}_U\mathcal{B}\xi) + g(\mathcal{T}_U\varphi V,\mathcal{C}\xi)$$

for any $U,V \in \Gamma(\mathrm{ker d}\varphi)$ and $\xi \in \Gamma(\mathrm{ker d}\varphi)^{\perp}$.

Proof It is clear that the vertical distribution defines a totally geodesic foliation if and only if $g(\overset{M}{\nabla}_U V,\xi) = 0$ for any $U,V \in \Gamma(\mathrm{ker d}\varphi)$ and $\xi \in \Gamma(\mathrm{ker d}\varphi)^{\perp}$. By using (3.5), (3.7) and (2.3), we get

$$g(\overset{M}{\nabla}_U V,\xi) = g(\hat{\nabla}_U\psi V,\mathcal{B}\xi) + g(\mathcal{T}_U\varphi V,\mathcal{C}\xi) - g(\omega V,\mathcal{T}_U\mathcal{B}\xi) + g(\overset{M}{\nabla}_U\omega V,\mathcal{C}\xi)$$

for any $U,V \in \Gamma(\mathrm{ker d}\varphi)$ and $\xi \in \Gamma(\mathrm{ker d}\varphi)^{\perp}$. Since φ is a conformal submersion, by using (2.7), we have

$$g(\overset{M}{\nabla}_U V,\xi) = g(\hat{\nabla}_U\psi V,\mathcal{B}\xi) + g(\mathcal{T}_U\varphi V,\mathcal{C}\xi) - g(\omega V,\mathcal{T}_U\mathcal{B}\xi) + \lambda^{-2}h(\mathrm{d}\varphi(\overset{M}{\nabla}_U\omega V),\mathrm{d}\varphi\mathcal{C}\xi)$$

which tells that

$$g(\overset{M}{\nabla}_U V,\xi) = g(\hat{\nabla}_U\psi V,\mathcal{B}\xi) + g(\mathcal{T}_U\varphi V,\mathcal{C}\xi) - g(\omega V,\mathcal{T}_U\mathcal{B}\xi) - \lambda^{-2}h((\nabla \mathrm{d}\varphi)(U,\omega V),\mathrm{d}\varphi(\mathcal{C}\xi))$$

which proves the assertion. □

From Theorems 3.20 and 3.22, we have the following result.

Theorem 3.23 *Let $\varphi : (M,g,J) \longrightarrow (B,h)$ be a conformal generic submersion from a Kähler manifold (M,g,J) onto a Riemannian manifold (B,h). Then the total space M is a generic product manifold of the leaves of $\mathrm{ker d}\varphi$ and $(\mathrm{ker d}\varphi)^{\perp}$, i.e. $M = M_{\mathrm{kerd}\varphi} \times M_{(\mathrm{kerd}\varphi)^{\perp}}$, if and only if*

$$\lambda^{-2}h((\nabla \mathrm{d}\varphi)(\xi,JV_1),\mathrm{d}\varphi(\eta)) = g(\eta,\mathcal{V}\overset{M}{\nabla}_\xi JV_1),$$

$$\begin{aligned}\lambda^{-2}h(\nabla^{\varphi}_\xi \mathrm{d}\varphi(\omega V_2),\mathrm{d}\varphi(\mathcal{C}\eta)) &= -g(\psi(\mathcal{A}_\xi\mathcal{C}\eta + \mathcal{V}\overset{M}{\nabla}_\xi\mathcal{B}\eta),V_2)\\ &+ g(\mathcal{A}_\xi\mathcal{B}\eta - \xi(\ln\lambda)\mathcal{C}\eta - \mathcal{C}\eta(\ln\lambda)\xi + g(\xi,\mathcal{C}\eta)\nabla\ln\lambda,\omega V_2)\end{aligned}$$

and

$$\lambda^{-2}h((\nabla \mathrm{d}\varphi)(U,\omega V),\mathrm{d}\varphi(\mathcal{C}\xi)) = g(\hat{\nabla}_U\psi V,\mathcal{B}\xi) - g(\omega V,\mathcal{T}_U\mathcal{B}\xi) + g(\mathcal{T}_U\varphi V,\mathcal{C}\xi)$$

for any $\xi,\eta \in \Gamma((\mathrm{ker d}\varphi)^{\perp})$, $U,V \in \Gamma(\mathrm{ker d}\varphi)$, $V_1 \in \Gamma(\mathcal{D})$ and $V_2 \in \Gamma(\mathcal{D}')$, where $M_{\mathrm{kerd}\varphi}$ and $M_{(\mathrm{kerd}\varphi)^{\perp}}$ are leaves of the distributions $\mathrm{ker d}\varphi$ and $(\mathrm{ker d}\varphi)^{\perp}$, respectively.

4. Totally geodesicity and harmonicity of conformal generic submersions

In this section, we investigate the necessary and sufficient conditions for such submersions to be totally geodesicity and harmonicity, respectively. We first give the following definition.

4.1. Totally geodesicity of $\varphi : (M, g, J) \longrightarrow (B, h)$

Definition 4.1 *Let φ be a conformal generic submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (B, h). Then φ is called a $(\omega\mathcal{D}', \mu)$-totally geodesic map if*

$$(\nabla \mathrm{d}\varphi)(\omega Z, \xi) = 0, \textit{for } Z \in \Gamma(\mathcal{D}') \textit{ and } \xi \in \Gamma(\mu).$$

The following result shows that the above definition has an important effect on the character of the conformal generic submersion.

Theorem 4.2 *Let φ be a conformal generic submersion from a Kähler manifold (M, g, J) to a Riemannian manifold (B, h). Then the following conditions are equivalent:*

(i) φ is a horizontally homothetic map.

(ii) φ is a $(\omega\mathcal{D}', \mu)$-totally geodesic map.

Proof Given $Z \in \Gamma(\mathcal{D}')$ and $\xi \in \Gamma(\mu)$, by Lemma 2.2, we have

$$\begin{aligned}(\nabla \mathrm{d}\varphi)(\omega Z, \xi) &= \omega Z(\ln\lambda)\mathrm{d}\varphi(\xi) + \xi(\ln\lambda)\mathrm{d}\varphi(\omega Z) - g(\omega Z, \xi)\mathrm{d}\varphi(\nabla \ln\lambda). \\ &= \omega Z(\ln\lambda)\mathrm{d}\varphi(\xi) + \xi(\ln\lambda)\mathrm{d}\varphi(\omega Z).\end{aligned}$$

From above equation, we easily get $(i) \Longrightarrow (ii)$. Conversely, if $(\nabla \mathrm{d}\varphi)(\omega Z, \xi) = 0$, we get

$$\omega Z(\ln\lambda)\mathrm{d}\varphi(\xi) + \xi(\ln\lambda)\mathrm{d}\varphi(\omega Z) = 0. \tag{4.1}$$

From above equation, since $\{\mathrm{d}\varphi(\xi), \mathrm{d}\varphi(\omega Z)\}$ is linearly independent for nonzero ξ, Z we have $\omega Z(\ln\lambda) = 0$ and $\xi(\ln\lambda) = 0$. It means that λ is a constant on $\Gamma(\mathcal{D}')$ and $\Gamma(\mu)$, which gives that $(i) \Longleftarrow (ii)$. This completes the proof of the theorem. □

We also have the following result.

Theorem 4.3 *Let $\varphi : (M, g, J) \longrightarrow (B, h)$ is a conformal generic submersion, where (M, g, J) is a Kähler manifold and (B, h) is a Riemannian manifold. φ is a totally geodesic map if and only if the following four conditions are satisfied:*

(i) $\mathcal{C}\mathcal{T}_U JV + \omega\hat{\nabla}_U JV = 0$ for $U, V \in \Gamma(\mathcal{D})$.

(ii) $\mathcal{T}_U \psi Z + \mathcal{A}_{\omega Z} U \in \Gamma(\omega\mathcal{D}')$ and $\hat{\nabla}_U \psi Z + \mathcal{T}_U \omega Z \in \Gamma(\mathcal{D})$, for $U \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\ker \mathrm{d}\varphi)$.

(iii) $\mathcal{C}(\mathcal{T}_W \psi Z + \mathcal{A}_{\omega Z} W) + \omega(\hat{\nabla}_W \psi Z + \mathcal{T}_W \omega Z) = 0$, for $W, Z \in \Gamma(\mathcal{D}')$.

(iv) $\mathcal{C}(\mathcal{T}_V \mathcal{B}\xi + \mathcal{H}\overset{M}{\nabla}_V \mathcal{C}\xi) + \omega(\hat{\nabla}_V \mathcal{B}\xi + \mathcal{T}_V \mathcal{C}\xi) = 0$ for $V \in \Gamma(\ker \mathrm{d}\varphi)$, $\xi \in \Gamma((\ker \mathrm{d}\varphi)^{\perp})$

(v) φ is a horizontally homotetic map.

Proof In view of Eqs. (3.1)(ii) and (2.7) we have

$$(\nabla \mathrm{d}\varphi)(U,V) = \mathrm{d}\varphi(J\overset{M}{\nabla}_U JV)$$

for any $U,V \in \Gamma(\mathcal{D})$. Then from Eq. (2.3) we arrive at

$$(\nabla \mathrm{d}\varphi)(U,V) = \mathrm{d}\varphi(J(\mathcal{T}_U JV + \hat{\nabla}_U JV)).$$

Using Eqs. (3.5) and (3.7) in above equation we obtain

$$(\nabla \mathrm{d}\varphi)(U,V) = \mathrm{d}\varphi(\mathcal{B}\mathcal{T}_U JV + \mathcal{C}\mathcal{T}_U JV + \psi\hat{\nabla}_U JV + \omega\hat{\nabla}_U JV).$$

So

$$(\nabla \mathrm{d}\varphi)(U,V) = 0 \Longleftrightarrow \mathcal{C}\mathcal{T}_U JV + \omega\hat{\nabla}_U JV = 0. \tag{4.2}$$

Given $U \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\ker \mathrm{d}\varphi)$, by Eqs. (3.1)(ii) and (2.7) we have

$$(\nabla \mathrm{d}\varphi)(U,Z) = \mathrm{d}\varphi(J\overset{M}{\nabla}_U JZ).$$

By Eqs. (2.3), (2.4), and (3.5) yields

$$(\nabla \mathrm{d}\varphi)(U,Z) = \mathrm{d}\varphi(J(\mathcal{T}_U\psi Z + \hat{\nabla}_U\psi Z + \mathcal{T}_U\omega Z + \mathcal{A}_{\omega Z}U)).$$

where we have used $\mathcal{H}\overset{M}{\nabla}_U\omega Z = \mathcal{A}_{\omega Z}U$. By using Eqs. (3.5) and (3.7) in above equation we obtain

$$\begin{aligned}(\nabla \mathrm{d}\varphi)(U,Z) = \mathrm{d}\varphi(&\mathcal{B}\mathcal{T}_U\psi Z + \mathcal{C}\mathcal{T}_U\psi Z + \psi\hat{\nabla}_U\psi Z + \omega\hat{\nabla}_U\psi Z \\ &+ \psi\mathcal{T}_U\omega Z + \omega\mathcal{T}_U\omega Z + \mathcal{B}\mathcal{A}_{\omega Z}U + \mathcal{C}\mathcal{A}_{\omega Z}U).\end{aligned}$$

So

$$(\nabla \mathrm{d}\varphi)(U,Z) = 0 \Longleftrightarrow \mathcal{C}(\mathcal{T}_U\psi Z + \mathcal{A}_{\omega Z}U) + \omega(\hat{\nabla}_U\psi Z + \mathcal{T}_U\omega Z) = 0 \tag{4.3}$$

which completes the proof of (ii).

Given $W,Z \in \Gamma(\mathcal{D}^{'})$, by Eqs. (3.1)(ii) and (2.7) we have

$$(\nabla \mathrm{d}\varphi)(W,Z) = \mathrm{d}\varphi(J\overset{M}{\nabla}_W JZ).$$

By Eqs. (2.3), (2.4), and (3.5) yields

$$(\nabla \mathrm{d}\varphi)(W,Z) = \mathrm{d}\varphi(J(\mathcal{T}_W\psi Z + \hat{\nabla}_W\psi Z + \mathcal{T}_W\omega Z + \mathcal{A}_{\omega Z}W)).$$

where we have used $\mathcal{H}\overset{M}{\nabla}_W\omega Z = \mathcal{A}_{\omega Z}W$. Taking into account of Eqs. (3.5) and (3.7) in above equation we obtain

$$\begin{aligned}(\nabla \mathrm{d}\varphi)(W,Z) = \mathrm{d}\varphi(&\mathcal{B}\mathcal{T}_W\psi Z + \mathcal{C}\mathcal{T}_W\psi Z + \psi\hat{\nabla}_W\psi Z + \omega\hat{\nabla}_W\psi Z \\ &+ \psi\mathcal{T}_W\omega Z + \omega\mathcal{T}_W\omega Z + \mathcal{B}\mathcal{A}_{\omega Z}W + \mathcal{C}\mathcal{A}_{\omega Z}W).\end{aligned}$$

So

$$(\nabla \mathrm{d}\varphi)(W,Z) = 0 \Longleftrightarrow \mathcal{C}(\mathcal{T}_W\psi Z + \mathcal{A}_{\omega Z}W) + \omega(\hat{\nabla}_W\psi Z + \mathcal{T}_W\omega Z) = 0 \tag{4.4}$$

Given $V \in \Gamma(\ker d\varphi),\ \xi \in \Gamma((\ker d\varphi)^{\perp})$, by Eqs. (3.1)(ii), (2.7), (2.3), (2.4), and (3.5) yields

$$
\begin{aligned}
(\nabla d\varphi)(V,\xi) &= d\varphi(J\overset{M}{\nabla}_V J\xi).\\
&= d\varphi(J(\overset{M}{\nabla}_V \mathcal{B}\xi + \overset{M}{\nabla}_V \mathcal{C}\xi)).\\
&= d\varphi(J(\mathcal{T}_V\mathcal{B}\xi + \hat{\nabla}_V\mathcal{B}\xi + \mathcal{T}_V\mathcal{C}\xi + \mathcal{H}\overset{M}{\nabla}_V\mathcal{C}\xi)).\\
&= d\varphi(\mathcal{C}(\mathcal{T}_V\mathcal{B}\xi + \mathcal{H}\overset{M}{\nabla}_V\mathcal{C}\xi) + \omega(\hat{\nabla}_V\mathcal{B}\xi + \mathcal{T}_V\mathcal{C}\xi)).
\end{aligned}
$$

So

$$
(\nabla d\varphi)(V,\xi) = 0 \Longleftrightarrow \mathcal{C}(\mathcal{T}_V\mathcal{B}\xi + \mathcal{H}\overset{M}{\nabla}_V\mathcal{C}\xi) + \omega(\hat{\nabla}_V\mathcal{B}\xi + \mathcal{T}_V\mathcal{C}\xi) = 0. \tag{4.5}
$$

Now, we will show that for any $\xi,\eta \in \Gamma((\ker d\varphi)^{\perp})$, $(\nabla d\varphi)(\xi,\eta) = 0 \Longleftrightarrow \varphi$ is a horizontally homothetic map. Given $\xi,\eta \in \Gamma(\mu)$, from Lemma 2.2, we have

$$
(\nabla d\varphi)(\xi,\eta) = \xi(\ln\lambda)d\varphi(\eta) + \eta(\ln\lambda)d\varphi(\xi) - g(\xi,\eta)d\varphi(\nabla\ln\lambda).
$$

Taking $\eta = J\xi,\ \xi \in \Gamma(\mu)$ in the above equation we get

$$
\begin{aligned}
(\nabla d\varphi)(\xi,J\xi) &= \xi(\ln\lambda)d\varphi(J\xi) + J\xi(\ln\lambda)d\varphi(\xi) - g(\xi,J\xi)d\varphi(\nabla\ln\lambda)\\
&= \xi(\ln\lambda)d\varphi(J\xi) + J\xi(\ln\lambda)d\varphi\xi.
\end{aligned}
$$

If $(\nabla d\varphi)(\xi,J\xi) = 0$, we get

$$
\xi(\ln\lambda)d\varphi(J\xi) + J\xi(\ln\lambda)d\varphi\xi = 0. \tag{4.6}
$$

Taking inner product in Eq. (4.6) with $d\varphi(\xi)$ and taking into account φ is a conformal submersion, we have

$$
g(\nabla\ln\lambda,\xi)h(d\varphi J\xi,d\varphi\xi) + g(\nabla\ln\lambda,J\xi)h(d\varphi\xi,d\varphi\xi) = 0.
$$

which implies that λ is a constant on $\Gamma(J\mu)$. On the other hand, taking inner product in Eq. (4.6) with $d\varphi(J\xi)$ we have

$$
g(\nabla\ln\lambda,\xi)h(d\varphi J\xi,d\varphi J\xi) + g_1(\nabla\ln\lambda,\xi)h(d\varphi\xi,d\varphi J\xi) = 0.
$$

which tells that λ is a constant on $\Gamma(\mu)$. In a similar way, for $U,V \in \Gamma(\mathcal{D}')$, by using Lemma 2.2 we have

$$
(\nabla d\varphi)(\omega U,\omega V) = \omega U(\ln\lambda)d\varphi(\omega V) + \omega V(\ln\lambda)d\varphi(\omega U) - g(\omega U,\omega V)d\varphi(\nabla\ln\lambda).
$$

From above equation, taking $V = U$ we obtain

$$
(\nabla d\varphi)(\omega U,\omega U) = 2\omega U(\ln\lambda)d\varphi(\omega U) - g(\omega U,\omega U)d\varphi(\nabla\ln\lambda). \tag{4.7}
$$

Taking inner product in Eq. (4.7) with $d\varphi(\omega U)$ and taking into account φ is a conformal submersion, we derive

$$
2g(\nabla\ln\lambda,\omega U)h(d\varphi(\omega U),d\varphi(\omega U)) - g(\omega U,\omega U)h(d\varphi(\nabla\ln\lambda),d\varphi(\omega U)) = 0
$$

which tells that λ is a constant on $\Gamma(\omega\mathcal{D}')$. Thus λ is a constant on $\Gamma((\ker d\varphi)^{\perp})$. By Eqs. (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7), we derive that φ is a totally geodesic map if and only if the relations (i)–(v) hold. This completes the proof of the theorem. □

4.2. Harmonicity of $\varphi : (M, g, J) \longrightarrow (B, h)$

Let $\varphi : N_1 \longrightarrow N_2$ be a C^∞ map between two Riemannian manifolds. We can naturally define a function $e(\varphi) = N_1 \longrightarrow [0, \infty]$ given by

$$e(\varphi)(x) = \frac{1}{2}|(d\varphi)_x|^2, x \in N_2$$

where $|(d\varphi)_x|$ denotes the Hilbert–Schmidt norm of $(d\varphi)_x$. We call $e(\varphi)$ the energy density of φ. Let Ω is the compact closure $\bar{U}$ of a nonempty connected open subset U of N_1. The energy integral of φ over Ω is the integral of its energy density:

$$E(\varphi; \Omega) = \int_\Omega e(\varphi) v_{g_N} = \int_\Omega \frac{1}{2}|(d\varphi)_x|^2 v_{g_N}$$

where v_{g_N} is the volume form on (N, g_N). Let $C^\infty(N_1, N_2)$ denote the space of all differentiable map from N_1 on N_2. A differentiable map $\varphi : N_1 \longrightarrow N_2$ is said to harmonic if it is a critical point of the energy functional $E(\varphi; \Omega) : C^\infty(N_1, N_2) \longrightarrow \mathbb{R}$ for any compact domain $\Omega \subset N_1$. By the result of Eells and Sampson [17], we know that the map φ is harmonic if and only if the tension field

$$\tau(\varphi) = trace(\nabla d\varphi) = 0.$$

Theorem 4.4 *Let $\varphi : (M, g, J) \longrightarrow (B, h)$ be a conformal generic submersion, where (M, g, J) is a Kähler manifold and (B, h) is a Riemannian manifold. Then φ is harmonic if and only if*

$$trace|_{(\mathcal{D})} d\varphi\Big(\mathcal{C}\mathcal{T}_{J(.)}(.) + \omega\hat{\nabla}_{J(.)}(.)\Big) - trace|_{(\mathcal{D}')}\ d\varphi\Big(\mathcal{C}\mathcal{T}_{(.)}\psi(.) + \omega\hat{\nabla}_{(.)}\psi(.) + \omega\mathcal{T}_{(.)}\omega(.) + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_{(.)}\omega(.)$$
$$+\, trace|_{(kerd\varphi)^\perp}\big(\nabla^\varphi_{(.)} d\varphi(\mathcal{C}^2(.) + \omega\mathcal{B}(.)) - d\varphi(\mathcal{C}\mathcal{A}_{(.)}\mathcal{B}(.) + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_{(.)}\mathcal{C}(.) + \omega\mathcal{A}_{(.)}\mathcal{C}(.) + \omega\mathcal{V}\overset{M}{\nabla}_{(.)}\mathcal{C}(.))\Big) = 0.$$

Proof For any $U \in \Gamma(\mathcal{D})$, $V \in \Gamma(\mathcal{D}')$ and $\xi \in \Gamma((\text{ker}d\varphi)^\perp)$, by using Eqs. (3.1)(i), (2.7), (3.7), (3.5) and Proposition 3.1 (f) we have

$$(\nabla d\varphi)(JU, JU) + (\nabla d\varphi)(V, V) + (\nabla d\varphi)(\xi, \xi) = -d\varphi(J\overset{M}{\nabla}_{JU} U)$$
$$+ d\varphi(J(\overset{M}{\nabla}_V \psi V + \overset{M}{\nabla}_V \omega V)) - \nabla^\varphi_\xi d\varphi(\mathcal{C}^2\xi + \omega\mathcal{B}\xi) + d\varphi(J(\overset{M}{\nabla}_\xi \mathcal{B}\xi + \overset{M}{\nabla}_\xi \mathcal{C}\xi)).$$

With a straight computation by using Eqs. (3.7), (3.5), and (2.3)-(2.6), we obtain

$$(\nabla d\varphi)(JU, JU) + (\nabla d\varphi)(V, V) + (\nabla d\varphi)(\xi, \xi) = -d\varphi(\mathcal{C}\mathcal{T}_{JU}U + \omega\hat{\nabla}_{JU}U)$$
$$+ d\varphi(\mathcal{C}\mathcal{T}_V\psi V + \omega\hat{\nabla}_V\psi V + \omega\mathcal{T}_V\omega V + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_V \omega V)$$
$$- \nabla^\varphi_\xi d\varphi(\mathcal{C}^2\xi + \omega\mathcal{B}\xi) + d\varphi(\mathcal{C}\mathcal{A}_\xi\mathcal{B}\xi + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_\xi \mathcal{C}\xi + \omega\mathcal{A}_\xi\mathcal{C}\xi + \omega\mathcal{V}\overset{M}{\nabla}_\xi \mathcal{C}\xi).$$

Now, by taking trace on the above equation, we obtain the proof of the theorem. □

Remark 4.5 *One can easily see that the maps defined in Examples 3.8 and 3.9 are examples of harmonic maps.*

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