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TUBA TAŞ ADIYAMAN

ZEYNEP ÖZKURT

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Automorphisms of free metabelian Leibniz algebras of rank three

Tuba TAŞ ADIYAMAN, Zeynep ÖZKURT*
Department of Mathematics, Faculty of Arts and Sciences, Çukurova University, Adana, Turkey

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Abstract: In this work, we determine the structure of the automorphism group of the free metabelian Leibniz algebra of rank three over a field K of characteristic zero.

Key words: Free metabelian Leibniz algebras, automorphisms, wreath product

1. Introduction

A Leibniz algebra L over a field K is a nonassociative algebra with multiplication called bracket $[,] : L \times L \longrightarrow L$ satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all $x, y, z \in L$. If the condition $[x, x] = 0$ for all $x \in L$ is satisfied, this identity is equivalent to the Jacobi identity. Let X be a set, $L(X)$ be the free nonassociative algebra on X over K and I_L be the two-sided ideal of $L(X)$ generated by the elements

$$[a, [b, c]] - [[a, b], c] + [[a, c], b]$$

for all $a, b, c \in L(X)$. Then the algebra $F(X) = L(X)/I_L$ is a free Leibniz algebra with the free generating set X.

In [5], the universal enveloping algebra of the Leibniz algebra L was defined. Let L^l and L^r be two copies of the Leibniz algebra L. We denote by l_x and r_x, the elements of L^l and L^r corresponding to the universal operators of left and right multiplication on x, respectively. Let I_T be the two-sided ideal of the associative tensor K-algebra $T(L^l \oplus L^r)$ with the identity element corresponding to the relations

$$\begin{aligned} r_{[x,y]} &= r_x.r_y - r_y.r_x \\ l_{[x,y]} &= l_x.r_y - r_y.l_x \\ (r_x + l_x).l_y &= 0 \end{aligned}$$

for any $x, y \in L$. Then the factor algebra $UL(L) = T(L^l \oplus L^r)/I_T$ is the universal enveloping algebra of Leibniz algebra L.

*Correspondence: zyapti@cu.edu.tr
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Let M_n be the free metabelian Leibniz algebra of finite rank n over a field K. Denote by $Aut(M_n)$ the automorphism group of M_n. The kernel of a natural homomorphism $\pi\colon\ Aut(M_n) \to Aut(M_n/M_n')$ consist of automorphisms which act identically modulo M_n', where M_n' derived subalgebra. It is called the group of IA-automorphisms and denoted by $IA(M_n)$.

In [6], IA-automorphisms of 2-generator metabelian Lie algebras were studied. In [8], the defining relations of the subgroup of IA-automorphisms of a free metabelian Lie algebra of rank 3 were given. In [7], generating sets of IA-automorphisms of a free metabelian Lie algebra of rank 3 were investigated. In 1973, Shmel'kin [9] noticed that all nonlinear automorphisms of a free metabelian Lie algebra of rank two are wild. In [2], an explicit matrix form for the IA-automorphisms of a free metabelian Lie algebra of rank n was obtained.

Leibniz algebras were first introduced by Loday [5] as a nonantisymmetric version of Lie algebras. In 2001 Abdykhalikov et al. [1] obtained a characterization of tame automorphisms of the free Leibniz algebra of rank 2. In [4], Papistas and Drensky proved that the automorphism group of free nilpotent Leibniz algebras of finite rank m, $m \geq 2$, is generated by the tame automorphisms and one more given nontame automorphism. In [3], a description of the free metabelian Leibniz algebras was given.

In the present article, we study the automorphisms of the free metabelian Leibniz algebra M_3. First, we obtain an explicit matrix form for the IA-automorphisms of M_3 and describe a set of generators of the group $IA(M_3)$, where our approache is via wreath products. Then our main result states that the automorphism group $Aut(M_3)$ is generated by the linear automorphisms and the set of generators of the group $IA(M_3)$.

2. Preliminaries

Let F be the free Leibniz algebra with the free generators x_1, x_2, and x_3. We denote by F' and F'' the derived subalgebra of F and F', respectively and F/F' is a free K-module. We fix the notation F/F'' for the free metabelian Leibniz algebra. Denote by $Aut(F/F'')$ the automorphism group of F/F''. By $Ann(F/F'')$ we denote the ideal of F/F'' generated by elements $\{[a,a] : a \in F/F''\}$. It is known (see [5]) that $r_a = 0 \Leftrightarrow a \in Ann(F/F'')$. The algebra $(F/F'')_{Lie} = (F/F'')/Ann(F/F'')$ is the free metabelian Lie algebra of rank three.

By [3], F/F'' has a basis

$$\{x_{i_1}, [x_{i_1}, x_{i_2}], [x_{i_1}, x_{i_2}, ..., x_{i_k}] \mid 1 \leq i_1, i_2 \leq 3, 1 \leq i_3 \leq ... \leq i_k \leq 3, k = 3, 4...\}.$$

We define the wreath product of abelian Leibniz algebras in a standard way, as in the case of Lie algebras [9]. Let W be the wreath product of F'/F'' and F/F' where F'/F'' is an abelian Leibniz algebra that is a free K-module with the free generating set $\{a_i\}_{i\in I}$ and F/F' is a Leibniz algebra over K. We write shortly $W = (F'/F'')wr(F/F')$ and it has the form $W = F/F' \oplus I_{F'/F''}$, where it is the semidirect sum of $F/F'$and free F/F'-module $I_{F'/F''}$ with the free generating set $\{a_i\}_{i\in I}$. In addition, F'/F'' is a module on F/F', is also a $U(F/F')-$module and the module action is given by

$$\begin{aligned} v * r_u &= [v,u] \\ v * l_u &= [u,v] \end{aligned}$$

for $v \in F'/F''$, $u \in F/F'$ and r_u, l_u $\in U(F/F')$. Denote $x_i + F'' \in F/F''$ by $\overline{x_i}$ and $x_i + F' \in F/F'$ by $\overline{\overline{x_i}}$.

For our aim, it is sufficient to restrict the automorphism group $AutW$ to its subgroup $\overline{AutW}$ whose elements leave $I_{F'/F''}$ and F/F' invariant. $\overline{AutW}$ contains a normal subgroup $IA(W)$ whose elements are identical on F/F' and a subgroup P whose elements leave each $a_i \in I_{F'/F''}$ invariant having trivial intersection. Thus, we have $\overline{AutW}$ is the semidirect product of P and $IA(W)$.

The proof of the next Lemma is the same as in the Lie algebra case [9].

Lemma 2.1 *Let L be a Leibniz algebra and S be an abelian ideal of L. Then there exists a semidirect sum $H = L/S + I$, where I is an abelian ideal, L/S is a subalgebra and a monomorphism $\mu : L \to H$ such that $\mu(L) \cap L = \mu(S)$.*

Corollary 2.2 *The mapping $\overline{x_i} \longrightarrow \overline{\overline{x_i}} + a_i$ extends to a monomorphism $\mu : F/F'' \longrightarrow (F'/F'')wr(F/F')$.*

Proof The map $\overline{x_i} \longrightarrow \overline{\overline{x_i}} + a_i$ extends to a homomorphism

$$\mu : F/F'' \to F/F' \oplus I_{F'/F''}.$$

By Lemma 2.1, μ is a monomorphism. □

3. IA-automorphisms of F/F''

Using similar arguments as in Lie algebras (see Bahturin and Nabiev [2]), we may prove the following proposition.

Proposition 3.1 *There exists an embedding $\vartheta : \overline{Aut(F/F'')} \to \overline{Aut((F'/F'')wr(F/F'))}$ such that if $\alpha \in \overline{Aut(F/F'')}$ which leaves F'/F'' invariant and $\widetilde{\alpha} = \vartheta(\alpha)$ then $\widetilde{\alpha}\mu = \mu\alpha$ where μ is the embedding of Corollary 2.2.*

Proof [2] First given an automorphism $\alpha : F/F'' \to F/F''$ with $\alpha(F'/F'') \subset F'/F''$, we define $\widetilde{\alpha}$ by first writing

$$\begin{aligned} \mu\alpha(x_i + F'') &= \mu(\alpha(x_i) + F'') \\ &= \mu(\overline{\alpha(x_i)}) \\ &= \overline{\overline{\alpha(x_i)}} + p_i, \end{aligned}$$

where $\overline{\overline{\alpha(x_i)}} \in F/F'$, $p_i \in I_{F'/F''}$. On the other hand,

$$\begin{aligned} \widetilde{\alpha}\mu(\overline{x_i}) &= \widetilde{\alpha}(\overline{\overline{x_i}} + a_i) \\ &= \widetilde{\alpha}(\overline{\overline{x_i}}) + \widetilde{\alpha}(a_i) \end{aligned}$$

and then setting $\widetilde{\alpha}(\overline{\overline{x_i}}) = \overline{\overline{\alpha(x_i)}}$, $\widetilde{\alpha}(a_i) = p_i$,

$$\widetilde{\alpha}(\mu(\overline{x_i})) = \mu(\alpha(\overline{x_i}))$$

satisfied. Since $\widetilde{\alpha}$ is induced by α which leaves F'/F'' invariant, then $\widetilde{\alpha} \in Aut(F/F')$. $\widetilde{\alpha}$ extends to a uniquely defined map of W by the definition of the wreath product. Let α^{-1} be the inverse of α. Applying α^{-1} to both side of the equality $\widetilde{\alpha}\mu = \mu\alpha$, it holds $\mu = \mu\alpha\alpha^{-1} = \widetilde{\alpha}\mu\alpha^{-1} = \widetilde{\alpha}\widetilde{\alpha^{-1}}\mu$ and $\widetilde{\alpha}\ \widetilde{\alpha^{-1}} = \widetilde{1}_{F/F''}$. Since $1_W = \widetilde{1}_{F/F''}$, we have $\widetilde{\alpha}\widetilde{\alpha^{-1}} = 1_W$. Hence, $\widetilde{\alpha}$ is an automorphism of W. □

Using this argument, the structure of $IA(F/F'')$ can be described in the following way.

Theorem 3.2 *Let F/F'' be a free metabelian Leibniz algebra of rank three generated by $\{\overline{x_1}, \overline{x_2}, \overline{x_3}\}$. Given the identity 3×3 matrix E , an arbitrary 3×9 matrix $Q = (q_{ij})$, both with coefficients in $U(F/F')$, where $1 \le i \le 3$, $1 \le j \le 9$ and a fix 3×9 matrix A which is defined below;*

$$\begin{bmatrix} l_{\overline{\overline{x_1}}} + r_{\overline{\overline{x_1}}} & r_{\overline{\overline{x_2}}} & l_{\overline{\overline{x_2}}} & 0 & r_{\overline{\overline{x_3}}} & l_{\overline{\overline{x_3}}} & 0 & 0 & 0 \\ 0 & l_{\overline{\overline{x_1}}} & r_{\overline{\overline{x_1}}} & l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_2}}} & 0 & 0 & r_{\overline{\overline{x_3}}} & l_{\overline{\overline{x_3}}} & 0 \\ 0 & 0 & 0 & 0 & l_{\overline{\overline{x_1}}} & r_{\overline{\overline{x_1}}} & l_{\overline{\overline{x_2}}} & r_{\overline{\overline{x_2}}} & l_{\overline{\overline{x_3}}} + r_{\overline{\overline{x_3}}} \end{bmatrix}_{3\times 9}.$$

Let G be the group of invertible matrices of the form $E + AQ$. Then $IA(F/F'') \cong G$.

Proof The elements of $IA(F/F'')$ are the automorphisms of F/F'' which are identical modulo F'/F''. The explicit matrix form for the elements of $IA(F/F'')$ will be found. F'/F'' is generated by $\overline{r_1} = [\overline{x_1}, \overline{x_1}]$, $\overline{r_2} = [\overline{x_1}, \overline{x_2}]$, $\overline{r_3} = [\overline{x_2}, \overline{x_1}]$, $\overline{r_4} = [\overline{x_2}, \overline{x_2}]$, $\overline{r_5} = [\overline{x_1}, \overline{x_3}]$, $\overline{r_6} = [\overline{x_3}, \overline{x_1}]$, $\overline{r_7} = [\overline{x_2}, \overline{x_3}]$, $\overline{r_8} = [\overline{x_3}, \overline{x_2}]$, $\overline{r_9} = [\overline{x_3}, \overline{x_3}]$, as an ideal, that is, as a F/F'-module. We denote the module action of $u \in F/F'$ (i.e. $u = \overline{\overline{w}}$ and $w \in F$) and r_u, $l_u \in U(F/F')$ on $\overline{r} \in F'/F''$ by

$$\begin{aligned} \overline{r} * r_u &= [\overline{r}\ , u] = \left[\overline{r}\ , \overline{\overline{w}}\right], \\ \overline{r} * l_u &= [u\ , \overline{r}] = \left[\overline{\overline{w}}\ , \overline{r}\right]. \end{aligned}$$

Consider $\alpha \in IA(F/F'')$ and $\alpha(\overline{x_i}) = \overline{x_i} + \sum_{j=1}^{9} \overline{r_j} * q_{ij}$ for $i = 1,2,3$ and $q_{ij} \in U(F/F')$. Then

$$\alpha(\overline{x_1}) = \overline{x_1} + \overline{r_1} * q_{11} + \overline{r_2} * q_{12} + \overline{r_3} * q_{13} + \overline{r_4} * q_{14} + \overline{r_5} * q_{15} + \overline{r_6} * q_{16} + \overline{r_7} * q_{17} + \overline{r_8} * q_{18} + \overline{r_9} * q_{19} \quad (1)$$

$$\alpha(\overline{x_2}) = \overline{x_2} + \overline{r_1} * q_{21} + \overline{r_2} * q_{22} + \overline{r_3} * q_{23} + \overline{r_4} * q_{24} + \overline{r_5} * q_{25} + \overline{r_6} * q_{26} + \overline{r_7} * q_{27} + \overline{r_8} * q_{28} + \overline{r_9} * q_{29} \quad (2)$$

$$\alpha(\overline{x_3}) = \overline{x_3} + \overline{r_1} * q_{31} + \overline{r_2} * q_{32} + \overline{r_3} * q_{33} + \overline{r_4} * q_{34} + \overline{r_5} * q_{35} + \overline{r_6} * q_{36} + \overline{r_7} * q_{37} + \overline{r_8} * q_{38} + \overline{r_9} * q_{39} \quad (3)$$

Now apply μ to both sides of $(1), (2)$, and (3),

$$\begin{aligned} \mu\alpha(\overline{x_1}) &= \mu(\overline{x_1} + \overline{r_1} * q_{11} + \overline{r_2} * q_{12} + \overline{r_3} * q_{13} + \overline{r_4} * q_{14} + \overline{r_5} * q_{15} + \overline{r_6} * q_{16} \\ &\quad + \overline{r_7} * q_{17} + \overline{r_8} * q_{18} + \overline{r_9} * q_{19}) \end{aligned}$$

$$\begin{aligned} \mu\alpha(\overline{x_1}) &= \mu\,(\overline{x_1} + [\overline{x_1}, \overline{x_1}] * q_{11} + [\overline{x_1}, \overline{x_2}] * q_{12} + [\overline{x_2}, \overline{x_1}] * q_{13} + [\overline{x_2}, \overline{x_2}] * q_{14}] + [\overline{x_1}, \overline{x_3}] * q_{15} \\ &\quad + [\overline{x_3}, \overline{x_1}] * q_{16} + [\overline{x_2}, \overline{x_3}] * q_{17} + [\overline{x_3}, \overline{x_2}] * q_{18} + [\overline{x_3}, \overline{x_3}] * q_{19}) \\ &= \overline{\overline{x_1}} + a_1 + \left[\overline{\overline{x_1}} + a_1, \overline{\overline{x_1}} + a_1\right] * q_{11} + \left[\overline{\overline{x_1}} + a_1, \overline{\overline{x_2}} + a_2\right] * q_{12} + \left[\ \overline{\overline{x_2}} + a_2, \overline{\overline{x_1}} + a_1\right] * q_{13} \\ &\quad + \left[\ \overline{\overline{x_2}} + a_2, \overline{\overline{x_2}} + a_2\right] * q_{14} + \left[\overline{\overline{x_1}} + a_1, \overline{\overline{x_3}} + a_3\right] * q_{15} + \left[\overline{\overline{x_3}} + a_3, \overline{\overline{x_1}} + a_1\right] * q_{16} \\ &\quad + \left[\ \overline{\overline{x_2}} + a_2, \overline{\overline{x_3}} + a_3\right] * q_{17} + \left[\overline{\overline{x_3}} + a_3, \overline{\overline{x_2}} + a_2\right] * q_{18} + \left[\ \overline{\overline{x_3}} + a_3, \overline{\overline{x_3}} + a_3\right] * q_{19} \end{aligned}$$

$$
\begin{aligned}
&= \overline{\overline{x_1}} + a_1 + ([\overline{\overline{x_1}}, a_1] + [a_1, \overline{\overline{x_1}}]) * q_{11} + ([\overline{\overline{x_1}}, a_2] + [a_1, \overline{\overline{x_2}}]) * q_{12} + ([\overline{\overline{x_2}}, a_1] + [a_2, \overline{\overline{x_1}}]) * q_{13} \\
&\quad +([\overline{\overline{x_2}}, a_2] + [a_2, \overline{\overline{x_2}}]) * q_{14} + ([\overline{\overline{x_1}}, a_3] + [a_1, \overline{\overline{x_3}}]) * q_{15} + ([\overline{\overline{x_3}}, a_1] + [a_3, \overline{\overline{x_1}}]) * q_{16} \\
&\quad +([\overline{\overline{x_2}}, a_3] + [a_2, \overline{\overline{x_3}}]) * q_{17} + ([\overline{\overline{x_3}}, a_2] + [a_3, \overline{\overline{x_2}}]) * q_{18} + ([\overline{\overline{x_3}}, a_3] + [a_3, \overline{\overline{x_3}}]) * q_{19} \\
&= \overline{\overline{x_1}} + a_1 + a_1 * (l_{\overline{\overline{x_1}}} q_{11} + r_{\overline{\overline{x_1}}} q_{11} + r_{\overline{\overline{x_2}}} q_{12} + l_{\overline{\overline{x_2}}} q_{13} + r_{\overline{\overline{x_3}}} q_{15} + l_{\overline{\overline{x_3}}} q_{16}) \\
&\quad +a_2 * (l_{\overline{\overline{x_1}}} q_{12} + r_{\overline{\overline{x_1}}} q_{13} + l_{\overline{\overline{x_2}}} q_{14} + r_{\overline{\overline{x_2}}} q_{14} + r_{\overline{\overline{x_3}}} q_{17} + l_{\overline{\overline{x_3}}} q_{18}) \\
&\quad +a_3 * (l_{\overline{\overline{x_1}}} q_{15} + r_{\overline{\overline{x_1}}} q_{16} + l_{\overline{\overline{x_1}}} q_{17} + r_{\overline{\overline{x_2}}} q_{18} + l_{\overline{\overline{x_3}}} q_{19} + r_{\overline{\overline{x_3}}} q_{19}),
\end{aligned}
$$

$$
\begin{aligned}
\mu\alpha(\overline{x_2}) &= \mu(\overline{x_2} + \overline{r_1} * q_{21} + \overline{r_2} * q_{22} + \overline{r_3} * q_{23} + \overline{r_4} * q_{24} + \overline{r_5} * q_{25} + \overline{r_6} * q_{26} + \overline{r_7} * q_{27} + \overline{r_8} * q_{28} + \overline{r_9} * q_{29}) \\
&= \mu(\overline{x_2} + [\overline{x_1}, \overline{x_1}] * q_{21} + [\overline{x_1}, \overline{x_2}] * q_{22} + [\overline{x_2}, \overline{x_1}] * q_{23} + [\overline{x_2}, \overline{x_2}] * q_{24} + [\overline{x_1}, \overline{x_3}] * q_{25} + \\
&\quad [\overline{x_3}, \overline{x_1}] * q_{26} + [\overline{x_2}, \overline{x_3}] * q_{27} + [\overline{x_3}, \overline{x_2}] * q_{28} + [\overline{x_3}, \overline{x_3}] * q_{29}) \\
&= \overline{\overline{x_2}} + a_2 + [\overline{\overline{x_1}} + a_1, \overline{\overline{x_1}} + a_1] * q_{21} + [\overline{\overline{x_1}} + a_1, \overline{\overline{x_2}} + a_2] * q_{22} + [\overline{\overline{x_2}} + a_2, \overline{\overline{x_1}} + a_1] * q_{23} \\
&\quad + [\overline{\overline{x_2}} + a_2, \overline{\overline{x_2}} + a_2] * q_{24} + [\overline{\overline{x_1}} + a_1, \overline{\overline{x_3}} + a_3] * q_{25} + [\overline{\overline{x_3}} + a_3, \overline{\overline{x_1}} + a_1] * q_{26} \\
&\quad + [\overline{\overline{x_2}} + a_2, \overline{\overline{x_3}} + a_3] * q_{27} + [\overline{\overline{x_3}} + a_3, \overline{\overline{x_2}} + a_2] * q_{28} + [\overline{\overline{x_3}} + a_3, \overline{\overline{x_3}} + a_3] * q_{29} \\
&= \overline{\overline{x_2}} + a_2 + ([\overline{\overline{x_1}}, a_1] + [a_1, \overline{\overline{x_1}}]) * q_{21} + ([\overline{\overline{x_1}}, a_2] + [a_1, \overline{\overline{x_2}}]) * q_{22} + ([\overline{\overline{x_2}}, a_1] + [a_2, \overline{\overline{x_1}}]) * q_{23} \\
&\quad +([\overline{\overline{x_2}}, a_2] + [a_2, \overline{\overline{x_2}}]) * q_{24} + ([\overline{\overline{x_1}}, a_3] + [a_1, \overline{\overline{x_3}}]) * q_{25} + ([\overline{\overline{x_3}}, a_1] + [a_3, \overline{\overline{x_1}}]) * q_{26} \\
&\quad +([\overline{\overline{x_2}}, a_3] + [a_2, \overline{\overline{x_3}}]) * q_{27} + ([\overline{\overline{x_3}}, a_2] + [a_3, \overline{\overline{x_2}}]) * q_{28} + ([\overline{\overline{x_3}}, a_3] + [a_3, \overline{\overline{x_3}}]) * q_{29} \\
&= \overline{\overline{x_2}} + a_2 + a_1 * (l_{\overline{\overline{x_1}}} q_{21} + r_{\overline{\overline{x_1}}} q_{21} + r_{\overline{\overline{x_2}}} q_{22} + l_{\overline{\overline{x_2}}} q_{23} + r_{\overline{\overline{x_3}}} q_{25} + l_{\overline{\overline{x_3}}} q_{26}) \\
&\quad +a_2 * (l_{\overline{\overline{x_1}}} q_{22} + r_{\overline{\overline{x_1}}} q_{23} + l_{\overline{\overline{x_2}}} q_{24} + r_{\overline{\overline{x_2}}} q_{24} + r_{\overline{\overline{x_3}}} q_{27} + l_{\overline{\overline{x_3}}} q_{28}) \\
&\quad +a_3 * (l_{\overline{\overline{x_1}}} q_{25} + r_{\overline{\overline{x_1}}} q_{26} + l_{\overline{\overline{x_2}}} q_{27} + r_{\overline{\overline{x_2}}} q_{28} + l_{\overline{\overline{x_3}}} q_{29} + r_{\overline{\overline{x_3}}} q_{29}),
\end{aligned}
$$

$$
\begin{aligned}
\mu\alpha(\overline{x_3}) &= \mu(\overline{x_3} + \overline{r_1} * q_{31} + \overline{r_2} * q_{32} + \overline{r_3} * q_{33} + \overline{r_4} * q_{34} + \overline{r_5} * q_{35} + \overline{r_6} * q_{36} + \overline{r_7} * q_{37} + \overline{r_8} * q_{38} + \overline{r_9} * q_{39}) \\
&= \mu(\overline{x_3} + [\overline{x_1}, \overline{x_1}] * q_{31} + [\overline{x_1}, \overline{x_2}] * q_{32} + [\overline{x_2}, \overline{x_1}] * q_{33} + [\overline{x_2}, \overline{x_2}] * q_{34} + [\overline{x_1}, \overline{x_3}] * q_{35} \\
&\quad + [\overline{x_3}, \overline{x_1}] * q_{36} + [\overline{x_2}, \overline{x_3}] * q_{37} + [\overline{x_3}, \overline{x_2}] * q_{38} + [\overline{x_3}, \overline{x_3}] * q_{39}) \\
&= \overline{\overline{x_3}} + a_3 + [\overline{\overline{x_1}} + a_1, \overline{\overline{x_1}} + a_1] * q_{31} + [\overline{\overline{x_1}} + a_1, \overline{\overline{x_2}} + a_2] * q_{32} + [\overline{\overline{x_2}} + a_2, \overline{\overline{x_1}} + a_1] * q_{33} \\
&\quad + [\overline{\overline{x_2}} + a_2, \overline{\overline{x_2}} + a_2] * q_{34} + [\overline{\overline{x_1}} + a_1, \overline{\overline{x_3}} + a_3] * q_{35} + [\overline{\overline{x_3}} + a_3, \overline{\overline{x_1}} + a_1] * q_{36} \\
&\quad + [\overline{\overline{x_2}} + a_2, \overline{\overline{x_3}} + a_3] * q_{37} + [\overline{\overline{x_3}} + a_3, \overline{\overline{x_2}} + a_2] * q_{38} + [\overline{\overline{x_3}} + a_3, \overline{\overline{x_3}} + a_3] * q_{39} \\
&= \overline{\overline{x_3}} + a_3 + ([\overline{\overline{x_1}}, a_1] + [a_1, \overline{\overline{x_1}}]) * q_{31} + ([\overline{\overline{x_1}}, a_2] + [a_1, \overline{\overline{x_2}}]) * q_{32} + ([\overline{\overline{x_2}}, a_1] + [a_2, \overline{\overline{x_1}}]) * q_{33} \\
&\quad +([\overline{\overline{x_2}}, a_2] + [a_2, \overline{\overline{x_2}}]) * q_{34} + ([\overline{\overline{x_1}}, a_3] + [a_1, \overline{\overline{x_3}}]) * q_{35} + ([\overline{\overline{x_3}}, a_1] + [a_3, \overline{\overline{x_1}}]) * q_{36} \\
&\quad +([\overline{\overline{x_2}}, a_3] + [a_2, \overline{\overline{x_3}}]) * q_{37} + ([\overline{\overline{x_3}}, a_2] + [a_3, \overline{\overline{x_2}}]) * q_{38} + ([\overline{\overline{x_3}}, a_3] + [a_3, \overline{\overline{x_3}}]) * q_{39} \\
&= \overline{\overline{x_3}} + a_3 + a_1 * (l_{\overline{\overline{x_1}}} q_{31} + r_{\overline{\overline{x_1}}} q_{31} + r_{\overline{\overline{x_2}}} q_{32} + l_{\overline{\overline{x_2}}} q_{33} + r_{\overline{\overline{x_3}}} q_{35} + l_{\overline{\overline{x_3}}} q_{36}) \\
&\quad +a_2 * (l_{\overline{\overline{x_1}}} q_{32} + r_{\overline{\overline{x_1}}} q_{33} + l_{\overline{\overline{x_2}}} q_{34} + r_{\overline{\overline{x_2}}} q_{34} + r_{\overline{\overline{x_3}}} q_{37} + l_{\overline{\overline{x_3}}} q_{38}) \\
&\quad +a_3 * (l_{\overline{\overline{x_1}}} q_{35} + r_{\overline{\overline{x_1}}} q_{36} + l_{\overline{\overline{x_2}}} q_{37} + r_{\overline{\overline{x_2}}} q_{38} + l_{\overline{\overline{x_3}}} q_{39} + r_{\overline{\overline{x_3}}} q_{39}).
\end{aligned}
$$

Hence, we obtain the following equalities;

$$
\begin{aligned}
\mu\alpha(\overline{x_1}) &= \overline{\overline{x_1}} + a_1 + a_1 * (l_{\overline{\overline{x_1}}}q_{11} + r_{\overline{\overline{x_1}}}q_{11} + r_{\overline{\overline{x_2}}}q_{12} + l_{\overline{\overline{x_2}}}q_{13} + r_{\overline{\overline{x_3}}}q_{15} + l_{\overline{\overline{x_3}}}q_{16}) \\
&\quad + a_2 * (l_{\overline{\overline{x_1}}}q_{12} + r_{\overline{\overline{x_1}}}q_{13} + l_{\overline{\overline{x_2}}}q_{14} + r_{\overline{\overline{x_2}}}q_{14} + r_{\overline{\overline{x_3}}}q_{17} + l_{\overline{\overline{x_3}}}q_{18}) \\
&\quad + a_3 * (l_{\overline{\overline{x_1}}}q_{15} + r_{\overline{\overline{x_1}}}q_{16} + l_{\overline{\overline{x_1}}}q_{17} + r_{\overline{\overline{x_2}}}q_{18} + l_{\overline{\overline{x_3}}}q_{19} + r_{\overline{\overline{x_3}}}q_{19}),
\end{aligned}
$$

$$
\begin{aligned}
\mu\alpha(\overline{x_2}) &= \overline{\overline{x_2}} + a_2 + a_1 * (l_{\overline{\overline{x_1}}}q_{21} + r_{\overline{\overline{x_1}}}q_{21} + r_{\overline{\overline{x_2}}}q_{22} + l_{\overline{\overline{x_2}}}q_{23} + r_{\overline{\overline{x_3}}}q_{25} + l_{\overline{\overline{x_3}}}q_{26}) \\
&\quad + a_2 * (l_{\overline{\overline{x_1}}}q_{22} + r_{\overline{\overline{x_1}}}q_{23} + l_{\overline{\overline{x_2}}}q_{24} + r_{\overline{\overline{x_2}}}q_{24} + r_{\overline{\overline{x_3}}}q_{27} + l_{\overline{\overline{x_3}}}q_{28}) \\
&\quad + a_3 * (l_{\overline{\overline{x_1}}}q_{25} + r_{\overline{\overline{x_1}}}q_{26} + l_{\overline{\overline{x_2}}}q_{27} + r_{\overline{\overline{x_2}}}q_{28} + l_{\overline{\overline{x_3}}}q_{29} + r_{\overline{\overline{x_3}}}q_{29}),
\end{aligned}
$$

$$
\begin{aligned}
\mu\alpha(\overline{x_3}) &= \overline{\overline{x_3}} + a_3 + a_1 * (l_{\overline{\overline{x_1}}}q_{31} + r_{\overline{\overline{x_1}}}q_{31} + r_{\overline{\overline{x_2}}}q_{32} + l_{\overline{\overline{x_2}}}q_{33} + r_{\overline{\overline{x_3}}}q_{35} + l_{\overline{\overline{x_3}}}q_{36}) \\
&\quad + a_2 * (l_{\overline{\overline{x_1}}}q_{32} + r_{\overline{\overline{x_1}}}q_{33} + l_{\overline{\overline{x_2}}}q_{34} + r_{\overline{\overline{x_2}}}q_{34} + r_{\overline{\overline{x_3}}}q_{37} + l_{\overline{\overline{x_3}}}q_{38}) \\
&\quad + a_3 * (l_{\overline{\overline{x_1}}}q_{35} + r_{\overline{\overline{x_1}}}q_{36} + l_{\overline{\overline{x_2}}}q_{37} + r_{\overline{\overline{x_2}}}q_{38} + l_{\overline{\overline{x_3}}}q_{39} + r_{\overline{\overline{x_3}}}q_{39}).
\end{aligned}
$$

By the equality $\mu\alpha = \widetilde{\alpha}\mu$ from Proposition 3.1 and the definition of the $\overline{AutW}$, we find that $\widetilde{\alpha}$ restricted to $I_{F'/F''}$ has a corresponding matrix M of the form

$$
\begin{bmatrix} 1+B_{11} & B_{12} & B_{13} \\ B_{21} & 1+B_{22} & B_{23} \\ B_{31} & B_{32} & 1+B_{33} \end{bmatrix}_{3\times 3}
$$

where

$$
\begin{aligned}
B_{11} &= l_{\overline{\overline{x_1}}}q_{11} + r_{\overline{\overline{x_1}}}q_{11} + r_{\overline{\overline{x_2}}}q_{12} + l_{\overline{\overline{x_2}}}q_{13} + r_{\overline{\overline{x_3}}}q_{15} + l_{\overline{\overline{x_3}}}q_{16}, \\
B_{12} &= l_{\overline{\overline{x_1}}}q_{12} + r_{\overline{\overline{x_1}}}q_{13} + l_{\overline{\overline{x_2}}}q_{14} + r_{\overline{\overline{x_2}}}q_{14} + r_{\overline{\overline{x_3}}}q_{17} + l_{\overline{\overline{x_3}}}q_{18}, \\
B_{13} &= l_{\overline{\overline{x_1}}}q_{15} + r_{\overline{\overline{x_1}}}q_{16} + l_{\overline{\overline{x_1}}}q_{17} + r_{\overline{\overline{x_2}}}q_{18} + l_{\overline{\overline{x_3}}}q_{19} + r_{\overline{\overline{x_3}}}q_{19}, \\
B_{21} &= l_{\overline{\overline{x_1}}}q_{21} + r_{\overline{\overline{x_1}}}q_{21} + r_{\overline{\overline{x_2}}}q_{22} + l_{\overline{\overline{x_2}}}q_{23} + r_{\overline{\overline{x_3}}}q_{25} + l_{\overline{\overline{x_3}}}q_{26}, \\
B_{22} &= l_{\overline{\overline{x_1}}}q_{22} + r_{\overline{\overline{x_1}}}q_{23} + l_{\overline{\overline{x_2}}}q_{24} + r_{\overline{\overline{x_2}}}q_{24} + r_{\overline{\overline{x_3}}}q_{27} + l_{\overline{\overline{x_3}}}q_{28}, \\
B_{23} &= l_{\overline{\overline{x_1}}}q_{25} + r_{\overline{\overline{x_1}}}q_{26} + l_{\overline{\overline{x_2}}}q_{27} + r_{\overline{\overline{x_2}}}q_{28} + l_{\overline{\overline{x_3}}}q_{29} + r_{\overline{\overline{x_3}}}q_{29}, \\
B_{31} &= l_{\overline{\overline{x_1}}}q_{31} + r_{\overline{\overline{x_1}}}q_{31} + r_{\overline{\overline{x_2}}}q_{32} + l_{\overline{\overline{x_2}}}q_{33} + r_{\overline{\overline{x_3}}}q_{35} + l_{\overline{\overline{x_3}}}q_{36}, \\
B_{32} &= l_{\overline{\overline{x_1}}}q_{32} + r_{\overline{\overline{x_1}}}q_{33} + l_{\overline{\overline{x_2}}}q_{34} + r_{\overline{\overline{x_2}}}q_{34} + r_{\overline{\overline{x_3}}}q_{37} + l_{\overline{\overline{x_3}}}q_{38}, \\
B_{33} &= l_{\overline{\overline{x_1}}}q_{35} + r_{\overline{\overline{x_1}}}q_{36} + l_{\overline{\overline{x_2}}}q_{37} + r_{\overline{\overline{x_2}}}q_{38} + l_{\overline{\overline{x_3}}}q_{39} + r_{\overline{\overline{x_3}}}q_{39}.
\end{aligned}
$$

Since $\tilde{\alpha}$ is an automorphism, M is two-sided invertible. We write the transpose of M of the form $E + AQ$, where

$$
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3},
$$

$$A=\begin{bmatrix} l_{\overline{\overline{x_1}}}+r_{\overline{\overline{x_1}}} & r_{\overline{\overline{x_2}}} & l_{\overline{\overline{x_2}}} & 0 & 0 & r_{\overline{\overline{x_3}}} & l_{\overline{\overline{x_3}}} & 0 & 0 \\ 0 & l_{\overline{\overline{x_1}}} & r_{\overline{\overline{x_1}}} & l_{\overline{\overline{x_2}}}+r_{\overline{\overline{x_2}}} & 0 & 0 & r_{\overline{\overline{x_3}}} & l_{\overline{\overline{x_3}}} & 0 \\ 0 & 0 & 0 & 0 & l_{\overline{\overline{x_1}}} & r_{\overline{\overline{x_1}}} & l_{\overline{\overline{x_2}}} & r_{\overline{\overline{x_2}}} & l_{\overline{\overline{x_3}}}+r_{\overline{\overline{x_3}}} \end{bmatrix}_{3\times 9},$$

$$Q=\begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \\ q_{14} & q_{24} & q_{34} \\ q_{15} & q_{25} & q_{35} \\ q_{16} & q_{26} & q_{36} \\ q_{17} & q_{27} & q_{37} \\ q_{18} & q_{28} & q_{38} \\ q_{19} & q_{29} & q_{39} \end{bmatrix}_{9\times 3}.$$

Conversely, let $C=E+AQ$ be invertible which defines as above form. There is an automorphism $\widetilde{\alpha}$ of W that is if $\widetilde{\alpha}$ is restricted to $I_{F'/F''}$ then the transpose matrix C^T of C determines this automorphism. Hence, we obtain

$$\widetilde{\alpha}(\overline{\overline{x_i}})=\overline{\overline{x_i}},$$

$$\widetilde{\alpha}(a_i)=a_i+\sum\sum a_k * a_{kj}q_{ij}.$$

By the equality $\widetilde{\alpha}\mu=\mu\alpha$, it yields

$$\begin{aligned} \widetilde{\alpha}(\mu(\overline{x_i})) &= \widetilde{\alpha}(\overline{\overline{x_i}}+a_i) \\ &= \overline{\overline{x_i}}+a_i+\sum\sum a_k * a_{kj}q_{ij} \\ &= \mu(\overline{x_i}+\sum\overline{r_j} * q_{ij}). \end{aligned}$$

Therefore, there is an automorphism α of F/F'' which is identical modulo F'/F'' defined by $\alpha(\overline{x_i}) = \overline{x_i}+\sum\overline{r_j} * q_{ij}$. □

As an application of the Theorem 3.2, we give the following corollary.

Corollary 3.3 *Let $M=E+AQ$ be as in the proof of Theorem 3.2.*
i) If $q_{14}=r_{\overline{\overline{x_1}}}$ and all other $q_{ij}=l_{\overline{\overline{x_1}}}$, then there is an automorphism

$$\begin{aligned} \zeta_1 \quad &: \quad \overline{x_1}\to\overline{x_1}+[[\overline{x_2},\overline{x_2}],\overline{x_1}] \\ \overline{x_2} \quad &\to \quad \overline{x_2} \\ \overline{x_3} \quad &\to \quad \overline{x_3} \end{aligned}$$

whose associated matrix is M^T.
ii) If $q_{31}=r_{\overline{\overline{x_3}}}$, $q_{21}=r_{\overline{\overline{x_2}}}$ and all other $q_{ij}=l_{\overline{\overline{x_1}}}$ then there is an automorphism

$$\begin{aligned} \zeta_2 \quad &: \quad \overline{x_1}\to\overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2}+[[\overline{x_2},\overline{x_2}],\overline{x_2}] \\ \overline{x_3} \quad &\to \quad \overline{x_3}+[[\overline{x_1},\overline{x_1}],\overline{x_3}] \end{aligned}$$

whose associated matrix is M^T.

iii) If $q_{24} = r_{\overline{\overline{x_1}}}$*,* $q_{21} = 1$*,* $q_{14} = 1$ *and all other* $q_{ij} = l_{\overline{\overline{x_1}}}$ *then* M^T *is the associated matrix of the following automorphism.*

$$\begin{aligned} \zeta_3 \quad &: \quad \overline{x_1} \to \overline{x_1} + [\overline{x_2}, \overline{x_2}] \\ \overline{x_2} \quad &\to \quad \overline{x_2} + [\overline{x_1}, \overline{x_1}] + [[\overline{x_2}, \overline{x_2}], \overline{x_1}] \\ \overline{x_3} \quad &\to \quad \overline{x_3} \end{aligned}$$

Proof *i)* If $q_{14} = r_{\overline{\overline{x_1}}}$ and all other $q_{ij} = l_{\overline{\overline{x_1}}}$, then

$$M = \begin{bmatrix} 1 & 0 & 0 \\ (l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_2}}}) r_{\overline{\overline{x_1}}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M^T = \begin{bmatrix} 1 & (l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_2}}}) r_{\overline{\overline{x_1}}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $det M^T = 1$, then there is an automorphism whose associated matrix is M^T, by Theorem 3.2 it is

$$\begin{aligned} \zeta_1 \quad &: \quad \overline{x_1} \to \overline{x_1} + [[\overline{x_2}, \overline{x_2}], \overline{x_1}] \\ \overline{x_2} \quad &\to \quad \overline{x_2} \\ \overline{x_3} \quad &\to \quad \overline{x_3}. \end{aligned}$$

ii) If $q_{31} = r_{\overline{\overline{x_3}}}$, $q_{21} = r_{\overline{\overline{x_2}}}$ and all other $q_{ij} = l_{\overline{\overline{x_1}}}$, then M is of the form

$$\begin{bmatrix} 1 & (r_{\overline{\overline{x_1}}} + l_{\overline{\overline{x_1}}}) r_{\overline{\overline{x_2}}} & (r_{\overline{\overline{x_1}}} + l_{\overline{\overline{x_1}}}) r_{\overline{\overline{x_3}}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M^T = \begin{bmatrix} 1 & 0 & 0 \\ (r_{\overline{\overline{x_1}}} + l_{\overline{\overline{x_1}}}) r_{\overline{\overline{x_2}}} & 1 & 0 \\ (r_{\overline{\overline{x_1}}} + l_{\overline{\overline{x_1}}}) r_{\overline{\overline{x_3}}} & 0 & 1 \end{bmatrix}.$$

Since $det M^T = 1$, then there is an automorphism whose associated matrix is M^T and by Theorem 3.2 it is

$$\begin{aligned} \zeta_2 \quad &: \quad \overline{x_1} \to \overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2} + [[\overline{x_1}, \overline{x_1}], \overline{x_2}] \\ \overline{x_3} \quad &\to \quad \overline{x_3} + [[\overline{x_1}, \overline{x_1}], \overline{x_3}]. \end{aligned}$$

iii) If $q_{24} = r_{\overline{\overline{x_1}}}$, $q_{21} = 1, q_{14} = 1$ and all other $q_{ij} = l_{\overline{\overline{x_1}}}$ then

$$M = \begin{bmatrix} 1 & l_{\overline{\overline{x_1}}} + r_{\overline{\overline{x_1}}} & 0 \\ l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_2}}} & 1 + (l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_2}}}) r_{\overline{\overline{x_1}}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M^T = \begin{bmatrix} 1 & l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_2}}} & 0 \\ l_{\overline{\overline{x_1}}} + r_{\overline{\overline{x_1}}} & 1 + (l_{\overline{\overline{x_2}}} + r_{\overline{\overline{x_2}}})r_{\overline{\overline{x_1}}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then there is an automorphism whose associated matrix has the above form which is

$$\begin{aligned} \zeta_3 \quad &: \quad \overline{x_1} \to \overline{x_1} + [\overline{x_2}, \overline{x_2}] \\ \overline{x_2} \quad &\to \quad \overline{x_2} + [\overline{x_1}, \overline{x_1}] + [[\overline{x_2}, \overline{x_2}], \overline{x_1}] \\ \overline{x_3} \quad &\to \quad \overline{x_3}. \end{aligned}$$

□

The following theorem gives the set of generators of $IA(F/F'')$.

Theorem 3.4 *Let* F/F'' *be the free metabelian Leibniz algebra of rank three generated by* $\{\overline{x_1}, \overline{x_2}, \overline{x_3}\}$. $IA(F/F'')$ *is generated by an inner automorphism and the following automorphisms;*

$$\begin{aligned} \tau_1 \quad &: \quad \overline{x_1} \to \overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2} + [z, \overline{x_2}] \\ \overline{x_3} \quad &\to \quad \overline{x_3} + [z, \overline{x_3}] \end{aligned}$$

where $z = [\overline{x_1}, \overline{x_1}] * z_1,\ z_1 \in U(F/F')$,

$$\begin{aligned} \tau_2 \quad &: \quad \overline{x_1} \to \overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2} + u \\ \overline{x_3} \quad &\to \quad \overline{x_3} \end{aligned}$$

where $u = [\overline{x_t}, \overline{x_t}] * u_1,\ u_1 \in U(F/F'),\ t \neq 2$ *and*

$$\begin{aligned} \tau_3 \quad &: \quad \overline{x_1} \to \overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2} \\ \overline{x_3} \quad &\to \quad \overline{x_3} + v \end{aligned}$$

where $v = [\overline{x_1}, \overline{x_2}] * v_1, v_1 \in U(F/F')$.

Proof Let

$$\varphi : F/F'' \to F/F''$$

be an automorphism of the free metabelian Leibniz algebra F/F'' with the free generators $\overline{x_1}$, $\overline{x_2}$, and $\overline{x_3}$ which acts identically modulo F'/F''. For $h \in F/F''$ and $x \in F'/F''$, $\varphi[h,x] = [\varphi(h), x] = [h,x]$, hence $[\varphi(h) - h, x] = 0$ and $\varphi(h) - h \in F'/F''$. Then $\varphi(h) - h = 0$ on $(F/F'')/(F'/F'')$. Therefore, φ is also the identity on F/F'.

By Corollary 2.2, F/F'' is embedded in the wreath product $(F'/F'')wr(F/F')$. Hence, we see F/F'' is a free metabelian Leibniz algebra with free generators $g_i = \overline{\overline{x_i}} + a_i, i = 1,2,3$ which are elements of the wreath product $(F'/F'')wr(F/F')$. Therefore, $\varphi(g_i) = g_i + u_i$, $u_i \in F'/F'', i = 1,2,3$. We have $\varphi[g_1, g_2] = [g_1, g_2]$ and

$$\begin{aligned}\varphi[g_1,g_2] &= [\varphi(g_1),\varphi(g_2)] = [g_1+u_1, g_2+u_2]\\ &= [g_1,g_2]+[u_1,g_2]+[g_1,u_2]+[u_1,u_2].\end{aligned}$$

The element $[u_1, u_2]$ is equal zero in the algebra F/F''. Hence, $[u_1, g_2] + [g_1, u_2] = 0$. In the equality $[u_1, g_2] + [g_1, u_2] = 0$ substituting $\overline{\overline{x_1}} + a_1$ for g_1, we obtain

$$\left[u_1, \overline{\overline{x_2}}\right] + [u_1, a_2] + \left[\overline{\overline{x_1}}, u_2\right] + [a_1, u_2] = 0.$$

The elements $[a_1, u_2]$ and $[u_1, a_2]$ are zero in the algebra F/F''. Thus, we have

$$\left[u_1, \overline{\overline{x_2}}\right] + \left[\overline{\overline{x_1}}, u_2\right] = 0.$$

Case 1: Let $u_2 = z * l_{\overline{\overline{x_2}}}$, where $z \in F'/F''$, $l_{\overline{\overline{x_2}}} \in U(F/F')$. Substituting $u_2 = z * l_{\overline{\overline{x_2}}}$ in the equality $\left[u_1, \overline{\overline{x_2}}\right] + \left[\overline{\overline{x_1}}, u_2\right] = 0$, we obtain

$$\begin{aligned}\left[u_1,\overline{\overline{x_2}}\right] &= -\left[\overline{\overline{x_1}},u_2\right]\\ &= -\left[\overline{\overline{x_1}}, z*l_{\overline{\overline{x_2}}}\right]\\ &= -\left[\overline{\overline{x_1}},\left[\overline{\overline{x_2}},z\right]\right]\\ &= \left[\overline{\overline{x_1}},\left[z,\overline{\overline{x_2}}\right]\right].\end{aligned}$$

By the Leibniz identity $\left[\overline{\overline{x_1}},\left[z,\overline{\overline{x_2}}\right]\right] = \left[\left[\overline{\overline{x_1}},z\right],\overline{\overline{x_2}}\right] - \left[\left[\overline{\overline{x_1}},\overline{\overline{x_2}}\right],z\right] = \left[\left[\overline{\overline{x_1}},z\right],\overline{\overline{x_2}}\right]$. Hence, $\left[u_1,\overline{\overline{x_2}}\right] = \left[\left[\overline{\overline{x_1}},z\right],\overline{\overline{x_2}}\right]$ and

$$u_1 = \left[\overline{\overline{x_1}}, z\right] = z * l_{\overline{\overline{x_1}}} \text{ or } u_1 = [g_1, z].$$

Thus, we obtain $u_1 = z * l_{\overline{\overline{x_1}}}$. Also if $\varphi([g_1, g_3]) = [g_1, g_3]$, we get $u_3 = z * l_{\overline{\overline{x_3}}}$. Hence, we obtain that for every $i \in I$

$$\varphi(g_i) = g_i + [g_i, z].$$

Since z lies in F'/F'', φ is an inner automorphism.

Case 2 : Let $u_2 = z * r_{\overline{\overline{x_2}}}$, where $z \in F'/F''$, $r_{\overline{\overline{x_2}}} \in U(F/F')$. From the equality $\left[u_1, \overline{\overline{x_2}}\right] + \left[\overline{\overline{x_1}}, u_2\right] = 0$,

$$\begin{aligned} \left[u_1, \overline{\overline{x_2}}\right] &= -\left[\overline{\overline{x_1}}, u_2\right] \\ &= -\left[\overline{\overline{x_1}}, z * r_{\overline{\overline{x_2}}}\right] \\ &= -\left[\overline{\overline{x_1}}, \left[z, \overline{\overline{x_2}}\right]\right] \\ &= -\left[\left[\overline{\overline{x_1}}, z\right], \overline{\overline{x_2}}\right] + \left[\left[\overline{\overline{x_1}}, \overline{\overline{x_2}}\right], z\right] \\ &= -\left[\left[\overline{\overline{x_1}}, z\right], \overline{\overline{x_2}}\right] \end{aligned}$$

and we obtain that $u_1 = -\left[\overline{\overline{x_1}}, z\right]$ or $u_1 = -[g_1, z]$. Now we will find u_3. There are two cases:

i) If $\varphi([g_1, g_3]) = [g_1, g_3]$, we get that $u_3 = \left[z, \overline{\overline{x_3}}\right]$ or $u_3 = [z, g_3]$. Hence, we obtain

$$\begin{array}{rcl} \varphi & : & g_1 \to g_1 - [g_1, z] \\ g_2 & \to & g_2 + [z, g_2] \\ g_3 & \to & g_3 + [z, g_3]\,. \end{array}$$

By Theorem 3.2, z is $[g_1, g_1] * z_1, z_1 \in U(F/F')$. Therefore, the automorphism is

$$\begin{array}{rcl} \varphi & : & g_1 \to g_1 \\ g_2 & \to & g_2 + [z, g_2] \\ g_3 & \to & g_3 + [z, g_3]\,. \end{array}$$

ii) If $\varphi([g_3, g_1]) = [g_3, g_1]$, we get that $u_3 = -\left[\overline{\overline{x_3}}, z\right]$ or $u_3 = -[g_3, z]$. Then we see that

$$\begin{array}{rcl} \varphi & : & g_1 \to g_1 - [g_1, z] \\ g_2 & \to & g_2 + [z, g_2] \\ g_3 & \to & g_3 - [g_3, z] \end{array}$$

and by Theorem 3.2, $z \in Ann(F/F'')$. It is obvious that $[g_1, z] = [g_3, z] = 0$. We know that if φ is an automorphism, $\{g_1, g_2 + [z, g_2], g_3\}$ is a free generating set and

$$\begin{aligned} F/F'' &= g_1 U\left(F/F'\right) \oplus g_2 U\left(F/F'\right) \oplus g_3 U\left(F/F'\right) \\ &= g_1 U\left(F/F'\right) \oplus \left(g_2 + [z, g_2]\right) U\left(F/F'\right) \oplus g_3 U\left(F/F'\right). \end{aligned}$$

Thus, $[z, g_2] \in g_1 U\left(F/F'\right)$ or $[z, g_2] \in g_3 U\left(F/F'\right)$. As a result of this $z \in Ann(F/F'') \cap g_1 U\left(F/F'\right)$ or $z \in Ann(F/F'') \cap g_3 U\left(F/F'\right)$. Hence, the automorphism is of the form

$$\begin{array}{rcl} \varphi & : & g_1 \to g_1 \\ g_2 & \to & g_2 + [z, g_2] \\ g_3 & \to & g_3 \end{array}$$

where $z = [g_t, g_t] * z_1$, $z_1 \in U(F/F')$, $t \neq 2$.

Case 3 : For every $u_2 \in [g_t, g_t] * U(F/F'), t \neq 2$, since $\left[\overline{\overline{x_1}}, u_2\right] = 0$ by the Leibniz identity, then $\left[u_1, \overline{\overline{x_2}}\right] = 0$ and it yields $u_1 = 0$. Now let us calculate u_3. If also $\varphi([g_3, g_1]) = [g_3, g_1]$, we get that $\left[\overline{\overline{x_3}}, u_1\right] = 0$ by the Leibniz identity, then $\left[u_3, \overline{\overline{x_1}}\right] = 0$ and it yields $u_3 = 0$. Therefore, we get

$$\begin{aligned} \varphi \quad &: \quad g_1 \to g_1 \\ g_2 \quad &\to \quad g_2 + u_2 \\ g_3 \quad &\to \quad g_3, \end{aligned}$$

where $u_2 = [g_t, g_t] * u_2',\ u_2' \in U(F/F'), t \neq 2$. Given

$$\begin{aligned} \varphi^{-1} \quad &: \quad g_1 \to g_1 \\ g_2 \quad &\to \quad g_2 - u_2 \\ g_3 \quad &\to \quad g_3. \end{aligned}$$

Since $\varphi^{-1} \circ \varphi = 1$ and $\varphi \circ \varphi^{-1} = 1$, φ is an automorphism.

Case 4 : Let $u_2 = 0$, then $u_1 = 0$. Now lets determine u_3.

i) Let $u_3 = [\overline{x_1}, \overline{x_2}] * u_3'$ or $u_3 = [\overline{x_2}, \overline{x_1}] * u_3',\ u_3' \in U(F/F')$, then φ is an automorphism by Theorem 3.2 and the automorphism is

$$\begin{aligned} \varphi \quad &: \quad g_1 \to g_1 \\ g_2 \quad &\to \quad g_2 \\ g_3 \quad &\to \quad g_3 + u_3, \end{aligned}$$

where $u_3 = [g_1, g_2] * u_3'$ or $u_3 = [g_2, g_1] * u_3', u_3' \in U(F/F')$. This automorphism is an elementary automorphism.

ii) If we take u_3 as one of $[\overline{x_1}, \overline{x_3}] * u_3'$, $[\overline{x_3}, \overline{x_1}] * u_3'$, $[\overline{x_3}, \overline{x_2}] * u_3'$, $[\overline{x_2}, \overline{x_3}] * u_3'$ or $[\overline{x_3}, \overline{x_3}] * u_3'$, $u_3' \in U(F/F')$, by the Theorem 3.2, φ is not an automorphism.

iii) For $u_3 = [\overline{x_t}, \overline{x_t}] * u_3',\ u_3' \in U(F/F')$, $t \neq 3$, we get the same result as in Case 3. □

4. Automorphisms of F/F''

For every $g = (g_{ij}) \in GL(3, K)$, the general lineer group over K, the mapping

$$g : x_j \ \longrightarrow \sum g_{ij}.x_i, j = 1, 2, 3$$

extends uniquely to an algebra automorphism of F/F'' and $GL(3, K)$ acts on $U\left(F/F'\right)$ as a group of algebra automorphism. We write $g \cdot f$ for the action where $g \in GL(3, K)$, $f \in U\left(F/F'\right)$ and

$$g \cdot x_j = \sum g_{ij}.x_i.$$

Thus, we consider $GL(3, K)$ as a subgroup of $Aut(F/F'')$. Since $IA(F/F'')$ is a normal subgroup of $Aut(F/F'')$ and $GL(3, K) \cap IA(F/F'') = \{1\}$, we obtain that $Aut(F/F'')$ is semidirect product of $IA(F/F'')$ by $GL(3, K)$. Thus, every automorphism α of F/F'' is uniquely written as $\varphi \circ g$, where $\varphi \in IA(F/F'')$ and $g \in GL(3, K)$. By Theorem 3.2, $IA(F/F'')$ is isomorphic to the group G. Denote this isomorphism by $\eta :\ IA(F/F'') \longrightarrow G$. The action of $GL(3, K)$ on $IA(F/F'')$ is given by

$$b \circ \varphi \circ b^{-1} = \eta^{-1}\left(E + b(b \cdot (\varphi_{ij}))b^{-1}\right),$$

where $b \in GL(3,K)$, $E+(\varphi_{ij}) = E+AQ$ is the corresponding matrix of the automorphism φ of $IA(F/F'')$ and η^{-1} is inverse of η. Hence $IA(F/F'')$ is a $GL(3,K)-$ module. By Theorem 3.4 we know the generators of the $IA(F/F'')$. Thus, we have proved the following theorem.

Theorem 4.1 *Let F/F'' be the free metabelian Leibniz algebra of rank three generated by $\overline{x_1}, \overline{x_2}$, and $\overline{x_3}$. The automorphism group of F/F'' is generated by the general lineer group $GL(3,K)$ together with the inner automorphism e^{adv} $(v \in F'/F'')$ and the following automorphisms;*

$$\begin{aligned} \tau_1 \quad &: \quad \overline{x_1} \to \overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2} + [z, \overline{x_2}] \\ \overline{x_3} \quad &\to \quad \overline{x_3} + [z, \overline{x_3}]\,, \end{aligned}$$

where $z = [\overline{x_1}, \overline{x_1}] * z_1$, $z_1 \in U(F/F')$,

$$\begin{aligned} \tau_2 \quad &: \quad \overline{x_1} \to \overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2} + u \\ \overline{x_3} \quad &\to \quad \overline{x_3} \end{aligned}$$

where $u = [\overline{x_t}, \overline{x_t}] * u_1$, $u_1 \in U(F/F')$, $t \neq 2$ and

$$\begin{aligned} \tau_3 \quad &: \quad \overline{x_1} \to \overline{x_1} \\ \overline{x_2} \quad &\to \quad \overline{x_2} \\ \overline{x_3} \quad &\to \quad \overline{x_3} + v \end{aligned}$$

where $v = [\overline{x_1}, \overline{x_2}] * v_1, v_1 \in U(F/F')$.

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