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## Lattice ordered semigroups and hypersemigroups

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**Abstract:** This paper shows that many results on hypersemigroups do not need any proof as can be obtained from lattice ordered semigroups.

**Key words:**  $le$ -Semigroup, hypersemigroup, regular, intraregular, right (left) ideal element, bi-ideal element, quasi-ideal element, right (left) ideal, bi-ideal, quasi-ideal

### 1. Introduction

According to Mittas [18, 19], the concept of the hypergroup introduced by the French mathematician Marty at the 8th Congress of Scandinavian Mathematicians in 1934 [17] is the following: a hypergroup is a nonempty set  $H$  endowed with a multiplication  $xy$  such that (i)  $xy \subseteq H$ ; (ii)  $x(yz) = (xy)z$ ; (iii)  $xH = Hx = H$  for every  $x, y, z \in H$ . This concept was first studied in the years 1969 and 1972 by Mittas [18] and Corsini [4]. Since then, hundreds of papers on hyperstructures appeared, and in recent years many groups in the world investigate hypersemigroups in research programs based on the definition given by Marty. While in a semigroup the result between two elements of the support set is an element, in a hypersemigroup the result between two elements of the support set is a set. This is the difference between semigroups and hypersemigroups (and the other hyperstructures, as well). In the present paper we emphasize the natural correspondence that exists between lattice ordered semigroups (shortly  $le$ -semigroups) and hypersemigroups.

An  $le$ -semigroup is a semigroup  $S$ , which is at the same time a lattice having a greatest element with respect to the order of  $S$ , usually denoted by  $e$ , such that  $a(b \vee c) = ab \vee ac$  and  $(a \vee b)c = ac \vee bc$  for all  $a, b, c \in S$ . In the above definition if  $S$  is not a lattice but only an upper semilattice ( $\vee$ -semilattice), then  $S$  is said to be a  $\vee e$ -semigroup. A  $poe$ -semigroup is a semigroup  $S$  with an order " $\leq$ " and a greatest element  $e$  ( $e \geq a \forall a \in S$ ) in which the multiplication is compatible with the ordering. This paper shows that from every result of an  $le$ -semigroup,  $\vee e$ -semigroup, or  $poe$ -semigroup based on right (left) ideal elements, ideal elements, bi-ideal elements, or quasi-ideal elements a corresponding result on a hypersemigroup can be obtained. From results on regular, intraregular  $le$ -semigroups (and not only), corresponding results on regular, intraregular hypersemigroups can be obtained. Illustrative examples are given. Thus, many results on hypersemigroups do not need any proof as they follow from more general statements about ordered semigroups.

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## 2. Prerequisites

An ordered groupoid, shortly a *po*-groupoid, is a groupoid  $S$  in which the multiplication is compatible with the ordering; that is,  $a \leq b$  implies  $ac \leq bc$  and  $ca \leq cb$  for every  $a, b, c \in S$  [1, 5, 6]. An element  $a$  of a *po*-groupoid is called idempotent if  $a^2 = a$  [5]. An element  $a$  of a *po*-groupoid  $S$  is called a right (resp. left) ideal element if  $ax \leq a$  (resp.  $xa \leq a$ ) for all  $x \in S$ ; it is called an ideal element if it is both a right and a left ideal element [2]. A  $\vee$ -groupoid is a groupoid  $S$  and at the same time an upper semilattice ( $\vee$ -semilattice) such that  $a(b \vee c) = ab \vee ac$  and  $(a \vee b)c = ac \vee bc$  for every  $a, b, c \in S$ . In the above definition if  $S$  is not only a  $\vee$ -semilattice but a lattice, then it is called an *l*-semigroup (lattice ordered semigroup). One can find this definition first by Dubreil-Jacotin et al. [5] and later by Birkhoff and Fuchs [1, 2, 6]. Every *l*-groupoid is a *po*-groupoid [2, 5]; moreover clearly, every *l*-groupoid is a  $\vee$ -groupoid. An *l*-groupoid,  $\vee$ -groupoid, or *po*-groupoid possessing a greatest element usually denoted by  $e$  ( $e \geq a \forall a \in S$ ) is called an *le*-groupoid,  $\vee e$ -groupoid, or *poe*-groupoid, respectively. An element  $a$  of a *poe*-groupoid  $S$  is a right (resp. left) ideal element of  $S$  if and only if  $ae \leq a$  (resp.  $ea \leq a$ ). Thus, in a *poe*-groupoid  $S$ , the right (resp. left) ideal element is defined as an element  $a$  of  $S$  such that  $ae \leq a$  (resp.  $ea \leq a$ ) [7, 8]. An element that is both a right and a left ideal element is called an ideal element. For a *poe*-groupoid, we denote by  $F_r$  (resp.  $F_l$ ) the set of its right (resp. left) ideal elements. For an element  $a$  of  $S$  we denote by  $r(a)$  the right ideal element of  $S$  generated by  $a$  (that is,  $r(a)$  is a right ideal element of  $S$  such that  $r(a) \geq a$  and if  $t$  is a right ideal element of  $S$  such that  $t \geq a$ , then  $r(a) \leq t$ ) and by  $l(a)$  the left ideal element of  $S$  generated by  $a$ . An element  $a$  of a *poe*-groupoid  $S$  is called a quasi-ideal element if  $ae \wedge ea$  exists in  $S$  and  $ae \wedge ea \leq a$  [7, 10].

If the multiplication on an ordered groupoid is associative, then it is called an ordered semigroup ( $\vee$ -*po*-semigroup). Every *po*-semigroup possessing a greatest element is called a *poe*-semigroup. An element  $a$  of a *poe*-semigroup is called a bi-ideal element if  $aea \leq a$  [9]. We denote by  $\mathcal{B}$  the set of bi-ideal elements and by  $\mathcal{Q}$  the set of quasi-ideal elements of  $S$ . For a  $\vee e$ -semigroup  $S$ , we have  $r(a) = a \vee ae$  and  $l(a) = a \vee ea$ . A *poe*-semigroup  $S$  is called regular if  $a \leq aea$  for every  $a \in S$ ; it is called intraregular if  $a \leq ea^2e$  for every  $a \in S$  [7, 8].

For a nonempty set  $H$ , we denote by  $\mathcal{P}^*(H)$  the set of all nonempty subsets of  $H$ . A hypergroupoid is a nonempty set  $H$  with a hyperoperation “ $\circ$ ” on  $H$  ( $a, b \in H \Rightarrow a \circ b \subseteq H$ ) and an operation “ $*$ ” on  $\mathcal{P}^*(H)$  defined by  $A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$  and it is denoted by  $(H, \circ, *)$  or just by  $(H, \circ)$  [12, 15].

If  $H$  is a hypergroupoid, then  $\{x\} * \{y\} = x \circ y$  for every  $x, y \in H$ . In a hypergroupoid  $H$ ,  $A \subseteq B$  implies  $A * C \subseteq B * C$  and  $C * A \subseteq C * B$  for every nonempty subset  $A, B, C$  of  $H$ . For nonempty subsets  $A, B$  of  $H$ , we have  $x \in A * B$  if and only if  $x \in a \circ b$  for some  $a \in A, b \in B$ , from which the implication “ $a \in A$  and  $b \in B \Rightarrow a \circ b \subseteq A * B$ ” also holds. A nonempty subset  $A$  of  $H$  is called idempotent if  $A * A = A$ ; that is,  $x \in a \circ b$  for some  $a, b \in A$  if and only if  $x \in A$ . A nonempty subset  $A$  of a hypergroupoid  $(H, \circ)$  is called a right ideal of  $H$  if  $A * H \subseteq A$ , and equivalently if  $a \circ h \subseteq A$  for every  $a \in A$  and every  $h \in H$ ; it is called left ideal of  $H$  if  $H * A \subseteq A$ , equivalently if  $h \circ a \subseteq A$  for every  $a \in A$  and every  $h \in H$ . If  $A$  is both a right and a left ideal of  $H$ , then it is called an ideal of  $H$ . A nonempty subset  $A$  of a hypergroupoid  $H$  is called a quasi-ideal of  $H$  if  $(A * H) \cap (H * A) \subseteq A$  [12, 15], which means that if  $x \in a \circ h$  and  $x \in k \circ b$  for some  $a, b \in A, h, k \in H$ , then  $x \in A$ .

A hypergroupoid  $(H, \circ)$  such that  $\{x\} * (y \circ z) = (x \circ y) * \{z\}$  for every  $x, y, z \in H$  is called a hypersemigroup [12, 15]. In a hypersemigroup, the operation “ $*$ ” is associative; that is,  $(A * B) * C = A * (B * C)$

for every nonempty subset  $A, B, C$  of  $H$  [15]. A nonempty subset  $A$  of a hypersemigroup  $H$  is called a bi-ideal of  $H$  if  $A * H * A \subseteq A$  [12, 15]; that is, if  $x \in u \circ a$  and  $u \in b \circ h$  for some  $a, b \in A$ ,  $h \in H$ , then  $x \in A$ . A hypersemigroup  $H$  is called regular if for any  $a \in H$  there exists  $x \in H$  such that  $a \in (a \circ x) * \{a\}$  ( $= \{a\} * (x \circ a)$ ) and equivalently if for any nonempty subset  $A$  of  $H$ , we have  $A \subseteq A * H * A$  [14, 15]. A hypersemigroup  $H$  is called intraregular if for any  $a \in H$  there exist  $x, y \in H$  such that  $a \in (x \circ a) * (a \circ y)$  and equivalently if for any nonempty subset  $A$  of  $H$ , we have  $A \subseteq H * A * A * H$  [14, 15].

In the present note we show that from many results on  $le$ ,  $\forall e$ , or  $poe$ -semigroups corresponding results on hypersemigroups can be obtained. In particular, from every result of an  $le$ -semigroup,  $\forall e$ -semigroup, or  $poe$ -semigroup based on right (left) ideal elements, ideal elements, bi-ideal elements, or quasi-ideal elements a corresponding result on a hypersemigroup can be obtained. From results on regular  $le$ -semigroups, intraregular  $le$ -semigroups (and not only), corresponding results on regular, intraregular hypersemigroups can be obtained. Illustrative examples are given. The present paper thus addresses what we have already said in the Turkish Journal of Mathematics [13]: “the study of  $le$ -semigroups plays an essential role in the theory of fuzzy semigroups and the theory of hypersemigroups”.

### 3. Main results

In this section, we consider the main theorems 2.9 and 2.12 presented in [12] and get them as corollaries to  $le$ -semigroups. Our investigation is based on the following lemma.

**Lemma 3.1** (see also [15, Proposition 7]) *If  $(H, \circ)$  is a hypergroupoid then, for any nonempty subsets  $A, B, C$  of  $H$ , we have:*

- (1)  $(A \cup B) * C = (A * C) \cup (B * C)$  and
- (2)  $A * (B \cup C) = (A * B) \cup (A * C)$ .

**Proof** Let  $x \in (A \cup B) * C$ . Then  $x \in u \circ c$  for some  $u \in A \cup B$  and  $c \in C$ . If  $u \in A$ , then  $x \in u \circ c = \{u\} * \{c\} \subseteq A * C$ . If  $u \in B$ , then  $x \in \{u\} * \{c\} \subseteq B * C$ , so we have  $(A \cup B) * C \subseteq (A * C) \cup (B * C)$ . Similarly,  $(A * C) \cup (B * C) \subseteq (A \cup B) * C$  and property (1) holds. The proof of (2) is similar.  $\square$

**Remark 3.2** According to Lemma 3.1, if  $(H, \circ)$  is a hypergroupoid (or a hypersemigroup), then the set  $\mathcal{P}^*(H)$  of all nonempty subsets of  $H$  endowed with the operation “ $*$ ” and the inclusion relation “ $\subseteq$ ” is an  $le$ -semigroup,  $H$  being the greatest element of  $\mathcal{P}^*(H)$  and  $A \cup B$ ,  $A \cap B$  the supremum and the infimum of  $A$  and  $B$ , respectively, for any  $A, B \in \mathcal{P}^*(H)$ .

**Lemma 3.3** *Let  $S$  be a  $poe$ -semigroup. If  $c$  is a right (resp. left) ideal element of  $S$  then, for any  $d \in S$ , the element  $cd$  is a bi-ideal element of  $S$ .*

**Proof** Let  $c \in F_r$  and  $d \in S$ . Then  $(cd)e(cd) = c(dec)d \leq (ce)d \leq cd$ , so  $cd$  is a bi-ideal element of  $S$ . If  $c \in F_l$  and  $d \in S$ , then  $(cd)e(cd) = (cde)cd \leq (ec)d \leq cd$  and again  $cd \in \mathcal{B}$ .  $\square$

**Proposition 3.4** *Let  $S$  be a regular  $\forall e$ -semigroup and  $b$  a bi-ideal element of  $S$ . Then there exists a right ideal element  $c$  and a left ideal element  $d$  of  $S$  such that  $b = cd$ .*

**Proof** The element  $b \vee be$  is a right ideal element of  $S$ . This is because  $(b \vee be)e = be \vee be^2 = be$  ( $e^2 \leq e$  as  $e$  is the greatest element of  $S$ ). Clearly  $b \vee be \geq b$ . If now  $t$  is a right ideal element of  $S$  such that  $t \geq b$ , then  $b \vee be \leq t \vee te = t$  and so  $b \vee be \leq t$ . Thus, the element  $b \vee be$  is the right ideal element of  $S$  generated by  $b$ ;

that is,  $r(b) = b \vee be$ . Similarly, the left ideal element of  $S$  generated by  $b$  is the element  $b \vee eb$ , which means that  $l(b) = b \vee eb$ .

Since  $r(b) = b \vee be$  and  $l(b) = b \vee eb$ , we have

$$r(b)l(b) = (b \vee be)(b \vee eb) = b^2 \vee beb \vee be^2b = b^2 \vee beb.$$

Since  $b$  is a bi-ideal element of  $S$ , we have  $beb \leq b$ ; since  $S$  is regular we have  $b \leq beb$  and so  $beb = b$ , and then  $b^2 = (beb)b = b(eb)b \leq beb$ . Thus, we have  $r(b)l(b) = beb = b$ , where  $r(b)$  is a right ideal element and  $l(b)$  is a left ideal element of  $S$ .  $\square$

By Lemma 3.3 and Proposition 3.4 we have the following theorem.

**Theorem 3.5** (see also [9, Lemma 2]) *Let  $S$  be a regular  $\vee e$ -semigroup. Then  $b$  is a bi-ideal element of  $S$  if and only if there exist a right ideal element  $c$  and a left ideal element  $d$  of  $S$  such that  $b = cd$ .*

**Corollary 3.6** [12, Theorem 2.9] *If  $(H, \circ)$  is a regular hypersemigroup, then  $B$  is a bi-ideal of  $H$  if and only if there exists a right ideal  $C$  and a left ideal  $D$  of  $H$  such that  $B = C * D$ .*

**Proof**  $\implies$ . Let  $H$  be a regular hypersemigroup and  $B$  a bi-ideal of  $H$ . Then  $(\mathcal{P}^*(H), *, \subseteq)$  is a regular  $\vee e$ -semigroup and  $B$  is a bi-ideal element of  $\mathcal{P}^*(H)$ . Then, by Theorem 3.5, there exist a right ideal element  $C$  and a left ideal element  $D$  of  $\mathcal{P}^*(H)$  such that  $B = C * D$ . Then  $C$  is a right ideal of  $H$ ,  $D$  is a left ideal of  $H$ , and  $B = C * D$ .

$\impliedby$ . Let  $H$  be a regular hypersemigroup,  $C$  a right ideal, and  $D$  a left ideal of  $H$ . Then  $(\mathcal{P}^*(H), *, \subseteq)$  is a regular  $\vee e$ -semigroup,  $C$  is a right ideal element, and  $D$  is a left ideal element of  $\mathcal{P}^*(H)$ . By Theorem 3.5, the set  $B := C * D$  is a bi-ideal element of the  $\vee e$ -semigroup  $(\mathcal{P}^*(H), *, \subseteq)$ , and so  $B$  is a bi-ideal of the hypersemigroup  $(H, \circ)$ .  $\square$

**Proposition 3.7** *Let  $S$  be a regular poe-semigroup. Then the right ideal elements and the left ideal elements of  $S$  are idempotent. If  $S$  is at the same time semilattice under " $\wedge$ ", then for every right ideal element  $a$  and every left ideal element  $b$  of  $S$  the product  $ab$  is a quasi-ideal element of  $S$ .*

**Proof** Let  $a$  be a right ideal element of  $S$ . Since  $S$  is regular, we have  $a \leq (ae)a \leq a^2 \leq ae \leq a$ , and so  $a^2 = a$ . If  $b$  is a left ideal element of  $S$ , then  $b \leq b(eb) \leq b^2 \leq eb \leq b$ , and so  $b^2 = b$ . Suppose now that  $S$  is a  $\wedge$ -semilattice,  $a$  is a right ideal element, and  $b$  is a left ideal element of  $S$ . Then  $(a \wedge b)e \wedge e(a \wedge b) \leq ae \wedge eb \leq a \wedge b$ , so  $a \wedge b$  is a quasi-ideal element of  $S$ . On the other hand,  $a \wedge b = ab$ . Indeed, since  $S$  is regular, we have  $a \wedge b \leq (a \wedge b)e(a \wedge b) \leq aeb \leq ab \leq ae \wedge eb \leq a \wedge b$ , so  $a \wedge b = ab$ ; thus,  $ab$  is a quasi-ideal element of  $S$ .  $\square$

**Proposition 3.8** *Let  $S$  be an  $le$ -semigroup. Suppose that the right ideal elements and the left ideal elements of  $S$  are idempotent and for every right ideal element  $a$  and every left ideal element  $b$  of  $S$  the product  $ab$  is a quasi-ideal element of  $S$ . Then  $S$  is regular.*

**Proof** Let  $a \in S$ . Since  $r(a)$  is a right ideal element of  $S$ , by hypothesis, we have

$$a \leq r(a) = (r(a))^2 = (a \vee ae)(a \vee ae) = a^2 \vee aea \vee a^2e \vee aea \leq ae.$$

Since  $l(a)$  is a left ideal element of  $S$ , in a similar way, we have  $a \leq ea$ . Thus, we have  $a \leq ae \wedge ea$ . Since  $ae$  is a right ideal element and  $ea$  is a left ideal element of  $S$ , by hypothesis, they are idempotent and we have

$$ae \wedge ea = (ae)^2 \wedge (ea)^2 = (aea)e \wedge e(aea).$$

On the other hand,  $aea$  is a quasi-ideal element of  $S$ . Indeed, since  $e$  is a right ideal element of  $S$ , we have  $aea = ae^2a = (ae)(ea)$ . Since  $ae$  is a right ideal element and  $ea$  is a left ideal element of  $S$ , by hypothesis,  $(ae)(ea)$  is a quasi-ideal element of  $S$ , so  $aea$  is a quasi-ideal element of  $S$ , which means that  $(aea)e \wedge e(aea) \leq aea$ . Hence, we obtain  $a \leq aea$  for any  $a \in S$  and so  $S$  is regular.  $\square$

By Propositions 3.7 and 3.8 the following theorem holds.

**Theorem 3.9** (see also [10, Theorem]) *An  $le$ -semigroup  $S$  is regular if and only if the right ideal elements and the left ideal elements of  $S$  are idempotent and for every right ideal element  $a$  and every left ideal element  $b$  of  $S$  the product  $ab$  is a quasi-ideal element of  $S$ .*

**Corollary 3.10** [12, Theorem 2.12] *A hypersemigroup  $(H, \circ)$  is regular if and only if the right ideals and the left ideals of  $H$  are idempotent, and for every right ideal  $A$  and every left ideal  $B$  of  $H$ , the product  $A * B$  is a quasi-ideal of  $H$ .*

**Proof**  $\implies$ . Let  $H$  be a regular hypersemigroup and  $A$  be a right ideal of  $H$ . Then  $(\mathcal{P}^*(H), *, \subseteq)$  is a regular  $le$ -semigroup and  $A$  is a right ideal element of  $\mathcal{P}^*(H)$ . Then, by Theorem 3.9,  $A$  is idempotent. Similarly, the left ideals of  $H$  are idempotent. Let now  $H$  be a regular hypersemigroup,  $A$  a right ideal, and  $B$  a left ideal of  $H$ . Then  $(\mathcal{P}^*(H), *, \subseteq)$  is a regular  $le$ -semigroup,  $A$  is a right ideal element, and  $B$  is a left ideal element of  $\mathcal{P}^*(H)$ . By Theorem 3.9, the product  $A * B$  is a quasi-ideal element of the  $le$ -semigroup  $(\mathcal{P}^*(H), *, \subseteq)$ , and so  $A * B$  is a quasi-ideal of the hypersemigroup  $(H, \circ)$ .

$\impliedby$ . Suppose that the right ideals and the left ideals of  $H$  are idempotent and for every right ideal  $A$  and every left ideal  $B$  of  $H$ ,  $A * B$  is a quasi-ideal of  $H$ . Then the right ideal elements and the left ideal elements of the  $le$ -semigroup  $(\mathcal{P}^*(H), *, \subseteq)$  are idempotent and for every right ideal element  $A$  and every left ideal element  $B$  of  $H$ ,  $A * B$  is a quasi-ideal element of  $\mathcal{P}^*(H)$ . By Theorem 3.9, the  $le$ -semigroup  $(\mathcal{P}^*(H), *, \subseteq)$  is regular; that is,  $A \subseteq A * H * A$  for every  $A \in \mathcal{P}^*(H)$ , and so the hypersemigroup  $(H, \circ)$  is regular as well.  $\square$

We apply Theorems 3.5 and 3.9 to the following example.

**Example 3.11** We consider the  $le$ -semigroup  $S = \{a, b, c, d, e\}$  defined by Table 1 and Figure 1.

Table 1: Multiplication table of Example 3.11.

$\cdot$	$a$	$b$	$c$	$d$	$e$
$a$	$a$	$a$	$d$	$d$	$d$
$b$	$a$	$b$	$e$	$d$	$e$
$c$	$a$	$b$	$e$	$d$	$e$
$d$	$a$	$a$	$d$	$d$	$d$
$e$	$a$	$b$	$e$	$d$	$e$

Theorem 3.5 can be applied. This is a regular  $\vee e$ -semigroup.

The elements  $a, b, d, e$  are the bi-ideal elements of  $S$ .

The elements  $d, e$  are the right ideal elements of  $S$ .

The elements  $a, b, d, e$  are the left ideal elements of  $S$ .

According to Theorem 3.5, for every bi-ideal element  $t$  of  $S$  there exist a right ideal element  $x$  and a left ideal element  $y$  of  $S$  such that  $t = xy$ .

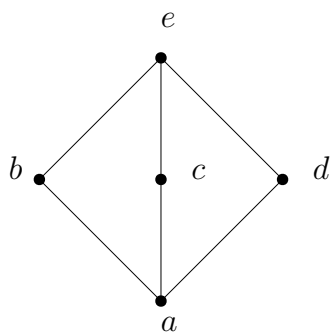


Figure 1: Figure corresponding to the order of Example 3.11.

Independently, we have  $a = da = db = ea$ , where  $d, e \in F_r$  and  $a, b \in F_l$ ;  $b = eb$ , where  $e \in F_r$  and  $b \in F_l$ ;  $d = dd = de = ed$ , where  $d, e \in F_r$  and  $d, e \in F_l$ ; and  $e = ee$ , where  $e \in F_r$  and  $e \in F_l$ .

Theorem 3.9 can be also applied. This is an  $le$ -semigroup; the elements  $a, b, d, e$  are the quasi-ideal elements of  $S$ . According to Theorem 3.9, the right and the left ideal elements of  $S$  are idempotent and for every right ideal element  $a$  and every left ideal element  $b$  of  $S$  the element  $ab$  is a quasi-ideal element of  $S$ . Independently we can check that this is true.

In a similar way, from every result in Turk J Math [13], a result on hypersemigroups can be obtained. As an example, let us consider one of the main theorems in [13], Theorem 16. We first have to give the definition of a prime element of a  $po$ -groupoid and the definition of a prime subset of a hypergroupoid. If  $S$  is a  $po$ -groupoid, an element  $m$  of  $S$  is said to be prime if for any elements  $a, b$  of  $S$  such that  $ab \leq m$  we have  $a \leq m$  or  $b \leq m$  [8]. If  $S$  is a hypersemigroup, a nonempty subset  $M$  of  $S$  is called prime if, for any nonempty subsets  $A, B$  of  $S$  such that  $A * B \subseteq M$ , we have  $A \subseteq M$  or  $B \subseteq M$  [16]. A prime ideal of a  $po$ -groupoid (resp. hypersemigroup)  $S$  is an ideal of  $S$  that is at the same time a prime subset of  $S$ .

**Theorem 3.12** [13, Theorem 16] *Let  $S$  be an  $le$ -semigroup. The ideal elements of  $S$  are prime if and only if they form a chain and  $S$  is intraregular.*

From this theorem, in the way indicated above, we have the following corollary.

**Corollary 3.13** *Let  $H$  be an hypersemigroup. The ideals of  $H$  are prime if and only if they form a chain and  $H$  is intraregular.*

We apply Theorem 3.12 to Example 3.11 given above. The  $le$ -semigroup considered in Example 3.11 is intraregular (as well), the elements  $d$  and  $e$  are the ideal elements of  $S$ , they are prime, and they form a chain.

Let us now give an example of an  $le$ -semigroup that is intraregular but not regular. Theorems 3.5 and 3.9 cannot be applied while Theorem 3.12 can be applied to this example.

**Example 3.14** We consider the  $le$ -semigroup  $S = \{a, b, c, d, e\}$  defined by Table 2 and Figure 2.

This is not regular as  $c \not\leq cdc$ .

The elements  $a, b, c, d$  are the right ideal elements of  $S$ .

The elements  $a, d$  are the left ideal elements of  $S$ .

The elements  $a, d$  are the ideal elements of  $S$ .

The elements  $a, b, c, d$  are the bi-ideal elements of  $S$ .

Table 2: Multiplication table of Example 3.14.

$\cdot$	$a$	$b$	$c$	$d$	$e$
$a$	$a$	$a$	$a$	$a$	$a$
$b$	$a$	$b$	$b$	$b$	$b$
$c$	$a$	$b$	$b$	$b$	$b$
$d$	$a$	$d$	$d$	$d$	$d$
$e$	$a$	$d$	$d$	$d$	$d$

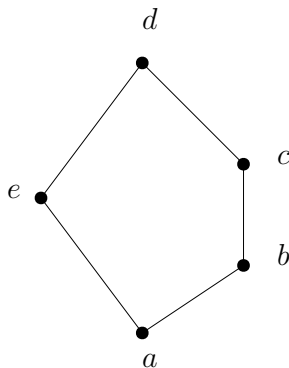


Figure 2: Figure corresponding to the order of Example 3.14.

For the bi-ideal element  $c$  there is no a right ideal element  $x$  and a left ideal element  $y$  of  $S$  such that  $c = xy$ , so Theorem 3.5 cannot be applied. The right ideal elements of  $S$  are not idempotent as  $c^2 \neq c$ , so Theorem 3.9 cannot be applied as well. On the other hand, this is an intraregular  $le$ -semigroup. According to Theorem 3.12 the ideal elements of  $S$  are prime and they form a chain. One can independently check that this is true.

Let us finally give an example of the  $le$ -semigroup  $(\mathcal{P}^*(H), *, \subseteq)$  constructed by a hypersemigroup  $(H, \circ)$  in the way indicated in the present paper.

**Example 3.15** We consider the hypersemigroup  $S = \{a, b, c\}$  with the hyperoperation “ $\circ$ ” given by Table 3.

Table 3: The hypersemigroup of Example 3.15.

$\circ$	$a$	$b$	$c$
$a$	$\{a\}$	$\{a, b\}$	$\{a, c\}$
$b$	$\{a, b\}$	$\{b\}$	$\{a, b, c\}$
$c$	$\{a, c\}$	$\{b, c\}$	$\{c\}$

We have

$$\mathcal{P}^*(S) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

According to Remark 3.2, the set  $\mathcal{P}^*(S)$ , with the operation given by Table 4 and the inclusion relation “ $\subseteq$ ”, is an  $le$ -semigroup.



Table 4: The operation of  $\mathcal{P}^*(S)$  of Example 3.15.

*	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
{a}	{a}	{a,b}	{a,c}	{a,b}	{a,c}	{a,b,c}	{a,b,c}
{b}	{a,b}	{b}	{a,b,c}	{a,b}	{a,b,c}	{a,b,c}	{a,b,c}
{c}	{a,c}	{b,c}	{c}	{a,b,c}	{a,c}	{b,c}	{a,b,c}
{a,b}	{a,b}	{a,b}	{a,b,c}	{a,b}	{a,b,c}	{a,b,c}	{a,b,c}
{a,c}	{a,c}	{a,b,c}	{a,c}	{a,b,c}	{a,c}	{a,b,c}	{a,b,c}
{b,c}	{a,b,c}	{b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}
{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}

$$\subseteq = \left\{ \left( \{a\}, \{a\} \right), \left( \{a\}, \{a,b\} \right), \left( \{a\}, \{a,c\} \right), \left( \{a\}, \{a,b,c\} \right), \left( \{b\}, \{b\} \right), \right. \\ \left( \{b\}, \{a,b\} \right), \left( \{b\}, \{b,c\} \right), \left( \{b\}, \{a,b,c\} \right), \left( \{c\}, \{c\} \right), \left( \{c\}, \{a,c\} \right), \\ \left( \{c\}, \{b,c\} \right), \left( \{c\}, \{a,b,c\} \right), \left( \{a,b\}, \{a,b\} \right), \left( \{a,b\}, \{a,b,c\} \right), \\ \left( \{a,c\}, \{a,c\} \right), \left( \{a,c\}, \{a,b,c\} \right), \left( \{b,c\}, \{b,c\} \right), \left( \{b,c\}, \{a,b,c\} \right), \\ \left. \left( \{a,b,c\}, \{a,b,c\} \right) \right\}.$$

If we write for short  $\{a\} = a$ ,  $\{b\} = b$ ,  $\{c\} = c$ ,  $\{a,b\} = d$ ,  $\{a,c\} = e$ ,  $\{b,c\} = f$ ,  $\{a,b,c\} = g$ , and “.” instead of “\*”, then we get the *le*-semigroup  $T = \{a,b,c,d,e,f,g\}$  with the multiplication given by Table 5 and the order below:

Table 5: The *le*-semigroup of Example 3.15.

·	a	b	c	d	e	f	g
a	a	d	e	b	e	g	g
b	d	b	g	d	g	g	g
c	e	f	c	g	e	f	g
d	d	d	g	d	g	g	g
e	e	g	e	g	e	g	g
f	g	f	g	g	g	g	g
g	g	g	g	g	g	g	g

$$\leq = \{(a,a), (a,d), (a,e), (a,g), (b,b), (b,d), (b,f), (b,g), (c,c), (c,e), \\ (c,f), (c,g), (d,d), (d,g), (e,e), (e,g), (f,f), (f,g), (g,g)\}.$$

We give the covering relation  $\prec$  and the figure of  $S$ .

$$\prec = \{(a,d), (a,e), (b,d), (b,f), (c,e), (c,f), (d,g), (e,g), (f,g)\}.$$

Using the Light’s associativity test [3], one can independently check that the set  $S = \{a,b,c,d,e,f,g\}$  with the multiplication given by Table 5 and the order of Figure 3 is an *le*-semigroup ( $g$  being the greatest element of  $S$ ); the methodology used in [11, Example 1], for example, also helps.

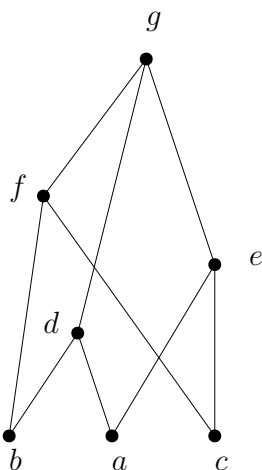


Figure 3: Figure of the  $le$ -semigroup of Example 3.15.

We observe that  $(S, \circ)$  is a regular hypersemigroup (that is, for every  $a \in S$  there exists  $x \in S$  such that  $a \in (a \circ x) * \{a\}$ ), and  $\mathcal{P}^*(S)$  is a regular  $le$ -semigroup. Indeed, looking at Table 5 and Figure 3, we see that  $x \leq xgx$  for every  $x \in \{a, b, c, d, e, f, g\}$ , as it was expected to be. The same holds if we replace the word “regular” with “intra-regular”.

#### 4. Conclusion

From every result of an  $le$ -semigroup based on right (left) ideal elements, ideal elements, bi-ideal elements, quasi-ideal elements, a corresponding result on a hypersemigroup based on right (left) ideals, ideals, bi-ideals, quasi-ideals can be obtained. From many results on regular or intra-regular  $le$ -semigroups, analogous results for hypersemigroups can be obtained. This is because if  $(H, \circ)$  is a hypersemigroup, then the set  $\mathcal{P}^*(H)$  of nonempty subsets of  $H$  endowed with the operation “ $*$ ” and the inclusion relation “ $\subseteq$ ” is an  $le$ -semigroup. Moreover,

- (1)  $A$  is a right (resp. left) ideal of  $(H, \circ)$  if and only if it is a right (resp. left) ideal element of  $(\mathcal{P}^*(H), *, \subseteq)$ .
- (2)  $A$  is a quasi-ideal of  $(H, \circ)$  if and only if it is a quasi-ideal element of  $(\mathcal{P}^*(H), *, \subseteq)$ .
- (3)  $A$  is a bi-ideal of  $(H, \circ)$  if and only if it is a bi-ideal element of  $(\mathcal{P}^*(H), *, \subseteq)$ .
- (4) The hypersemigroup  $(H, \circ)$  is regular if and only if the  $le$ -semigroup  $(\mathcal{P}^*(H), *, \subseteq)$  is regular.
- (5) The hypersemigroup  $(H, \circ)$  is intra-regular if and only if the  $le$ -semigroup  $(\mathcal{P}^*(H), *, \subseteq)$  is intra-regular.

We can say the same if we replace the word “ $le$ -semigroup” with “ $\vee e$ -semigroup” or “ $poe$ -semigroup”.

Many results on hypersemigroups do not need any proof; automatically hold. We can prove them independently just to show how an independent proof works, but even in that case this independent proof goes along the lines of the  $le$ ,  $\vee e$ , or  $poe$ -semigroups. If  $(H, \circ)$  is a hypersemigroup, then  $(\mathcal{P}^*(H), *, \subseteq)$  is a  $poe$ -semigroup and a  $\vee e$ -semigroup as well. This is not the case for ordered hypersemigroups, but even in that case the main idea comes from the  $le$ ,  $\vee e$ , or  $poe$ -semigroups and the independent proofs go along the lines of the  $le$ ,  $\vee e$ , or  $poe$ -semigroups.

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