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## Quasi-idempotent ranks of the proper ideals in finite symmetric inverse semigroups

Leyla BUGAY\* 

Department of Mathematics, Faculty of Science, Çukurova University, Adana, Turkey

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**Abstract:** Let  $I_n$  and  $S_n$  be the symmetric inverse semigroup and the symmetric group on a finite chain  $X_n = \{1, \dots, n\}$ , respectively. Also, let  $I_{n,r} = \{\alpha \in I_n : |\text{im}(\alpha)| \leq r\}$  for  $1 \leq r \leq n-1$ . For any  $\alpha \in I_n$ , if  $\alpha \neq \alpha^2 = \alpha^4$  then  $\alpha$  is called a quasi-idempotent. In this paper, we show that the quasi-idempotent rank of  $I_{n,r}$  (both as a semigroup and as an inverse semigroup) is  $\binom{n}{2}$  if  $r = 2$ , and  $\binom{n}{r} + 1$  if  $r \geq 3$ . The quasi-idempotent rank of  $I_{n,1}$  is  $n$  (as a semigroup) and  $n-1$  (as an inverse semigroup).

**Key words:** Symmetric inverse semigroup, symmetric group, quasi-idempotent, rank

### 1. Introduction

Let  $I_n$  and  $S_n$  be the symmetric inverse semigroup and the symmetric group on a finite chain  $X_n = \{1, \dots, n\}$ , respectively. Throughout this paper, we assume that  $n \geq 2$  unless otherwise stated. It is well known that  $I_n$  is an inverse semigroup, and that every finite inverse semigroup  $S$  is embeddable in  $I_n$ . Hence, the importance of  $I_n$  in inverse semigroup theory is similar to the importance of  $S_n$  in group theory. Although the semigroup  $I_n$  has been extensively studied (see, for example, [1, 5, 10, 14]), there are still many interesting problems concerning  $I_n$  to be investigated.

For any  $\alpha \in I_n$ , for a suitable  $n$ , if  $\alpha^2 = \alpha$  then  $\alpha$  is called an idempotent; and if  $\alpha \neq \alpha^2 = \alpha^4$  then  $\alpha$  is called a quasi-idempotent. Throughout this paper we use the notation  $Q(U)$  to denote the set of all quasi-idempotents in any subset  $U$  of any semigroup.

Let  $S$  be a semigroup and let  $\emptyset \neq A \subseteq S$ . The smallest subsemigroup of  $S$  containing  $A$  is called the subsemigroup generated by  $A$  and denoted by  $\langle A \rangle$ . Clearly  $\langle A \rangle$  is the set of all finite products of elements of  $A$ . If there exists a nonempty subset  $A$  of  $S$  such that  $S = \langle A \rangle$ , then  $A$  is called a generating set of  $S$ . Also, the rank of a finitely generated semigroup  $S$  is defined by

$$\text{rank}(S) = \min\{|A| : \langle A \rangle = S\}. \quad (1.1)$$

There are studies of various ranks of semigroups, such as idempotent-rank,  $(m, r)$  rank and nilpotent rank, as well as of minimal generating sets of elements of a given kind (see, for example, [2, 3, 6, 8, 13, 17, 18]). In particular, if there exists a generating set  $A$  of  $S$  consisting entirely of quasi-idempotents, then  $A$  is called

\*Correspondence: ltanguler@cu.edu.tr

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a quasi-idempotent generating set of  $S$ , and the quasi-idempotent rank of  $S$  is defined by

$$\text{qrang}(S) = \min\{|A| : \langle A \rangle = S, A \subseteq Q(S)\}. \quad (1.2)$$

Let  $S$  be an inverse semigroup and let  $\emptyset \neq A \subseteq S$ . The smallest inverse subsemigroup of  $S$  containing  $A$  is called the subsemigroup generated by  $A$  as an inverse semigroup. To avoid confusion, as in [11], we shall use the notation  $\langle\langle A \rangle\rangle$  for this inverse subsemigroup. It is clear that  $\langle\langle A \rangle\rangle$  is the set of all finite products of elements of  $A$  and their inverses. If there exists a nonempty subset  $A$  of  $S$  such that  $S = \langle\langle A \rangle\rangle$ , then  $A$  is called a generating set of  $S$  as an inverse semigroup. Also, the inverse rank of a finitely generated semigroup  $S$  is defined by

$$\text{rank}^*(S) = \min\{|A| : \langle\langle A \rangle\rangle = S\}. \quad (1.3)$$

Similarly, if there exists a generating set  $A$  of  $S$  as an inverse semigroup such that  $A$  consists of quasi-idempotents, then  $A$  is called a quasi-idempotent generating set of  $S$  as an inverse semigroup, and if  $S$  is finitely generated, then the quasi-idempotent inverse rank of  $S$  is defined by

$$\text{qrang}^*(S) = \min\{|A| : \langle\langle A \rangle\rangle = S, A \subseteq Q(S)\}. \quad (1.4)$$

For every transformation  $\alpha \in I_n$ , we denote by  $\text{dom}(\alpha)$  and  $\text{im}(\alpha)$  the *domain* and the *image* of  $\alpha$ , respectively. Let  $a_1, \dots, a_k$ , where  $k \geq 1$ , be distinct elements of  $X_n$ . Then  $\sigma \in I_n$  is called a *cycle* of length  $k$  (or a  $k$ -cycle), and denoted by  $\sigma = (a_1 \dots a_k)$ , if  $a_i\sigma = a_{i+1}$  ( $1 \leq i \leq k-1$ ),  $a_k\sigma = a_1$ , and  $x\sigma$  is undefined if  $x \notin \{a_1, \dots, a_k\}$ . Similarly,  $\lambda \in I_n$  is called a *chain* of length  $k$  (or a  $k$ -chain), and denoted by  $\lambda = [a_1 \dots a_k]$ , if  $a_i\lambda = a_{i+1}$  ( $1 \leq i \leq k-1$ ), and  $x\lambda$  is undefined if  $x \notin \{a_1, \dots, a_{k-1}\}$ . Note that  $[a_1]$  is the zero element of  $I_n$ . We say that  $\alpha, \beta \in I_n$  are disjoint if  $(\text{dom}(\alpha) \cup \text{im}(\alpha)) \cap (\text{dom}(\beta) \cup \text{im}(\beta)) = \emptyset$ . Every nonzero  $\alpha \in I_n$  can be written uniquely (up to the order) as a join (set theoretical union) of disjoint cycles and chains [14]. We write the join of  $\alpha$  and  $\beta$  as  $\alpha\beta$  (only if  $\alpha$  and  $\beta$  are disjoint), and the product as  $\alpha \circ \beta$ .

If  $\alpha \in I_n$  with  $\text{dom}(\alpha) = A$  and  $\text{im}(\alpha) = B$ , we will write  $\alpha : A \mapsto B$ . Moreover, if  $\text{dom}(\alpha)$  is specified, we will skip 1-cycles in the notation of  $\alpha$ . With some abuse of the definition of a cycle, if  $\alpha : A \mapsto A$  and  $\alpha = (a_1 \dots a_k)$ , with  $k \geq 1$  and 1-cycles omitted (except one 1-cycle if  $k = 1$ ), we will refer to  $\alpha$  as a  $k$ -cycle on  $A$ . For example, the map  $\alpha : \{1, 2, 3, 4, 5\} \mapsto \{1, 2, 3, 4, 6\}$  defined by tabular form

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 6 & - & - & - \end{pmatrix} \in I_{8,5}$$

can be written as  $\alpha = (12)[56]$  if we know that  $\text{dom}(\alpha) = \{1, 2, 3, 4, 5\}$ .

It is well known that  $S_2 = \langle(12)\rangle$ ,  $S_3 = \langle(13), (23)\rangle = \langle(12), (123)\rangle$ , and  $S_n = \langle(12), (12 \dots n)\rangle$  for  $n \geq 4$ ;  $I_2 = \langle(12), (1)\rangle$  and  $I_n = \langle(12), (12 \dots n), (1)(2) \dots (n-1)\rangle$  for  $n \geq 3$ . Furthermore,

$$\text{rank}(S_n) = \begin{cases} 1 & \text{for } n = 2 \\ 2 & \text{for } n \geq 3 \end{cases}, \text{ and } \text{rank}(I_n) = \begin{cases} 2 & \text{for } n = 2 \\ 3 & \text{for } n \geq 3 \end{cases}. \quad (1.5)$$

Umar showed in [15] that rank and quasi-idempotent rank of  $L^-(n, n-1)$  are both equal to  $\frac{n(n+1)}{2}$ , where  $L^-(n, r)$ ,  $1 \leq r \leq n-1$  is the subsemigroup of  $I_n$  consisting of all decreasing partial one-to-one maps  $\alpha$  (including the empty or zero map) for which  $|\text{im}(\alpha)| \leq r$ . Also, Umar [16] calculated the quasi-idempotent rank of  $L^-(n, r)$  for  $1 \leq r \leq n-1$ .

For any element  $a$  of a semigroup  $S$ , the smallest left ideal of  $S$  containing  $a$  is  $Sa \cup \{a\}$ , which is denoted by  $S^1a$ . We shall call it the principal left ideal of  $S$  generated by  $a$ . Principal right ideal of  $S$  generated by  $a$ ,  $aS^1$ , can be defined similarly. An equivalence  $\mathcal{L}$  on  $S$  is defined by the rule that  $a\mathcal{L}b$  if and only if  $S^1a = S^1b$ , and an equivalence  $\mathcal{R}$  on  $S$  is defined by the rule that  $a\mathcal{R}b$  if and only if  $aS^1 = bS^1$ . It is well known that  $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$  (see, for example, [12]). Also, an equivalence  $\mathcal{D}$  on  $S$  is defined by  $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ , and an equivalence  $\mathcal{H}$  on  $S$  is defined by  $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ . An element  $a$  of a semigroup  $S$  is called regular if there exists  $x \in S$  such that  $axa = a$ . If all elements of a semigroup  $S$  are regular then  $S$  is called regular. It follows from [12, Proposition 2.3.1] that, for any  $\mathcal{D}$ -class  $D$ , either every element of  $D$  is regular or no element of  $D$  is regular. If all elements of  $D$  are regular then we call the  $\mathcal{D}$ -class  $D$  regular. For more information see, for example, [12].

Now let

$$I_{n,r} = \{\alpha \in I_n : |\text{im}(\alpha)| \leq r\} \tag{1.6}$$

for  $1 \leq r \leq n - 1$ . Clearly  $I_{n,r}$  is an ideal of  $I_n$  for each  $1 \leq r \leq n - 1$ , and notice that  $I_{n,n-1} = I_n \setminus S_n$ . Moreover, it is easy to see that Green's equivalences  $\mathcal{L}, \mathcal{R}, \mathcal{D}$ , and  $\mathcal{H}$  on  $I_{n,r}$  are characterized by

$$\begin{aligned} (\alpha, \beta) \in \mathcal{L} &\Leftrightarrow \text{im}(\alpha) = \text{im}(\beta) \\ (\alpha, \beta) \in \mathcal{R} &\Leftrightarrow \text{dom}(\alpha) = \text{dom}(\beta) \\ (\alpha, \beta) \in \mathcal{D} &\Leftrightarrow |\text{im}(\alpha)| = |\text{im}(\beta)| \\ (\alpha, \beta) \in \mathcal{H} &\Leftrightarrow \text{dom}(\alpha) = \text{dom}(\beta) \text{ and } \text{im}(\alpha) = \text{im}(\beta) \end{aligned}$$

for any  $\alpha, \beta \in I_{n,r}$ . Hence, there exist  $r + 1$   $\mathcal{D}$ -classes in  $I_{n,r}$  as follows:

$$D_k = \{\alpha \in I_{n,r} : |\text{im}(\alpha)| = k\} \quad \text{for } 0 \leq k \leq r. \tag{1.7}$$

All  $\mathcal{D}$ -classes of  $I_{n,r}$  are regular, since  $I_{n,r}$  is a regular semigroup. For  $\alpha \in I_{n,r}$  with  $\text{dom}(\alpha) = A$  and  $\text{im}(\alpha) = B$  we will denote the  $\mathcal{H}$ -class of  $\alpha$  by  $H_{A,B}$ . The rank and inverse rank of  $I_{n,r}$  have been determined. By [11, Theorem 3.7] and [7, Theorem 2.2],

$$\text{rank}^*(I_{n,r}) = \begin{cases} n - 1 & \text{for } r = 1 \\ \binom{n}{2} & \text{for } r = 2 \\ \binom{n}{r} + 1 & \text{for } 3 \leq r \leq n - 1 \end{cases}. \tag{1.8}$$

Nearly twenty years after the introduction of  $\text{rank}^*(I_{n,r})$  into the literature, Zhao and Fernandes showed in [18, Theorem 4.2] that

$$\text{rank}(I_{n,r}) = \begin{cases} n & \text{for } r = 1 \\ \binom{n}{2} & \text{for } r = 2 \\ \binom{n}{r} + 1 & \text{for } 3 \leq r \leq n - 1 \end{cases}. \tag{1.9}$$

Note that the rank of  $I_{n,r}$  coincides with its inverse rank for  $2 \leq r \leq n - 1$ . However,  $\text{rank}(I_{n,1}) = n$  while  $\text{rank}^*(I_{n,1}) = n - 1$ . In the next section, we show that  $\text{qrnk}(I_{n,r}) = \text{rank}(I_{n,r})$  and  $\text{qrnk}^*(I_{n,r}) = \text{rank}^*(I_{n,r})$  for each  $1 \leq r \leq n - 1$ .

**2. Quasi-idempotent rank of  $I_{n,r}$**

It is clear that  $\alpha \in I_n$  is a quasi-idempotent if and only if it is a join of some (perhaps none) 1-cycles and at least one 2-cycle or 2-chain.

**Lemma 2.1** *Let  $n \geq 4$  and  $3 \leq r \leq n - 1$ . Order  $1, \dots, r$  arbitrarily as  $a_1, \dots, a_r$ , and let  $a_{r+1} = r + 1$ . Consider the subsets  $A_1, \dots, A_{r+1}$  of  $\{a_1, \dots, a_{r+1}\}$  defined by  $A_i = \{a_1, \dots, a_{r+1}\} \setminus \{a_{r+2-i}\}$ ,  $1 \leq i \leq r + 1$ . Then there are quasi-idempotents  $\alpha_1, \dots, \alpha_r, \beta_1, \beta_2 \in I_n$  such that  $\alpha_i : A_i \mapsto A_{i+1}$ ,  $1 \leq i \leq r$ ,  $\beta_1, \beta_2 : A_{r+1} \mapsto A_1$ ,  $\alpha_1 \circ \dots \circ \alpha_r \circ \beta_1 = (a_1 \cdots a_r)$ , and  $\alpha_1 \circ \dots \circ \alpha_r \circ \beta_2 = (a_1 \cdots a_{r-2} a_r)$ .*

**Proof** Recall that we skip 1-cycles in  $\alpha \in I_n$  when  $\text{dom}(\alpha)$  is specified. Let  $\alpha_i = [a_{r+1-i} a_{r+2-i}]$ ,  $1 \leq i \leq r$ ,  $\beta_1 = [a_{r+1} a_1]$ , and  $\beta_2 = [a_{r+1} a_1](a_{r-1} a_r)$ . Then the result is clear.  $\square$

**Lemma 2.2** *Let  $n \geq 4$  and  $3 \leq r \leq n - 2$ . Let  $A_1 = \{1, \dots, r\}$ . Order the  $r$ -elements subsets of  $X_n$  that are not included in  $\{1, \dots, r + 1\}$ ,  $A_{r+2}, \dots, A_{m_r}$ , where  $m_r = \binom{n}{r}$ , with  $A_{m_r} = \{1, \dots, r - 1, n\}$ . Then there are quasi-idempotents  $\gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \in I_n$  such that  $\gamma : A_1 \mapsto A_{r+2}$ ,  $\alpha_i : A_i \mapsto A_{i+1}$ ,  $r + 2 \leq i \leq m_r - 1$ ,  $\delta : A_{m_r} \mapsto A_1$ , and  $\langle \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$  contains a 2-cycle, or an  $(r - 1)$ -cycle, or an  $r$ -cycle on  $A_1$ .*

**Proof** Define  $\gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}$  arbitrarily, and let  $\eta = \gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ [nr]$ , where  $\text{dom}([nr]) = A_{m_r}$ . If  $\eta$  is the identity on  $A_1$ , take  $\delta = [nr](12)$ .

Otherwise, let  $(c_1^1 \dots c_{i_1}^1), \dots, (c_1^k \dots c_{i_k}^k)$ ,  $k \geq 1$  be the cycles in  $\eta$  of length at least 2. If  $r$  occurs in any of these cycles, we may assume that  $r = c_1^1$ . Define  $\theta : A_{m_r} \mapsto A_1$  by  $\theta : [nr]$  if  $k = 1$ , and  $\theta = [nr][c_{i_1}^1 c_1^2] \cdots [c_{i_{k-1}}^{k-1} c_1^k]$ , if  $k \geq 2$ . Then  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \theta$  is an  $l$ -cycle on  $A_1$ . If  $l \in \{2, r - 1, r\}$ , take  $\delta = \theta$ .

If  $l = r - 2$ , then for  $\delta = \theta(c_{i_k}^k d)$ , where  $d$  is a fixed point of  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \theta$  with  $d \neq r$ ,  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \delta$  is an  $(r - 1)$ -cycle on  $A_1$ .

Suppose  $l \leq r - 3$  and there are distinct fixed points  $d_1, d_2, d_3$  of  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \theta$  such that  $r \notin \{d_1, d_2, d_3\}$ . If  $l$  is odd, then for  $\delta = \theta(d_1 d_2)$ , we have  $(d_1 d_2) = (\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \delta)^l \in \langle \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ . If  $l$  is even, then for  $\delta = \theta(c_{i_k}^k d_3)(d_1 d_2)$ , we have  $(d_1 d_2) = (\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \delta)^{l+1} \in \langle \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ .

Suppose  $l \leq r - 3$  and there are only two distinct fixed points  $d_1, d_2$  of  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \theta$  such that  $r \notin \{d_1, d_2\}$ . Then  $l = r - 3$  and  $r$  is a fixed point of  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \theta$ . If  $i_1 = \dots = i_k = 2$ , then taking  $\delta = [nr](c_1^1 c_2^1) \cdots (c_1^k c_2^k)(d_1 d_2)$ , we obtain  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \delta = (d_1 d_2)$ . Otherwise, we may assume that  $i_1 \geq 3$ . Then taking  $\delta = \theta(c_1^1 d_1)(c_2^1 d_2)$ , we obtain  $\gamma \circ \alpha_{r+2} \circ \dots \circ \alpha_{m_r-1} \circ \delta$  to be an  $(r - 1)$ -cycle.  $\square$

**Lemma 2.3** *Let  $S$  be a finite semigroup, and let  $D$  be a regular  $\mathcal{D}$ -class of  $S$ . Let  $G$  be a group  $\mathcal{H}$ -class of  $D$ , and let  $W$  be a subset of  $S$  that intersects each  $\mathcal{H}$ -class of  $D$ . Then  $D \subseteq \langle G, W \rangle$ .*

**Proof** This is Lemma 4.9 in [18].  $\square$

**Lemma 2.4** *Let  $n \geq 2$  and  $1 \leq r \leq n - 1$ . Then the ideal  $I_{n,r}$  is generated by its elements of rank  $r$ .*

**Proof** This is Lemma 4.7 in [18].

**Lemma 2.5** Let  $n \geq 2$  and  $1 \leq r \leq n - 1$ . Suppose  $\alpha_1, \dots, \alpha_k \in I_n$  are elements of rank  $r$  such that there exists a group  $\mathcal{H}$ -class  $G$  of  $D_r$  such that  $G \subseteq \langle \alpha_1, \dots, \alpha_k \rangle$ , and  $\langle \alpha_1, \dots, \alpha_k \rangle$  intersects each  $\mathcal{H}$ -class of  $D_r$ . Then  $\langle \alpha_1, \dots, \alpha_k \rangle = I_{n,r}$ .

**Proof** The result follows immediately from Lemmas 2.3 and 2.4. □

**Lemma 2.6** For  $n \geq 2$ , let  $A_i = \{i\}$  for  $1 \leq i \leq n$ , let  $\alpha_i : A_i \mapsto A_{i+1}$  be the quasi-idempotent  $\alpha_i = [i \ i + 1]$  for  $1 \leq i \leq n - 1$ , and let  $\alpha_n : A_n \mapsto A_1$  be the quasi-idempotent  $\alpha_n = [n \ 1]$ . Then  $I_{n,1} = \langle \alpha_1, \dots, \alpha_n \rangle$ .

**Proof** For any  $\alpha \in D_1$  there exist  $1 \leq i, j \leq n$  such that  $\text{dom}(\alpha) = \{i\}$  and  $\text{im}(\alpha) = \{j\}$ , so the result follows from the equality  $\alpha = \alpha_i \circ \dots \circ \alpha_n \circ \alpha_1 \circ \dots \circ \alpha_{j-1}$  if  $j \leq i$  and  $\alpha = \alpha_i \circ \alpha_{i+1} \circ \dots \circ \alpha_{j-1}$  if  $j > i$ . □

**Lemma 2.7** For  $n \geq 3$ , let  $A_{i,j} = \{i, j\}$  for  $1 \leq i < j \leq n$ , and let  $\alpha_{i,j} : A_{i,j} \mapsto A_{i,j+1}$  for  $1 \leq i < j \leq n - 1$ ,  $\alpha_{i,n} : A_{i,n} \mapsto A_{i+1,i+2}$  for  $1 \leq i \leq n - 2$ , and  $\alpha_{n-1,n} : A_{n-1,n} \mapsto A_{1,2}$  be the quasi-idempotents

$$\begin{aligned} \alpha_{i,j} &= [j \ j + 1] && \text{for } 1 \leq i < j \leq n - 1, \\ \alpha_{i,n} &= [i \ i + 1][n \ i + 2] && \text{for } 1 \leq i \leq n - 3, \\ \alpha_{n-2,n} &= [n - 2 \ n - 1] && \text{and} \\ \alpha_{n-1,n} &= [n - 1 \ 2][n \ 1]. \end{aligned}$$

Then  $I_{n,2} = \langle \alpha_{1,2}, \alpha_{1,3}, \dots, \alpha_{1,n}, \alpha_{2,3}, \dots, \alpha_{2,n}, \dots, \alpha_{n-1,n} \rangle$ .

**Proof** For convenience, order the sets  $A_{1,2}, A_{1,3}, \dots, A_{1,n}, A_{2,3}, \dots, A_{2,n}, \dots, A_{n-1,n}$ , respectively, by  $A_1, A_2, \dots, A_{m_2}$ , and order the elements  $\alpha_{1,2}, \alpha_{1,3}, \dots, \alpha_{1,n}, \alpha_{2,3}, \dots, \alpha_{2,n}, \dots, \alpha_{n-1,n}$ , respectively, by  $\alpha_1, \alpha_2, \dots, \alpha_{m_2}$  where  $m_2 = \binom{n}{2}$ .

It is easy to see that  $\alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_{m_2} = (12)$ . Then,  $(12) \in \langle \alpha_1, \alpha_2, \dots, \alpha_{m_2} \rangle$ , so  $H_{A_1, A_1} \subseteq \langle \alpha_1, \alpha_2, \dots, \alpha_{m_2} \rangle$ . Moreover, for any  $\alpha \in H_{A_i, A_j}$  ( $1 \leq i, j \leq m_2$ ), there exists  $\bar{\alpha} \in H_{A_1, A_1}$  such that

$$\alpha = \alpha_i \circ \dots \circ \alpha_{m_2} \circ \bar{\alpha} \circ \alpha_1 \circ \dots \circ \alpha_{j-1}.$$

Hence,  $I_{n,2} = \langle \alpha_1, \alpha_2, \dots, \alpha_{m_2} \rangle$ , by Lemma 2.5. □

**Theorem 2.8** For  $n \geq 2$  and  $1 \leq r \leq n - 1$ ,

$$\text{qrk}(I_{n,r}) = \text{rank}(I_{n,r}) = \begin{cases} n & \text{for } r = 1 \\ \binom{n}{2} & \text{for } r = 2 \\ \binom{n}{r} + 1 & \text{for } 3 \leq r \leq n - 1 \end{cases}.$$

**Proof** The cases  $r = 1$  and  $r = 2$  follow from Lemmas 2.6 and 2.7, (1.9), and the fact that  $\text{rank}(I_{n,r}) \leq \text{qrk}(I_{n,r})$ . Let  $n \geq 4$  and  $3 \leq r \leq n - 1$ .

Suppose  $r = n - 1$ . Let  $A_1, \dots, A_{r+1}$  be the  $r$ -element subsets of  $X_n$  as in Lemma 2.1, with  $a_i = i$ , and let  $\alpha_1, \dots, \alpha_r, \beta_1 \in I_n$  be the quasi-idempotents as in Lemma 2.1. Then  $(1 \dots r) \in \langle \alpha_1, \dots, \alpha_r, \beta_1 \rangle$ , and  $\langle \alpha_1, \dots, \alpha_r, \beta_1 \rangle$  intersects each  $\mathcal{H}$ -class of  $D_r$ . Consider  $(12)$  as a 2-cycle on  $A_1$ . Then  $H_{A_1, A_1} \subseteq \langle \alpha_1, \dots, \alpha_r, \beta_1, (12) \rangle$ . Hence, by Lemma 2.5,  $\langle \alpha_1, \dots, \alpha_r, \beta_1, (12) \rangle = I_{n,r}$ , so  $\text{qrk}(I_{n,r}) \leq r + 2 = \binom{n}{r} + 1$ .

Suppose  $r \leq n - 2$ . Let  $A_1, A_{r+2}, \dots, A_{m_r}$ , where  $m_r = \binom{n}{r}$ , be the  $r$ -element subsets of  $X_n$  as in Lemma 2.2, and let  $\gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \in I_n$  be the quasi-idempotents as in Lemma 2.2. Note that the set  $A_1$  in Lemma 2.1 is the same as the  $A_1$  in Lemma 2.2, namely  $\{1, \dots, r\}$ . Then  $\langle \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$  contains a 2-cycle, or an  $(r - 1)$ -cycle, or an  $r$ -cycle on  $A_1$ . We now have three cases to consider.

Suppose  $(b_1 \dots b_r) \in \langle \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ , where  $(b_1 \dots b_r)$  is an  $r$ -cycle on  $A_1$ . Let  $A_1, \dots, A_{r+1}$  be the  $r$ -element subsets of  $X_n$  as in Lemma 2.1, with  $a_1 = b_1, \dots, a_{r-2} = b_{r-2}$  and  $a_r = b_{r-1}$ , and let  $\alpha_1, \dots, \alpha_r, \beta_2 \in I_n$  be the quasi-idempotents as in Lemma 2.1. Then  $(b_1 \dots b_{r-1}) \in \langle \alpha_1, \dots, \alpha_r, \beta_2 \rangle$ , so  $H_{A_1, A_1} \subseteq \langle \alpha_1, \dots, \alpha_r, \beta_2, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ . Moreover,  $\langle \alpha_1, \dots, \alpha_r, \beta_2, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$  intersects each  $\mathcal{H}$ -class of  $D_r$ . Hence, by Lemma 2.5,  $\langle \alpha_1, \dots, \alpha_r, \beta_2, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle = I_{n,r}$ , so  $\text{qrang}(I_{n,r}) \leq m_r - 2 + 3 = \binom{n}{r} + 1$ .

Suppose  $(b_1 \dots b_{r-1}) \in \langle \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ , where  $(b_1 \dots b_{r-1})$  is an  $(r - 1)$ -cycle on  $A_1$ . Let  $A_1, \dots, A_{r+1}$  be the  $r$ -element subsets of  $X_n$  as in Lemma 2.1, with  $a_1 = b_1, \dots, a_{r-1} = b_{r-1}$  and  $\{b_r\} = A_1 \setminus \{b_1, \dots, b_{r-1}\}$ , and let  $\alpha_1, \dots, \alpha_r, \beta_1 \in I_n$  be the quasi-idempotents as in Lemma 2.1. Then  $(b_1 \dots b_r) \in \langle \alpha_1, \dots, \alpha_r, \beta_1 \rangle$ , so  $H_{A_1, A_1} \subseteq \langle \alpha_1, \dots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ . Moreover,  $\langle \alpha_1, \dots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$  intersects each  $\mathcal{H}$ -class of  $D_r$ . Hence, by Lemma 2.5,  $\langle \alpha_1, \dots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle = I_{n,r}$ , so  $\text{qrang}(I_{n,r}) \leq m_r - 2 + 3 = \binom{n}{r} + 1$ .

Suppose  $(b_1 b_2) \in \langle \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ , where  $(b_1 b_2)$  is a 2-cycle on  $A_1$ . Let  $A_1, \dots, A_{r+1}$  be the  $r$ -element subsets of  $X_n$  as in Lemma 2.1, with  $a_1 = b_1, a_2 = b_2$  and  $\{a_3, \dots, a_r\} = A_1 \setminus \{b_1, b_2\}$ , and let  $\alpha_1, \dots, \alpha_r, \beta_1 \in I_n$  be the quasi-idempotents as in Lemma 2.1. Then  $(b_1 \dots b_r) \in \langle \alpha_1, \dots, \alpha_r, \beta_1 \rangle$ , so  $H_{A_1, A_1} \subseteq \langle \alpha_1, \dots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$ . Moreover,  $\langle \alpha_1, \dots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle$  intersects each  $\mathcal{H}$ -class of  $D_r$ . Hence, by Lemma 2.5,  $\langle \alpha_1, \dots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \dots, \alpha_{m_r-1}, \delta \rangle = I_{n,r}$ , so  $\text{qrang}(I_{n,r}) \leq m_r - 2 + 3 = \binom{n}{r} + 1$ .

Therefore, the result follows immediately from the fact that  $\binom{n}{r} + 1 = \text{rank}(I_{n,r}) \leq \text{qrang}(I_{n,r}) \leq \binom{n}{r} + 1$ . □

**Theorem 2.9** For  $n \geq 2$  and  $1 \leq r \leq n - 1$ ,

$$\text{qrang}^*(I_{n,r}) = \text{rank}^*(I_{n,r}) = \begin{cases} n - 1 & \text{for } r = 1 \\ \binom{n}{2} & \text{for } r = 2 \\ \binom{n}{r} + 1 & \text{for } 3 \leq r \leq n - 1 \end{cases}.$$

**Proof** The case  $r = 1$  is easy to prove directly. Let  $n \geq 3$  and  $2 \leq r \leq n - 1$ . Then the result follows immediately from the facts that  $\text{rank}^*(I_{n,r}) \leq \text{qrang}^*(I_{n,r}) \leq \text{qrang}(I_{n,r})$  and  $\text{rank}^*(I_{n,r}) = \text{qrang}(I_{n,r})$  (by (1.8) and Theorem 2.8). □

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