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Automorphisms of free metabelian Leibniz algebras of rank three

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Abstract: In this work, we determine the structure of the automorphism group of the free metabelian Leibniz algebra of rank three over a field K of characteristic zero.

Key words: Free metabelian Leibniz algebras, automorphisms, wreath product

1. Introduction

A Leibniz algebra L over a field K is a nonassociative algebra with multiplication called bracket $[\cdot, \cdot] : L \times L \rightarrow L$ satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all $x, y, z \in L$. If the condition $[x, x] = 0$ for all $x \in L$ is satisfied, this identity is equivalent to the Jacobi identity. Let X be a set, $L(X)$ be the free nonassociative algebra on X over K and I_L be the two-sided ideal of $L(X)$ generated by the elements

$$[a, [b, c]] - [[a, b], c] + [[a, c], b]$$

for all $a, b, c \in L(X)$. Then the algebra $F(X) = L(X)/I_L$ is a free Leibniz algebra with the free generating set X .

In [5], the universal enveloping algebra of the Leibniz algebra L was defined. Let L^l and L^r be two copies of the Leibniz algebra L . We denote by l_x and r_x , the elements of L^l and L^r corresponding to the universal operators of left and right multiplication on x , respectively. Let I_T be the two-sided ideal of the associative tensor K -algebra $T(L^l \oplus L^r)$ with the identity element corresponding to the relations

$$\begin{aligned} r_{[x,y]} &= r_x \cdot r_y - r_y \cdot r_x \\ l_{[x,y]} &= l_x \cdot r_y - r_y \cdot l_x \\ (r_x + l_x) \cdot l_y &= 0 \end{aligned}$$

for any $x, y \in L$. Then the factor algebra $UL(L) = T(L^l \oplus L^r)/I_T$ is the universal enveloping algebra of Leibniz algebra L .

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Let M_n be the free metabelian Leibniz algebra of finite rank n over a field K . Denote by $Aut(M_n)$ the automorphism group of M_n . The kernel of a natural homomorphism $\pi: Aut(M_n) \rightarrow Aut(M_n/M'_n)$ consist of automorphisms which act identically modulo M'_n , where M'_n derived subalgebra. It is called the group of IA -automorphisms and denoted by $IA(M_n)$.

In [6], IA -automorphisms of 2-generator metabelian Lie algebras were studied. In [8], the defining relations of the subgroup of IA -automorphisms of a free metabelian Lie algebra of rank 3 were given. In [7], generating sets of IA -automorphisms of a free metabelian Lie algebra of rank 3 were investigated. In 1973, Shmel'kin [9] noticed that all nonlinear automorphisms of a free metabelian Lie algebra of rank two are wild. In [2], an explicit matrix form for the IA -automorphisms of a free metabelian Lie algebra of rank n was obtained.

Leibniz algebras were first introduced by Loday [5] as a nonantisymmetric version of Lie algebras. In 2001 Abdykhalikov et al. [1] obtained a characterization of tame automorphisms of the free Leibniz algebra of rank 2. In [4], Papistas and Drensky proved that the automorphism group of free nilpotent Leibniz algebras of finite rank m , $m \geq 2$, is generated by the tame automorphisms and one more given nontame automorphism. In [3], a description of the free metabelian Leibniz algebras was given.

In the present article, we study the automorphisms of the free metabelian Leibniz algebra M_3 . First, we obtain an explicit matrix form for the IA -automorphisms of M_3 and describe a set of generators of the group $IA(M_3)$, where our approach is via wreath products. Then our main result states that the automorphism group $Aut(M_3)$ is generated by the linear automorphisms and the set of generators of the group $IA(M_3)$.

2. Preliminaries

Let F be the free Leibniz algebra with the free generators x_1, x_2 , and x_3 . We denote by F' and F'' the derived subalgebra of F and F' , respectively and F/F' is a free K -module. We fix the notation F/F'' for the free metabelian Leibniz algebra. Denote by $Aut(F/F'')$ the automorphism group of F/F'' . By $Ann(F/F'')$ we denote the ideal of F/F'' generated by elements $\{[a, a] : a \in F/F''\}$. It is known (see [5]) that $r_a = 0 \Leftrightarrow a \in Ann(F/F'')$. The algebra $(F/F'')_{Lie} = (F/F'')/Ann(F/F'')$ is the free metabelian Lie algebra of rank three.

By [3], F/F'' has a basis

$$\{x_{i_1}, [x_{i_1}, x_{i_2}], [x_{i_1}, x_{i_2}, \dots, x_{i_k}] \mid 1 \leq i_1, i_2 \leq 3, 1 \leq i_3 \leq \dots \leq i_k \leq 3, k = 3, 4, \dots\}.$$

We define the wreath product of abelian Leibniz algebras in a standard way, as in the case of Lie algebras [9]. Let W be the wreath product of F'/F'' and F/F' where F'/F'' is an abelian Leibniz algebra that is a free K -module with the free generating set $\{a_i\}_{i \in I}$ and F/F' is a Leibniz algebra over K . We write shortly $W = (F'/F'')wr(F/F')$ and it has the form $W = F/F' \oplus I_{F'/F''}$, where it is the semidirect sum of F/F' and free F/F' -module $I_{F'/F''}$ with the free generating set $\{a_i\}_{i \in I}$. In addition, F'/F'' is a module on F/F' , is also a $U(F/F')$ -module and the module action is given by

$$\begin{aligned} v * r_u &= [v, u] \\ v * l_u &= [u, v] \end{aligned}$$

for $v \in F'/F''$, $u \in F/F'$ and $r_u, l_u \in U(F/F')$. Denote $x_i + F'' \in F/F''$ by $\overline{x_i}$ and $x_i + F' \in F/F'$ by $\overline{\overline{x_i}}$.

For our aim, it is sufficient to restrict the automorphism group $AutW$ to its subgroup \overline{AutW} whose elements leave $I_{F'/F''}$ and F/F' invariant. \overline{AutW} contains a normal subgroup $IA(W)$ whose elements are identical on F/F' and a subgroup P whose elements leave each $a_i \in I_{F'/F''}$ invariant having trivial intersection. Thus, we have \overline{AutW} is the semidirect product of P and $IA(W)$.

The proof of the next Lemma is the same as in the Lie algebra case [9].

Lemma 2.1 *Let L be a Leibniz algebra and S be an abelian ideal of L . Then there exists a semidirect sum $H = L/S + I$, where I is an abelian ideal, L/S is a subalgebra and a monomorphism $\mu : L \rightarrow H$ such that $\mu(L) \cap L = \mu(S)$.*

Corollary 2.2 *The mapping $\overline{x_i} \rightarrow \overline{\overline{x_i}} + a_i$ extends to a monomorphism $\mu : F/F'' \rightarrow (F'/F'')wr(F/F')$.*

Proof The map $\overline{x_i} \rightarrow \overline{\overline{x_i}} + a_i$ extends to a homomorphism

$$\mu : F/F'' \rightarrow F/F' \oplus I_{F'/F''}.$$

By Lemma 2.1, μ is a monomorphism. □

3. IA-automorphisms of F/F''

Using similar arguments as in Lie algebras (see Bahturin and Nabiev [2]), we may prove the following proposition.

Proposition 3.1 *There exists an embedding $\vartheta : \overline{Aut(F/F'')} \rightarrow \overline{Aut((F'/F'')wr(F/F'))}$ such that if $\alpha \in \overline{Aut(F/F'')}$ which leaves F'/F'' invariant and $\tilde{\alpha} = \vartheta(\alpha)$ then $\tilde{\alpha}\mu = \mu\alpha$ where μ is the embedding of Corollary 2.2.*

Proof [2] First given an automorphism $\alpha : F/F'' \rightarrow F/F''$ with $\alpha(F'/F'') \subset F'/F''$, we define $\tilde{\alpha}$ by first writing

$$\begin{aligned} \mu\alpha(x_i + F'') &= \mu(\alpha(x_i) + F'') \\ &= \mu(\overline{\alpha(x_i)}) \\ &= \overline{\overline{\alpha(x_i)}} + p_i, \end{aligned}$$

where $\overline{\overline{\alpha(x_i)}} \in F/F'$, $p_i \in I_{F'/F''}$. On the other hand,

$$\begin{aligned} \tilde{\alpha}\mu(\overline{x_i}) &= \tilde{\alpha}(\overline{\overline{x_i}} + a_i) \\ &= \tilde{\alpha}(\overline{\overline{x_i}}) + \tilde{\alpha}(a_i) \end{aligned}$$

and then setting $\tilde{\alpha}(\overline{\overline{x_i}}) = \overline{\overline{\alpha(x_i)}}$, $\tilde{\alpha}(a_i) = p_i$,

$$\tilde{\alpha}(\mu(\overline{x_i})) = \mu(\alpha(\overline{x_i}))$$

satisfied. Since $\tilde{\alpha}$ is induced by α which leaves F'/F'' invariant, then $\tilde{\alpha} \in Aut(F/F')$. $\tilde{\alpha}$ extends to a uniquely defined map of W by the definition of the wreath product. Let α^{-1} be the inverse of α . Applying α^{-1} to

both side of the equality $\tilde{\alpha}\mu = \mu\alpha$, it holds $\mu = \mu\alpha\alpha^{-1} = \tilde{\alpha}\mu\alpha^{-1} = \tilde{\alpha}\widetilde{\alpha^{-1}}\mu$ and $\tilde{\alpha}\widetilde{\alpha^{-1}} = \tilde{1}_{F/F''}$. Since $1_W = \tilde{1}_{F/F''}$, we have $\tilde{\alpha}\widetilde{\alpha^{-1}} = 1_W$. Hence, $\tilde{\alpha}$ is an automorphism of W . \square

Using this argument, the structure of $IA(F/F'')$ can be described in the following way.

Theorem 3.2 *Let F/F'' be a free metabelian Leibniz algebra of rank three generated by $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$. Given the identity 3×3 matrix E , an arbitrary 3×9 matrix $Q = (q_{ij})$, both with coefficients in $U(F/F')$, where $1 \leq i \leq 3, 1 \leq j \leq 9$ and a fix 3×9 matrix A which is defined below;*

$$\begin{bmatrix} l_{\bar{x}_1} + r_{\bar{x}_1} & r_{\bar{x}_2} & l_{\bar{x}_2} & 0 & r_{\bar{x}_3} & l_{\bar{x}_3} & 0 & 0 & 0 \\ 0 & l_{\bar{x}_1} & r_{\bar{x}_1} & l_{\bar{x}_2} + r_{\bar{x}_2} & 0 & 0 & r_{\bar{x}_3} & l_{\bar{x}_3} & 0 \\ 0 & 0 & 0 & 0 & l_{\bar{x}_1} & r_{\bar{x}_1} & l_{\bar{x}_2} & r_{\bar{x}_2} & l_{\bar{x}_3} + r_{\bar{x}_3} \end{bmatrix}_{3 \times 9}.$$

Let G be the group of invertible matrices of the form $E + AQ$. Then $IA(F/F'') \cong G$.

Proof The elements of $IA(F/F'')$ are the automorphisms of F/F'' which are identical modulo F'/F'' . The explicit matrix form for the elements of $IA(F/F'')$ will be found. F'/F'' is generated by $\bar{r}_1 = [\bar{x}_1, \bar{x}_1], \bar{r}_2 = [\bar{x}_1, \bar{x}_2], \bar{r}_3 = [\bar{x}_2, \bar{x}_1], \bar{r}_4 = [\bar{x}_2, \bar{x}_2], \bar{r}_5 = [\bar{x}_1, \bar{x}_3], \bar{r}_6 = [\bar{x}_3, \bar{x}_1], \bar{r}_7 = [\bar{x}_2, \bar{x}_3], \bar{r}_8 = [\bar{x}_3, \bar{x}_2], \bar{r}_9 = [\bar{x}_3, \bar{x}_3]$, as an ideal, that is, as a F/F' -module. We denote the module action of $u \in F/F'$ (i.e. $u = \bar{w}$ and $w \in F$) and $r_u, l_u \in U(F/F')$ on $\bar{r} \in F'/F''$ by

$$\begin{aligned} \bar{r} * r_u &= [\bar{r}, u] = [\bar{r}, \bar{w}], \\ \bar{r} * l_u &= [u, \bar{r}] = [\bar{w}, \bar{r}]. \end{aligned}$$

Consider $\alpha \in IA(F/F'')$ and $\alpha(\bar{x}_i) = \bar{x}_i + \sum_{j=1}^9 \bar{r}_j * q_{ij}$ for $i = 1, 2, 3$ and $q_{ij} \in U(F/F')$. Then

$$\alpha(\bar{x}_1) = \bar{x}_1 + \bar{r}_1 * q_{11} + \bar{r}_2 * q_{12} + \bar{r}_3 * q_{13} + \bar{r}_4 * q_{14} + \bar{r}_5 * q_{15} + \bar{r}_6 * q_{16} + \bar{r}_7 * q_{17} + \bar{r}_8 * q_{18} + \bar{r}_9 * q_{19} \tag{1}$$

$$\alpha(\bar{x}_2) = \bar{x}_2 + \bar{r}_1 * q_{21} + \bar{r}_2 * q_{22} + \bar{r}_3 * q_{23} + \bar{r}_4 * q_{24} + \bar{r}_5 * q_{25} + \bar{r}_6 * q_{26} + \bar{r}_7 * q_{27} + \bar{r}_8 * q_{28} + \bar{r}_9 * q_{29} \tag{2}$$

$$\alpha(\bar{x}_3) = \bar{x}_3 + \bar{r}_1 * q_{31} + \bar{r}_2 * q_{32} + \bar{r}_3 * q_{33} + \bar{r}_4 * q_{34} + \bar{r}_5 * q_{35} + \bar{r}_6 * q_{36} + \bar{r}_7 * q_{37} + \bar{r}_8 * q_{38} + \bar{r}_9 * q_{39} \tag{3}$$

Now apply μ to both sides of (1), (2), and (3),

$$\begin{aligned} \mu\alpha(\bar{x}_1) &= \mu(\bar{x}_1 + \bar{r}_1 * q_{11} + \bar{r}_2 * q_{12} + \bar{r}_3 * q_{13} + \bar{r}_4 * q_{14} + \bar{r}_5 * q_{15} + \bar{r}_6 * q_{16} \\ &\quad + \bar{r}_7 * q_{17} + \bar{r}_8 * q_{18} + \bar{r}_9 * q_{19}) \end{aligned}$$

$$\begin{aligned} \mu\alpha(\bar{x}_1) &= \mu(\bar{x}_1 + [\bar{x}_1, \bar{x}_1] * q_{11} + [\bar{x}_1, \bar{x}_2] * q_{12} + [\bar{x}_2, \bar{x}_1] * q_{13} + [\bar{x}_2, \bar{x}_2] * q_{14}) + [\bar{x}_1, \bar{x}_3] * q_{15} \\ &\quad + [\bar{x}_3, \bar{x}_1] * q_{16} + [\bar{x}_2, \bar{x}_3] * q_{17} + [\bar{x}_3, \bar{x}_2] * q_{18} + [\bar{x}_3, \bar{x}_3] * q_{19}) \\ &= \bar{x}_1 + a_1 + [\bar{x}_1 + a_1, \bar{x}_1 + a_1] * q_{11} + [\bar{x}_1 + a_1, \bar{x}_2 + a_2] * q_{12} + [\bar{x}_2 + a_2, \bar{x}_1 + a_1] * q_{13} \\ &\quad + [\bar{x}_2 + a_2, \bar{x}_2 + a_2] * q_{14} + [\bar{x}_1 + a_1, \bar{x}_3 + a_3] * q_{15} + [\bar{x}_3 + a_3, \bar{x}_1 + a_1] * q_{16} \\ &\quad + [\bar{x}_2 + a_2, \bar{x}_3 + a_3] * q_{17} + [\bar{x}_3 + a_3, \bar{x}_2 + a_2] * q_{18} + [\bar{x}_3 + a_3, \bar{x}_3 + a_3] * q_{19} \end{aligned}$$

$$\begin{aligned}
 &= \overline{x_1} + a_1 + (\overline{[x_1, a_1]} + [a_1, \overline{x_1}]) * q_{11} + (\overline{[x_1, a_2]} + [a_1, \overline{x_2}]) * q_{12} + (\overline{[x_2, a_1]} + [a_2, \overline{x_1}]) * q_{13} \\
 &\quad + (\overline{[x_2, a_2]} + [a_2, \overline{x_2}]) * q_{14} + (\overline{[x_1, a_3]} + [a_1, \overline{x_3}]) * q_{15} + (\overline{[x_3, a_1]} + [a_3, \overline{x_1}]) * q_{16} \\
 &\quad + (\overline{[x_2, a_3]} + [a_2, \overline{x_3}]) * q_{17} + (\overline{[x_3, a_2]} + [a_3, \overline{x_2}]) * q_{18} + (\overline{[x_3, a_3]} + [a_3, \overline{x_3}]) * q_{19} \\
 &= \overline{x_1} + a_1 + a_1 * (l_{\overline{x_1}}q_{11} + r_{\overline{x_1}}q_{11} + r_{\overline{x_2}}q_{12} + l_{\overline{x_2}}q_{13} + r_{\overline{x_3}}q_{15} + l_{\overline{x_3}}q_{16}) \\
 &\quad + a_2 * (l_{\overline{x_1}}q_{12} + r_{\overline{x_1}}q_{13} + l_{\overline{x_2}}q_{14} + r_{\overline{x_2}}q_{14} + r_{\overline{x_3}}q_{17} + l_{\overline{x_3}}q_{18}) \\
 &\quad + a_3 * (l_{\overline{x_1}}q_{15} + r_{\overline{x_1}}q_{16} + l_{\overline{x_1}}q_{17} + r_{\overline{x_2}}q_{18} + l_{\overline{x_3}}q_{19} + r_{\overline{x_3}}q_{19}), \\
 \\
 \mu\alpha(\overline{x_2}) &= \mu(\overline{x_2} + \overline{r_1} * q_{21} + \overline{r_2} * q_{22} + \overline{r_3} * q_{23} + \overline{r_4} * q_{24} + \overline{r_5} * q_{25} + \overline{r_6} * q_{26} + \overline{r_7} * q_{27} + \overline{r_8} * q_{28} + \overline{r_9} * q_{29}) \\
 &= \mu(\overline{x_2} + [\overline{x_1}, \overline{x_1}] * q_{21} + [\overline{x_1}, \overline{x_2}] * q_{22} + [\overline{x_2}, \overline{x_1}] * q_{23} + [\overline{x_2}, \overline{x_2}] * q_{24} + [\overline{x_1}, \overline{x_3}] * q_{25} + \\
 &\quad [\overline{x_3}, \overline{x_1}] * q_{26} + [\overline{x_2}, \overline{x_3}] * q_{27} + [\overline{x_3}, \overline{x_2}] * q_{28} + [\overline{x_3}, \overline{x_3}] * q_{29}) \\
 &= \overline{x_2} + a_2 + [\overline{x_1} + a_1, \overline{x_1} + a_1] * q_{21} + [\overline{x_1} + a_1, \overline{x_2} + a_2] * q_{22} + [\overline{x_2} + a_2, \overline{x_1} + a_1] * q_{23} \\
 &\quad + [\overline{x_2} + a_2, \overline{x_2} + a_2] * q_{24} + [\overline{x_1} + a_1, \overline{x_3} + a_3] * q_{25} + [\overline{x_3} + a_3, \overline{x_1} + a_1] * q_{26} \\
 &\quad + [\overline{x_2} + a_2, \overline{x_3} + a_3] * q_{27} + [\overline{x_3} + a_3, \overline{x_2} + a_2] * q_{28} + [\overline{x_3} + a_3, \overline{x_3} + a_3] * q_{29} \\
 &= \overline{x_2} + a_2 + (\overline{[x_1, a_1]} + [a_1, \overline{x_1}]) * q_{21} + (\overline{[x_1, a_2]} + [a_1, \overline{x_2}]) * q_{22} + (\overline{[x_2, a_1]} + [a_2, \overline{x_1}]) * q_{23} \\
 &\quad + (\overline{[x_2, a_2]} + [a_2, \overline{x_2}]) * q_{24} + (\overline{[x_1, a_3]} + [a_1, \overline{x_3}]) * q_{25} + (\overline{[x_3, a_1]} + [a_3, \overline{x_1}]) * q_{26} \\
 &\quad + (\overline{[x_2, a_3]} + [a_2, \overline{x_3}]) * q_{27} + (\overline{[x_3, a_2]} + [a_3, \overline{x_2}]) * q_{28} + (\overline{[x_3, a_3]} + [a_3, \overline{x_3}]) * q_{29} \\
 &= \overline{x_2} + a_2 + a_1 * (l_{\overline{x_1}}q_{21} + r_{\overline{x_1}}q_{21} + r_{\overline{x_2}}q_{22} + l_{\overline{x_2}}q_{23} + r_{\overline{x_3}}q_{25} + l_{\overline{x_3}}q_{26}) \\
 &\quad + a_2 * (l_{\overline{x_1}}q_{22} + r_{\overline{x_1}}q_{23} + l_{\overline{x_2}}q_{24} + r_{\overline{x_2}}q_{24} + r_{\overline{x_3}}q_{27} + l_{\overline{x_3}}q_{28}) \\
 &\quad + a_3 * (l_{\overline{x_1}}q_{25} + r_{\overline{x_1}}q_{26} + l_{\overline{x_2}}q_{27} + r_{\overline{x_2}}q_{28} + l_{\overline{x_3}}q_{29} + r_{\overline{x_3}}q_{29}), \\
 \\
 \mu\alpha(\overline{x_3}) &= \mu(\overline{x_3} + \overline{r_1} * q_{31} + \overline{r_2} * q_{32} + \overline{r_3} * q_{33} + \overline{r_4} * q_{34} + \overline{r_5} * q_{35} + \overline{r_6} * q_{36} + \overline{r_7} * q_{37} + \overline{r_8} * q_{38} + \overline{r_9} * q_{39}) \\
 &= \mu(\overline{x_3} + [\overline{x_1}, \overline{x_1}] * q_{31} + [\overline{x_1}, \overline{x_2}] * q_{32} + [\overline{x_2}, \overline{x_1}] * q_{33} + [\overline{x_2}, \overline{x_2}] * q_{34} + [\overline{x_1}, \overline{x_3}] * q_{35} \\
 &\quad + [\overline{x_3}, \overline{x_1}] * q_{36} + [\overline{x_2}, \overline{x_3}] * q_{37} + [\overline{x_3}, \overline{x_2}] * q_{38} + [\overline{x_3}, \overline{x_3}] * q_{39}) \\
 &= \overline{x_3} + a_3 + [\overline{x_1} + a_1, \overline{x_1} + a_1] * q_{31} + [\overline{x_1} + a_1, \overline{x_2} + a_2] * q_{32} + [\overline{x_2} + a_2, \overline{x_1} + a_1] * q_{33} \\
 &\quad + [\overline{x_2} + a_2, \overline{x_2} + a_2] * q_{34} + [\overline{x_1} + a_1, \overline{x_3} + a_3] * q_{35} + [\overline{x_3} + a_3, \overline{x_1} + a_1] * q_{36} \\
 &\quad + [\overline{x_2} + a_2, \overline{x_3} + a_3] * q_{37} + [\overline{x_3} + a_3, \overline{x_2} + a_2] * q_{38} + [\overline{x_3} + a_3, \overline{x_3} + a_3] * q_{39} \\
 &= \overline{x_3} + a_3 + (\overline{[x_1, a_1]} + [a_1, \overline{x_1}]) * q_{31} + (\overline{[x_1, a_2]} + [a_1, \overline{x_2}]) * q_{32} + (\overline{[x_2, a_1]} + [a_2, \overline{x_1}]) * q_{33} \\
 &\quad + (\overline{[x_2, a_2]} + [a_2, \overline{x_2}]) * q_{34} + (\overline{[x_1, a_3]} + [a_1, \overline{x_3}]) * q_{35} + (\overline{[x_3, a_1]} + [a_3, \overline{x_1}]) * q_{36} \\
 &\quad + (\overline{[x_2, a_3]} + [a_2, \overline{x_3}]) * q_{37} + (\overline{[x_3, a_2]} + [a_3, \overline{x_2}]) * q_{38} + (\overline{[x_3, a_3]} + [a_3, \overline{x_3}]) * q_{39} \\
 &= \overline{x_3} + a_3 + a_1 * (l_{\overline{x_1}}q_{31} + r_{\overline{x_1}}q_{31} + r_{\overline{x_2}}q_{32} + l_{\overline{x_2}}q_{33} + r_{\overline{x_3}}q_{35} + l_{\overline{x_3}}q_{36}) \\
 &\quad + a_2 * (l_{\overline{x_1}}q_{32} + r_{\overline{x_1}}q_{33} + l_{\overline{x_2}}q_{34} + r_{\overline{x_2}}q_{34} + r_{\overline{x_3}}q_{37} + l_{\overline{x_3}}q_{38}) \\
 &\quad + a_3 * (l_{\overline{x_1}}q_{35} + r_{\overline{x_1}}q_{36} + l_{\overline{x_2}}q_{37} + r_{\overline{x_2}}q_{38} + l_{\overline{x_3}}q_{39} + r_{\overline{x_3}}q_{39}).
 \end{aligned}$$

Hence, we obtain the following equalities;

$$\begin{aligned} \mu\alpha(\overline{x_1}) &= \overline{x_1} + a_1 + a_1 * (l_{\overline{x_1}}q_{11} + r_{\overline{x_1}}q_{11} + r_{\overline{x_2}}q_{12} + l_{\overline{x_2}}q_{13} + r_{\overline{x_3}}q_{15} + l_{\overline{x_3}}q_{16}) \\ &\quad + a_2 * (l_{\overline{x_1}}q_{12} + r_{\overline{x_1}}q_{13} + l_{\overline{x_2}}q_{14} + r_{\overline{x_2}}q_{14} + r_{\overline{x_3}}q_{17} + l_{\overline{x_3}}q_{18}) \\ &\quad + a_3 * (l_{\overline{x_1}}q_{15} + r_{\overline{x_1}}q_{16} + l_{\overline{x_1}}q_{17} + r_{\overline{x_2}}q_{18} + l_{\overline{x_3}}q_{19} + r_{\overline{x_3}}q_{19}), \end{aligned}$$

$$\begin{aligned} \mu\alpha(\overline{x_2}) &= \overline{x_2} + a_2 + a_1 * (l_{\overline{x_1}}q_{21} + r_{\overline{x_1}}q_{21} + r_{\overline{x_2}}q_{22} + l_{\overline{x_2}}q_{23} + r_{\overline{x_3}}q_{25} + l_{\overline{x_3}}q_{26}) \\ &\quad + a_2 * (l_{\overline{x_1}}q_{22} + r_{\overline{x_1}}q_{23} + l_{\overline{x_2}}q_{24} + r_{\overline{x_2}}q_{24} + r_{\overline{x_3}}q_{27} + l_{\overline{x_3}}q_{28}) \\ &\quad + a_3 * (l_{\overline{x_1}}q_{25} + r_{\overline{x_1}}q_{26} + l_{\overline{x_2}}q_{27} + r_{\overline{x_2}}q_{28} + l_{\overline{x_3}}q_{29} + r_{\overline{x_3}}q_{29}), \end{aligned}$$

$$\begin{aligned} \mu\alpha(\overline{x_3}) &= \overline{x_3} + a_3 + a_1 * (l_{\overline{x_1}}q_{31} + r_{\overline{x_1}}q_{31} + r_{\overline{x_2}}q_{32} + l_{\overline{x_2}}q_{33} + r_{\overline{x_3}}q_{35} + l_{\overline{x_3}}q_{36}) \\ &\quad + a_2 * (l_{\overline{x_1}}q_{32} + r_{\overline{x_1}}q_{33} + l_{\overline{x_2}}q_{34} + r_{\overline{x_2}}q_{34} + r_{\overline{x_3}}q_{37} + l_{\overline{x_3}}q_{38}) \\ &\quad + a_3 * (l_{\overline{x_1}}q_{35} + r_{\overline{x_1}}q_{36} + l_{\overline{x_2}}q_{37} + r_{\overline{x_2}}q_{38} + l_{\overline{x_3}}q_{39} + r_{\overline{x_3}}q_{39}). \end{aligned}$$

By the equality $\mu\alpha = \tilde{\alpha}\mu$ from Proposition 3.1 and the definition of the \overline{AutW} , we find that $\tilde{\alpha}$ restricted to $I_{F'/F''}$ has a corresponding matrix M of the form

$$\begin{bmatrix} 1 + B_{11} & B_{12} & B_{13} \\ B_{21} & 1 + B_{22} & B_{23} \\ B_{31} & B_{32} & 1 + B_{33} \end{bmatrix}_{3 \times 3}$$

where

$$\begin{aligned} B_{11} &= l_{\overline{x_1}}q_{11} + r_{\overline{x_1}}q_{11} + r_{\overline{x_2}}q_{12} + l_{\overline{x_2}}q_{13} + r_{\overline{x_3}}q_{15} + l_{\overline{x_3}}q_{16}, \\ B_{12} &= l_{\overline{x_1}}q_{12} + r_{\overline{x_1}}q_{13} + l_{\overline{x_2}}q_{14} + r_{\overline{x_2}}q_{14} + r_{\overline{x_3}}q_{17} + l_{\overline{x_3}}q_{18}, \\ B_{13} &= l_{\overline{x_1}}q_{15} + r_{\overline{x_1}}q_{16} + l_{\overline{x_1}}q_{17} + r_{\overline{x_2}}q_{18} + l_{\overline{x_3}}q_{19} + r_{\overline{x_3}}q_{19}, \\ B_{21} &= l_{\overline{x_1}}q_{21} + r_{\overline{x_1}}q_{21} + r_{\overline{x_2}}q_{22} + l_{\overline{x_2}}q_{23} + r_{\overline{x_3}}q_{25} + l_{\overline{x_3}}q_{26}, \\ B_{22} &= l_{\overline{x_1}}q_{22} + r_{\overline{x_1}}q_{23} + l_{\overline{x_2}}q_{24} + r_{\overline{x_2}}q_{24} + r_{\overline{x_3}}q_{27} + l_{\overline{x_3}}q_{28}, \\ B_{23} &= l_{\overline{x_1}}q_{25} + r_{\overline{x_1}}q_{26} + l_{\overline{x_2}}q_{27} + r_{\overline{x_2}}q_{28} + l_{\overline{x_3}}q_{29} + r_{\overline{x_3}}q_{29}, \\ B_{31} &= l_{\overline{x_1}}q_{31} + r_{\overline{x_1}}q_{31} + r_{\overline{x_2}}q_{32} + l_{\overline{x_2}}q_{33} + r_{\overline{x_3}}q_{35} + l_{\overline{x_3}}q_{36}, \\ B_{32} &= l_{\overline{x_1}}q_{32} + r_{\overline{x_1}}q_{33} + l_{\overline{x_2}}q_{34} + r_{\overline{x_2}}q_{34} + r_{\overline{x_3}}q_{37} + l_{\overline{x_3}}q_{38}, \\ B_{33} &= l_{\overline{x_1}}q_{35} + r_{\overline{x_1}}q_{36} + l_{\overline{x_2}}q_{37} + r_{\overline{x_2}}q_{38} + l_{\overline{x_3}}q_{39} + r_{\overline{x_3}}q_{39}. \end{aligned}$$

Since $\tilde{\alpha}$ is an automorphism, M is two-sided invertible. We write the transpose of M of the form $E + AQ$, where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3},$$

$$A = \begin{bmatrix} l_{\overline{x_1}} + r_{\overline{x_1}} & r_{\overline{x_2}} & l_{\overline{x_2}} & 0 & 0 & r_{\overline{x_3}} & l_{\overline{x_3}} & 0 & 0 \\ 0 & l_{\overline{x_1}} & r_{\overline{x_1}} & l_{\overline{x_2}} + r_{\overline{x_2}} & 0 & 0 & r_{\overline{x_3}} & l_{\overline{x_3}} & 0 \\ 0 & 0 & 0 & 0 & l_{\overline{x_1}} & r_{\overline{x_1}} & l_{\overline{x_2}} & r_{\overline{x_2}} & l_{\overline{x_3}} + r_{\overline{x_3}} \end{bmatrix}_{3 \times 9},$$

$$Q = \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \\ q_{14} & q_{24} & q_{34} \\ q_{15} & q_{25} & q_{35} \\ q_{16} & q_{26} & q_{36} \\ q_{17} & q_{27} & q_{37} \\ q_{18} & q_{28} & q_{38} \\ q_{19} & q_{29} & q_{39} \end{bmatrix}_{9 \times 3}.$$

Conversely, let $C = E + AQ$ be invertible which defines as above form. There is an automorphism $\tilde{\alpha}$ of W that is if $\tilde{\alpha}$ is restricted to $I_{F'/F''}$ then the transpose matrix C^T of C determines this automorphism. Hence, we obtain

$$\tilde{\alpha}(\overline{x_i}) = \overline{x_i},$$

$$\tilde{\alpha}(a_i) = a_i + \sum \sum a_k * a_{kj} q_{ij}.$$

By the equality $\tilde{\alpha}\mu = \mu\alpha$, it yields

$$\begin{aligned} \tilde{\alpha}(\mu(\overline{x_i})) &= \tilde{\alpha}(\overline{x_i} + a_i) \\ &= \overline{x_i} + a_i + \sum \sum a_k * a_{kj} q_{ij} \\ &= \mu(\overline{x_i} + \sum \overline{r_j} * q_{ij}). \end{aligned}$$

Therefore, there is an automorphism α of F/F'' which is identical modulo F'/F'' defined by $\alpha(\overline{x_i}) = \overline{x_i} + \sum \overline{r_j} * q_{ij}$. □

As an application of the Theorem 3.2, we give the following corollary.

Corollary 3.3 *Let $M = E + AQ$ be as in the proof of Theorem 3.2.*

i) If $q_{14} = r_{\overline{x_1}}$ and all other $q_{ij} = l_{\overline{x_1}}$, then there is an automorphism

$$\begin{aligned} \zeta_1 &: \overline{x_1} \rightarrow \overline{x_1} + [[\overline{x_2}, \overline{x_2}], \overline{x_1}] \\ \overline{x_2} &\rightarrow \overline{x_2} \\ \overline{x_3} &\rightarrow \overline{x_3} \end{aligned}$$

whose associated matrix is M^T .

ii) If $q_{31} = r_{\overline{x_3}}$, $q_{21} = r_{\overline{x_2}}$ and all other $q_{ij} = l_{\overline{x_1}}$ then there is an automorphism

$$\begin{aligned} \zeta_2 &: \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} &\rightarrow \overline{x_2} + [[\overline{x_2}, \overline{x_2}], \overline{x_2}] \\ \overline{x_3} &\rightarrow \overline{x_3} + [[\overline{x_1}, \overline{x_1}], \overline{x_3}] \end{aligned}$$

whose associated matrix is M^T .

iii) If $q_{24} = r_{\overline{x_1}}$, $q_{21} = 1$, $q_{14} = 1$ and all other $q_{ij} = l_{\overline{x_1}}$ then M^T is the associated matrix of the following automorphism.

$$\begin{aligned} \zeta_3 & : \quad \overline{x_1} \rightarrow \overline{x_1} + [\overline{x_2}, \overline{x_2}] \\ \overline{x_2} & \rightarrow \overline{x_2} + [\overline{x_1}, \overline{x_1}] + [[\overline{x_2}, \overline{x_2}], \overline{x_1}] \\ \overline{x_3} & \rightarrow \overline{x_3} \end{aligned}$$

Proof i) If $q_{14} = r_{\overline{x_1}}$ and all other $q_{ij} = l_{\overline{x_1}}$, then

$$M = \begin{bmatrix} 1 & 0 & 0 \\ (l_{\overline{x_2}} + r_{\overline{x_2}})r_{\overline{x_1}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M^T = \begin{bmatrix} 1 & (l_{\overline{x_2}} + r_{\overline{x_2}})r_{\overline{x_1}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\det M^T = 1$, then there is an automorphism whose associated matrix is M^T , by Theorem 3.2 it is

$$\begin{aligned} \zeta_1 & : \quad \overline{x_1} \rightarrow \overline{x_1} + [[\overline{x_2}, \overline{x_2}], \overline{x_1}] \\ \overline{x_2} & \rightarrow \overline{x_2} \\ \overline{x_3} & \rightarrow \overline{x_3}. \end{aligned}$$

ii) If $q_{31} = r_{\overline{x_3}}$, $q_{21} = r_{\overline{x_2}}$ and all other $q_{ij} = l_{\overline{x_1}}$, then M is of the form

$$\begin{bmatrix} 1 & (r_{\overline{x_1}} + l_{\overline{x_1}})r_{\overline{x_2}} & (r_{\overline{x_1}} + l_{\overline{x_1}})r_{\overline{x_3}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M^T = \begin{bmatrix} 1 & 0 & 0 \\ (r_{\overline{x_1}} + l_{\overline{x_1}})r_{\overline{x_2}} & 1 & 0 \\ (r_{\overline{x_1}} + l_{\overline{x_1}})r_{\overline{x_3}} & 0 & 1 \end{bmatrix}.$$

Since $\det M^T = 1$, then there is an automorphism whose associated matrix is M^T and by Theorem 3.2 it is

$$\begin{aligned} \zeta_2 & : \quad \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} & \rightarrow \overline{x_2} + [[\overline{x_1}, \overline{x_1}], \overline{x_2}] \\ \overline{x_3} & \rightarrow \overline{x_3} + [[\overline{x_1}, \overline{x_1}], \overline{x_3}]. \end{aligned}$$

iii) If $q_{24} = r_{\overline{x_1}}$, $q_{21} = 1$, $q_{14} = 1$ and all other $q_{ij} = l_{\overline{x_1}}$ then

$$M = \begin{bmatrix} 1 & l_{\overline{x_1}} + r_{\overline{x_1}} & 0 \\ l_{\overline{x_2}} + r_{\overline{x_2}} & 1 + (l_{\overline{x_2}} + r_{\overline{x_2}})r_{\overline{x_1}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M^T = \begin{bmatrix} 1 & l_{\overline{x_2}} + r_{\overline{x_2}} & 0 \\ l_{\overline{x_1}} + r_{\overline{x_1}} & 1 + (l_{\overline{x_2}} + r_{\overline{x_2}})r_{\overline{x_1}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then there is an automorphism whose associated matrix has the above form which is

$$\begin{aligned} \zeta_3 & : \quad \overline{x_1} \rightarrow \overline{x_1} + [\overline{x_2}, \overline{x_2}] \\ \overline{x_2} & \rightarrow \overline{x_2} + [\overline{x_1}, \overline{x_1}] + [[\overline{x_2}, \overline{x_2}], \overline{x_1}] \\ \overline{x_3} & \rightarrow \overline{x_3}. \end{aligned}$$

□

The following theorem gives the set of generators of $IA(F/F'')$.

Theorem 3.4 *Let F/F'' be the free metabelian Leibniz algebra of rank three generated by $\{\overline{x_1}, \overline{x_2}, \overline{x_3}\}$. $IA(F/F'')$ is generated by an inner automorphism and the following automorphisms;*

$$\begin{aligned} \tau_1 & : \quad \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} & \rightarrow \overline{x_2} + [z, \overline{x_2}] \\ \overline{x_3} & \rightarrow \overline{x_3} + [z, \overline{x_3}] \end{aligned}$$

where $z = [\overline{x_1}, \overline{x_1}] * z_1, z_1 \in U(F/F')$,

$$\begin{aligned} \tau_2 & : \quad \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} & \rightarrow \overline{x_2} + u \\ \overline{x_3} & \rightarrow \overline{x_3} \end{aligned}$$

where $u = [\overline{x_t}, \overline{x_t}] * u_1, u_1 \in U(F/F'), t \neq 2$ and

$$\begin{aligned} \tau_3 & : \quad \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} & \rightarrow \overline{x_2} \\ \overline{x_3} & \rightarrow \overline{x_3} + v \end{aligned}$$

where $v = [\overline{x_1}, \overline{x_2}] * v_1, v_1 \in U(F/F')$.

Proof Let

$$\varphi : F/F'' \rightarrow F/F''$$

be an automorphism of the free metabelian Leibniz algebra F/F'' with the free generators $\overline{x_1}, \overline{x_2}$, and $\overline{x_3}$ which acts identically modulo F'/F'' . For $h \in F/F''$ and $x \in F'/F''$, $\varphi[h, x] = [\varphi(h), x] = [h, x]$, hence $[\varphi(h) - h, x] = 0$ and $\varphi(h) - h \in F'/F''$. Then $\varphi(h) - h = 0$ on $(F/F'')/(F'/F'')$. Therefore, φ is also the identity on F/F' .

By Corollary 2.2, F/F'' is embedded in the wreath product $(F'/F'')wr(F/F')$. Hence, we see F/F'' is a free metabelian Leibniz algebra with free generators $g_i = \overline{x_i} + a_i, i = 1, 2, 3$ which are elements of the wreath

product $(F'/F'')wr(F/F')$. Therefore, $\varphi(g_i) = g_i + u_i$, $u_i \in F'/F''$, $i = 1, 2, 3$. We have $\varphi[g_1, g_2] = [g_1, g_2]$ and

$$\begin{aligned} \varphi[g_1, g_2] &= [\varphi(g_1), \varphi(g_2)] = [g_1 + u_1, g_2 + u_2] \\ &= [g_1, g_2] + [u_1, g_2] + [g_1, u_2] + [u_1, u_2]. \end{aligned}$$

The element $[u_1, u_2]$ is equal zero in the algebra F/F'' . Hence, $[u_1, g_2] + [g_1, u_2] = 0$. In the equality $[u_1, g_2] + [g_1, u_2] = 0$ substituting $\overline{x_1} + a_1$ for g_1 , we obtain

$$[u_1, \overline{x_2}] + [u_1, a_2] + [\overline{x_1}, u_2] + [a_1, u_2] = 0.$$

The elements $[a_1, u_2]$ and $[u_1, a_2]$ are zero in the algebra F/F'' . Thus, we have

$$[u_1, \overline{x_2}] + [\overline{x_1}, u_2] = 0.$$

Case 1: Let $u_2 = z * l_{\overline{x_2}}$, where $z \in F'/F''$, $l_{\overline{x_2}} \in U(F/F')$. Substituting $u_2 = z * l_{\overline{x_2}}$ in the equality $[u_1, \overline{x_2}] + [\overline{x_1}, u_2] = 0$, we obtain

$$\begin{aligned} [u_1, \overline{x_2}] &= -[\overline{x_1}, u_2] \\ &= -[\overline{x_1}, z * l_{\overline{x_2}}] \\ &= -[\overline{x_1}, [\overline{x_2}, z]] \\ &= [\overline{x_1}, [z, \overline{x_2}]]. \end{aligned}$$

By the Leibniz identity $[\overline{x_1}, [z, \overline{x_2}]] = [[\overline{x_1}, z], \overline{x_2}] - [\overline{x_1}, \overline{x_2}], z] = [[\overline{x_1}, z], \overline{x_2}]$. Hence, $[u_1, \overline{x_2}] = [[\overline{x_1}, z], \overline{x_2}]$ and

$$u_1 = [\overline{x_1}, z] = z * l_{\overline{x_1}} \text{ or } u_1 = [g_1, z].$$

Thus, we obtain $u_1 = z * l_{\overline{x_1}}$. Also if $\varphi([g_1, g_3]) = [g_1, g_3]$, we get $u_3 = z * l_{\overline{x_3}}$. Hence, we obtain that for every $i \in I$

$$\varphi(g_i) = g_i + [g_i, z].$$

Since z lies in F'/F'' , φ is an inner automorphism.

Case 2 : Let $u_2 = z * r_{\overline{x_2}}$, where $z \in F'/F''$, $r_{\overline{x_2}} \in U(F/F')$. From the equality $[u_1, \overline{x_2}] + [\overline{x_1}, u_2] = 0$,

$$\begin{aligned} [u_1, \overline{x_2}] &= -[\overline{x_1}, u_2] \\ &= -[\overline{x_1}, z * r_{\overline{x_2}}] \\ &= -[\overline{x_1}, [z, \overline{x_2}]] \\ &= -[[\overline{x_1}, z], \overline{x_2}] + [[\overline{x_1}, \overline{x_2}], z] \\ &= -[[\overline{x_1}, z], \overline{x_2}] \end{aligned}$$

and we obtain that $u_1 = -[\overline{x_1}, z]$ or $u_1 = -[g_1, z]$. Now we will find u_3 . There are two cases:

i) If $\varphi([g_1, g_3]) = [g_1, g_3]$, we get that $u_3 = [z, \overline{x_3}]$ or $u_3 = [z, g_3]$. Hence, we obtain

$$\begin{aligned} \varphi &: g_1 \rightarrow g_1 - [g_1, z] \\ g_2 &\rightarrow g_2 + [z, g_2] \\ g_3 &\rightarrow g_3 + [z, g_3]. \end{aligned}$$

By Theorem 3.2, z is $[g_1, g_1] * z_1, z_1 \in U(F/F')$. Therefore, the automorphism is

$$\begin{aligned} \varphi &: g_1 \rightarrow g_1 \\ g_2 &\rightarrow g_2 + [z, g_2] \\ g_3 &\rightarrow g_3 + [z, g_3]. \end{aligned}$$

ii) If $\varphi([g_3, g_1]) = [g_3, g_1]$, we get that $u_3 = -[\overline{x_3}, z]$ or $u_3 = -[g_3, z]$. Then we see that

$$\begin{aligned} \varphi &: g_1 \rightarrow g_1 - [g_1, z] \\ g_2 &\rightarrow g_2 + [z, g_2] \\ g_3 &\rightarrow g_3 - [g_3, z] \end{aligned}$$

and by Theorem 3.2, $z \in Ann(F/F'')$. It is obvious that $[g_1, z] = [g_3, z] = 0$. We know that if φ is an automorphism, $\{g_1, g_2 + [z, g_2], g_3\}$ is a free generating set and

$$\begin{aligned} F/F'' &= g_1U(F/F') \oplus g_2U(F/F') \oplus g_3U(F/F') \\ &= g_1U(F/F') \oplus (g_2 + [z, g_2])U(F/F') \oplus g_3U(F/F'). \end{aligned}$$

Thus, $[z, g_2] \in g_1U(F/F')$ or $[z, g_2] \in g_3U(F/F')$. As a result of this $z \in Ann(F/F'') \cap g_1U(F/F')$ or $z \in Ann(F/F'') \cap g_3U(F/F')$. Hence, the automorphism is of the form

$$\begin{aligned} \varphi &: g_1 \rightarrow g_1 \\ g_2 &\rightarrow g_2 + [z, g_2] \\ g_3 &\rightarrow g_3 \end{aligned}$$

where $z = [g_t, g_t] * z_1, z_1 \in U(F/F'), t \neq 2$.

Case 3 : For every $u_2 \in [g_t, g_t] * U(F/F'), t \neq 2$, since $[\overline{x_1}, u_2] = 0$ by the Leibniz identity, then $[u_1, \overline{x_2}] = 0$ and it yields $u_1 = 0$. Now let us calculate u_3 . If also $\varphi([g_3, g_1]) = [g_3, g_1]$, we get that $[\overline{x_3}, u_1] = 0$ by the Leibniz identity, then $[u_3, \overline{x_1}] = 0$ and it yields $u_3 = 0$. Therefore, we get

$$\begin{aligned} \varphi & : g_1 \rightarrow g_1 \\ g_2 & \rightarrow g_2 + u_2 \\ g_3 & \rightarrow g_3, \end{aligned}$$

where $u_2 = [g_t, g_t] * u'_2, u'_2 \in U(F/F'), t \neq 2$. Given

$$\begin{aligned} \varphi^{-1} & : g_1 \rightarrow g_1 \\ g_2 & \rightarrow g_2 - u_2 \\ g_3 & \rightarrow g_3. \end{aligned}$$

Since $\varphi^{-1} \circ \varphi = 1$ and $\varphi \circ \varphi^{-1} = 1$, φ is an automorphism.

Case 4 : Let $u_2 = 0$, then $u_1 = 0$. Now lets determine u_3 .

i) Let $u_3 = [\overline{x_1}, \overline{x_2}] * u'_3$ or $u_3 = [\overline{x_2}, \overline{x_1}] * u'_3, u'_3 \in U(F/F')$, then φ is an automorphism by Theorem 3.2 and the automorphism is

$$\begin{aligned} \varphi & : g_1 \rightarrow g_1 \\ g_2 & \rightarrow g_2 \\ g_3 & \rightarrow g_3 + u_3, \end{aligned}$$

where $u_3 = [g_1, g_2] * u'_3$ or $u_3 = [g_2, g_1] * u'_3, u'_3 \in U(F/F')$. This automorphism is an elementary automorphism.

ii) If we take u_3 as one of $[\overline{x_1}, \overline{x_3}] * u'_3, [\overline{x_3}, \overline{x_1}] * u'_3, [\overline{x_3}, \overline{x_2}] * u'_3, [\overline{x_2}, \overline{x_3}] * u'_3$ or $[\overline{x_3}, \overline{x_3}] * u'_3, u'_3 \in U(F/F')$, by the Theorem 3.2, φ is not an automorphism.

iii) For $u_3 = [\overline{x_t}, \overline{x_t}] * u'_3, u'_3 \in U(F/F'), t \neq 3$, we get the same result as in Case 3. □

4. Automorphisms of F/F''

For every $g = (g_{ij}) \in GL(3, K)$, the general linear group over K , the mapping

$$g : x_j \longrightarrow \sum g_{ij} \cdot x_i, j = 1, 2, 3$$

extends uniquely to an algebra automorphism of F/F'' and $GL(3, K)$ acts on $U(F/F')$ as a group of algebra automorphism. We write $g \cdot f$ for the action where $g \in GL(3, K), f \in U(F/F')$ and

$$g \cdot x_j = \sum g_{ij} \cdot x_i.$$

Thus, we consider $GL(3, K)$ as a subgroup of $Aut(F/F'')$. Since $IA(F/F'')$ is a normal subgroup of $Aut(F/F'')$ and $GL(3, K) \cap IA(F/F'') = \{1\}$, we obtain that $Aut(F/F'')$ is semidirect product of $IA(F/F'')$ by $GL(3, K)$. Thus, every automorphism α of F/F'' is uniquely written as $\varphi \circ g$, where $\varphi \in IA(F/F'')$ and $g \in GL(3, K)$. By Theorem 3.2, $IA(F/F'')$ is isomorphic to the group G . Denote this isomorphism by $\eta : IA(F/F'') \longrightarrow G$. The action of $GL(3, K)$ on $IA(F/F'')$ is given by

$$b \circ \varphi \circ b^{-1} = \eta^{-1} (E + b \cdot (\varphi_{ij}) b^{-1}),$$

where $b \in GL(3, K)$, $E + (\varphi_{ij}) = E + AQ$ is the corresponding matrix of the automorphism φ of $IA(F/F'')$ and η^{-1} is inverse of η . Hence $IA(F/F'')$ is a $GL(3, K)$ -module. By Theorem 3.4 we know the generators of the $IA(F/F'')$. Thus, we have proved the following theorem.

Theorem 4.1 *Let F/F'' be the free metabelian Leibniz algebra of rank three generated by $\overline{x_1}, \overline{x_2}$, and $\overline{x_3}$. The automorphism group of F/F'' is generated by the general linear group $GL(3, K)$ together with the inner automorphism e^{adv} ($v \in F'/F''$) and the following automorphisms;*

$$\begin{aligned}\tau_1 & : \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} & \rightarrow \overline{x_2} + [z, \overline{x_2}] \\ \overline{x_3} & \rightarrow \overline{x_3} + [z, \overline{x_3}],\end{aligned}$$

where $z = [\overline{x_1}, \overline{x_1}] * z_1$, $z_1 \in U(F/F')$,

$$\begin{aligned}\tau_2 & : \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} & \rightarrow \overline{x_2} + u \\ \overline{x_3} & \rightarrow \overline{x_3}\end{aligned}$$

where $u = [\overline{x_t}, \overline{x_t}] * u_1$, $u_1 \in U(F/F')$, $t \neq 2$ and

$$\begin{aligned}\tau_3 & : \overline{x_1} \rightarrow \overline{x_1} \\ \overline{x_2} & \rightarrow \overline{x_2} \\ \overline{x_3} & \rightarrow \overline{x_3} + v\end{aligned}$$

where $v = [\overline{x_1}, \overline{x_2}] * v_1$, $v_1 \in U(F/F')$.

References

- [1] Abdykhalikov AT, Mikhalev AA, Umirbaev UU. Automorphism of two-generated free Leibniz algebras. Communications in Algebra 2001; 29 (7): 2953-2960.
- [2] Bahturin Y, Nabiyevev S. Automorphisms and derivations of abelian extensions of some Lie algebras. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 1992; 62: 43-57.
- [3] Drensky V, Cattaneo GMP. Varieties of metabelian Leibniz algebras. Journal of Algebra and Its Applications 2002; 1 (1): 31-50.
- [4] Drensky V, Papistas AI. Automorphisms of free left nilpotent Leibniz algebras and fixed points. Communications in Algebra 2005; 33: 2957-2975.
- [5] Loday JL, Prashvili T. Universal enveloping algebras of Leibniz algebras and (co)homology. Mathematische Annalen 1993; 296: 139-158.
- [6] Papistas IA. IA-automorphisms of 2-generator metabelian Lie algebras. Algebra Colloquium 1996; 3: 193-198.
- [7] Romankov V. On the automorphism group of a free metabelian Lie algebra. International Journal of Algebra and Computation 2008; 2(18): 209-226.
- [8] Shevelin MA. Periodic automorphisms of a free Lie algebra of rank 3. Siberian Mathematical Journal 2011; 52(3): 544-553.
- [9] Smel'kin AL. Wreath products of Lie algebras and their application in the theory of groups. Transactions of the Moscow Mathematical Society 1973; 29: 247-260.