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Univalence criteria for analytic functions obtained using fuzzy differential subordinations

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Abstract: Ever since Lotfi A. Zadeh published the paper "Fuzzy Sets" in 1965 setting the basis of a new theory named fuzzy sets theory, many scientists have developed this theory and its applications. Mathematicians were especially interested in extending classical mathematical results in the fuzzy context. Such an extension was also done relating fuzzy sets theory and geometric theory of analytic functions. The study begun in 2011 has many interesting published outcomes and the present paper follows the line of the previous research in the field. The aim of the paper is to give some references related to the connections already made between fuzzy sets theory and geometric theory of analytic functions and to present some new results that might prove interesting for mathematicians willing to enlarge their views on certain aspects of the merge between the two theories. Using the notions of fuzzy differential subordination and the classical notion of differential subordination for analytic functions, two criteria for the univalence of the analytic functions are stated in this work.

Key words: Analytic function, univalent function, fuzzy function, fuzzy differential subordination, dominant, best dominant, differential subordination

1. Introduction and preliminaries

The starting point for fuzzy sets theory is the paper published by Lotfi A. Zadeh in 1965 [34]. At first, it was received with skepticism but has reached now over 100,000 citations. As many researchers have tried to link this theory with different domains of mathematics, the connection between fuzzy sets theory and the branch of complex analysis that studies analytic functions in view of their geometric properties was established in 2011 [23].

The concept of differential subordination was introduced in two papers in 1978 [16] and 1981 [17], developed by many authors continuing the study and synthesized by S.S. Miller and P.T. Mocanu in [18]. The concept of fuzzy subordination was introduced by G.I. Oros and Gh. Oros in 2011 [23] and the notion of fuzzy differential subordination was introduced by the same authors in 2012 [24]. Both papers were well received among the researchers interested in the topic of differential subordinations and are cited over 20 times each.

In the paper published in 2017 [10], a nice review is done showing the history of the notion of fuzzy set and its connections to different branches of science and technique, citing the results obtained up to that point related

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to fuzzy differential subordination concept. The first results published consolidated the direction of the study adapting the classical theory of differential subordination to the new aspects of fuzzy differential subordination giving methods for finding dominants and best dominants of the fuzzy differential subordinations [25] without which the study could not continue. Then, the special case of Briot-Bouquet fuzzy differential subordinations was studied [26]. The idea was picked up by other researchers which have started to investigate the new outcomes being published on fuzzy differential subordinations [14]. In the next phase, fuzzy differential subordinations were associated with different operators [1], [3] giving a new direction to the study. The investigations using operators were continued in many papers [2, 28, 29], the study being prolific and of interest as it can be seen from the numerous papers published in 2020 [8, 11, 12, 30].

An interesting turn was marked by the introduction of the dual notion of fuzzy differential superordination in 2017 [9] following the general theory of differential superordination introduced by Miller and Mocanu in 2003 [19]. The idea was soon continued and the special case of first order fuzzy differential superordination was studied [7].

Applications of both concepts of fuzzy differential subordinations and fuzzy differential superordinations started to be investigated regarding their connection [15] and the study is pushed forward by the blend of the notion of fuzzy differential subordination with special functions like fractional integral associated with generalized Mittag-Leffler function [31], fractional derivative [32], and λ-pseudo starlike and λ-pseudo convex functions [33].

The study of fuzzy subordinations and superordinations continue to develop as it can be seen in very recent publications. New fuzzy differential subordinations were obtained using an integral operator for which classical differential subordination results had been previously introduced [20] and also using a newly introduced hypergeometric integral operator [21]. Fuzzy differential subordinations based upon the Mittag-Leffler type Borel distribution were obtained [27] and Atangana–Baleanu fractional integral was also considered for obtaining fuzzy differential subordinations [4]. Fuzzy differential subordinations and superordinations emerged as applications of the fractional calculus [5] and the two concepts were used together for obtaining fuzzy differential sandwich theorems involving the fractional integral of confluent hypergeometric function [6].

Fuzzy differential subordination concept is introduced without the use of concepts from Fuzzy Set Theory, such as fuzzy arithmetic, fuzzy analysis, fuzzy numbers, Zadeh's extension principle, stacking theorem since it is well-known that the approach on complex numbers is completely different from that on real numbers. The notion of subordination was introduced in the theory of complex valued functions in order to extend the idea of inequality which can be applied for real numbers but not for complex numbers. A short history of the emergence of the concept can be found in [22]. The introduction of the concept of fuzzy differential subordination is the first attempt to embed the fuzzy set notion into the study related to geometric theory of analytic functions.

It is now obvious that the link between the two distinct domains, fuzzy sets theory and geometric theory of analytic functions, is durable and generates studies which reach original and interesting results.

The investigation of the present paper uses means of the classical theory of differential subordination and notions already adapted to fuzzy differential subordination.

The main classes of univalent functions used in this paper are first presented.

The unit disc of the complex plane is denoted by U and defined as

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

$\mathcal{H}(U)$ is the class of analytic functions in the unit disc. For n a positive integer and a a complex number, the class $\mathcal{H}[a,n]$ is defined as consisting of all functions from class $\mathcal{H}(U)$ who have the serial development

$$f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots,$$

with $\mathcal{H}_0=\mathcal{H}[0,1]$. The classical definition of class A_n is considered,

$$A_n=\{f\in\mathcal{H}(U):\ f(z)=z+a_{n+1}z^{n+1}+\dots\},$$

with $z\in U$ and A_1 written simply as A. All the functions in class A which are univalent in U form the class denoted by S. In particular, the functions in class A that have the property that $\mathrm{Re}\frac{zf'(z)}{f(z)}>0$ are called starlike functions and their class is denoted by S^*. The special class of starlike functions of order α is defined as

$$S^*(\alpha)=\left\{f\in A:\ \mathrm{Re}\frac{zf'(z)}{f(z)}>\alpha\right\},\ \alpha<1.$$

The functions from class A which have the property that

$$\mathrm{Re}\frac{zf''(z)}{f'(z)}+1>0$$

represent the class of convex functions denoted by K.

The class of Carathéodory functions is defined as

$$\mathcal{P}=\{p\in\mathcal{H}(U):p(0)=1,\ \mathrm{Re}\ p(z)>0,\ z\in U\}.$$

Some definitions and lemmas from the theory of differential subordination are needed for the study to be conducted.

Definition 1.1 *[18, pp. 4] A function $\mathcal{L}(z,t)$, $z\in\overline{U}$, $t\geq0$, is a subordination chain if $\mathcal{L}(\cdot,t)$ is analytic and univalent in U, for all $t\geq0$, and*

$$L(z,t_1)\prec L(z,t_2),\ \textit{when}\ 0\leq t_1\leq t_2.$$

Definition 1.2 *[18, Definition 2.2.d] We denote by Q the set of functions q that are analytic and injective on $\overline{U}\setminus E(q)$, where*

$$E(q)=\left\{\zeta\in\partial U:\ \lim_{z\to\zeta}q(z)=\infty\right\}$$

and are such that $q'(\zeta)\neq0$, for $\zeta\in\partial U\setminus E(q)$.

The set $E(q)$ is called exception set.

Lemma A. [18, Lemma 2.2.d] *Let $q\in Q$, with $q(0)=a$ and let*

$$p(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots$$

be analytic in U, with $p(z)\neq a$ and $n\geq1$. If p is not subordinate to q, then there exist points $z_0=r_0e^{i\theta_0}\in U$ and $\zeta_0\in\partial U\setminus E(q)$ and an $m\geq n\geq1$ for which $p(U_{r_0})\subset q(U)$ such that *(i)* $p(z_0) = q(\zeta_0)$;

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ *and*

(iii) $\mathrm{Re}\,\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\left[\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

Lemma B. [18, p. 4] *The function*

$$\mathcal{L}(z,t) = a_1(t)z + a_2(t)z^2 + \dots$$

with $a_1(t) \neq 0$, *for* $t \geq 0$, *and* $\lim\limits_{t\to\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if there exist constants* $t \in (0,1]$ *and* $M > 0$ *such that:*

(j) $L(z,t)$ *is analytic in* $|z| < r$ *for each* $t \geq 0$, *locally absolutely continuous in* $t \geq 0$ *for each* $|z| < r$, *and satisfies*

$$|L(z,t)| \leq M|a_1(t)|, \text{ for } |z| < r \text{ and } t \geq 0.$$

(jj) there exists a function $p(z,t)$ *analytic in* U *for all* $t \in [0,\infty)$ *and measurable in* $[0,\infty)$ *for each* $z \in U$, *such that* $\mathrm{Re}\,p(z,t) > 0$ *for* $z \in U$, $t \in [0,\infty)$ *and*

$$\frac{\partial L(z,t)}{\partial t} = z \cdot \frac{\partial L(z,t)}{\partial z} p(z,t),$$

for $|z| < r$ *and for almost all* $t \in [0,\infty)$.

Note that the univalency of the function $L(z,t)$ can be extended from $|z| < r$ to all of U.

Definition 1.3 *[18, p. 4] Let* f *and* F *be analytic in* U. *The function* f *is said to be subordinate to* F, *written* $f \prec F$ *or* $f(z) \prec F(z)$, *if there exists a function* w *analytic in* U, *with* $w(0) = 0$ *and* $|w(z)| < 1$, *and such that* $f(z) = F(w(z))$. *If* F *is univalent, then* $f \prec F$ *if and only if* $f(0) = F(0)$ *and* $f(U) \subset F(U)$.

In order to use the concept of fuzzy differential subordination, we remember the following definitions.

Definition 1.4 *[13] Let* X *be a nonempty set. An application* $F : X \to [0,1]$ *is called fuzzy subset.*

An alternate definition, more precise, would be the following:

A Pair (A, F_A), where $F_A : X \to [0,1]$ and

$$A = \{x \in X : 0 < F_A(x) \leq 1\}$$

is called fuzzy subset of X.

The set A is called the support of the fuzzy set (A, F_A) and F_A is called membership function of the fuzzy set (A, F_A).

One can also denote $A = \mathrm{supp}(A, F_A)$.

Definition 1.5 *[13] Let* (M, F_M) *and* (N, F_N) *be two fuzzy subsets of* X. *For the fuzzy subsets* M *and* N, *we say that* $M = N$ *if and only if* $F_M(x) = F_N(x)$, $x \in X$ *and we denote by* $(M, F_M) = (N, F_N)$. *The fuzzy subset* (M, F_M) *is contained in the fuzzy subset* (N, F_N) *if and only if* $F_M(x) \leq F_N(x)$, $x \in X$ *and we denote the inclusion relation by* $(M, F_M) \subseteq (N, F_N)$.

In [23], the following definition was introduced for the notion of fuzzy differential subordination as a generalization of the classical notion of differential subordination:

Let $D \subset \mathbb{C}$ and $f, g \in \mathcal{H}(D)$ be holomorphic functions. We denote by

$$f(D) = \left\{ f(z) \,|\, 0 < F_{f(D)} f(z) \leq 1, \;\; z \in D \right\} = \operatorname{supp}\left(f(D), F_{f(D)}\right)$$

and

$$g(D) = \left\{ g(z) \,|\, 0 < F_{g(D)} g(z) \leq 1, \;\; z \in D \right\} = \operatorname{supp}\left(g(D), F_{g(D)}\right).$$

Definition 1.6 *[23] Let $D \subset \mathbb{C}$ and let $z_0 \in D$ be a fixed point. We take the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and we write $f \prec_F g$ or $f(z) \prec_F g(z)$, if there exists a function $F : \mathbb{C} \to [0,1]$, such that*

(t) $f(z_0) = g(z_0)$;

(tt) $F_{f(D)} f(z) \leq F_{g(D)} g(z)$, $z \in D$.

Remark 1.7 *a) For making it easy to write, we shall use the following form:*

(tt) $F(f(z)) \leq F(g(z))$ for all $z \in D$.

b) If g is univalent, then $f \prec_F g$ if and only if $f(z_0) = g(z_0)$ and $f(D) \subset g(D)$.

c) Such a function $F : \mathbb{C} \to [0,1]$ can be considered

$$F(z) = \frac{|z|}{1+|z|}, \quad F(z) = \frac{1}{1+|z|}.$$

d) If $D = U$ then inequalities (t) and (tt) become:

(t') $f(0) = g(0)$

(tt') $f(U) \subset g(U)$,

which is equivalent to Definition 1.3, where the classical definition of subordination is given.

Definition 1.8 *[24] Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, $a \in \mathbb{C}$, and let h be univalent in U, with $h(0) = \psi(a,0,0,0)$, q be univalent in U, with $q(0) = a$, and p be analytic in U, with $p(0) = a$. Also, $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U and $F : \mathbb{C} \to [0,1]$. If p is analytic in U and satisfies the (second-order) fuzzy differential subordination*

$$F(\psi(p(z), zp'^2p''(z); z)) \leq F(h(z)) \tag{1.1}$$

i.e.

$$\psi(p(z), zp'^2p''(z); z) \prec_F h(z), \;\; z \in U, \tag{1.2}$$

then p is called a fuzzy solution of the fuzzy differential subordination.

The univalent function q is called a fuzzy dominant of the fuzzy solution of the fuzzy differential subordination, or more simply a fuzzy dominant, if

$$p(z) \prec_F q(z), \;\; z \in U,$$

for all p satisfying (1.1) or (1.2). A fuzzy dominant $\widetilde{q}$ that satisfies

$$\widetilde{q(z)} \prec_F q(z), \;\; z \in U,$$

for all fuzzy dominants q of (1.1) or (1.2) is said to be fuzzy best dominant of (1.1). Note that the fuzzy best dominant is unique up to a rotation in U.

The present paper continues the idea of adapting the results from the theory of differential subordination to the new aspects of fuzzy differential subordination. Obtaining univalence criteria for analytic functions is an important topic in the study concerning those functions and many such criteria were obtained using the classical theory of differential subordination. The theorem stated and proven in the next section and the corollaries associated with it show how sufficient conditions for univalence can be found using fuzzy differential subordinations.

2. Main results

The original part of this paper consists of proving a theorem using fuzzy differential subordinations and stating two univalence criteria in two corollaries derived from the proof of this theorem.

Theorem 2.1 *Let p and h be analytic in U, with $p(0) = h(0)$, let $\varphi : D \subset \mathbb{C} \to \mathbb{C}$, be analytic in a domain D containing $p(U)$ and $F : \mathbb{C} \to [0, 1]$,*

$$F(z) = \frac{|z|}{1 + |z|}, \ z \in U.$$

If

(k) $\operatorname{Re} \varphi[p(z)] > 0, \ z \in U$, *and*

(kk) h *is convex, then*

$$\frac{|p(z) + zp'(z) \cdot \varphi[p(z)]|}{1 + |p(z) + zp'(z) \cdot \varphi[p(z)]|} \le \frac{|h(z)|}{1 + |h(z)|} \tag{2.1}$$

that is

$$F(p(z) + zp'(z) \cdot \varphi[p(z)]) \le F(h(z)), \tag{2.2}$$

i.e.

$$p(z) + zp'(z) \cdot \varphi[p(z)] \prec_F h(z), \ z \in U, \tag{2.3}$$

implies

$$\frac{|p(z)|}{1 + |p(z)|} \le \frac{|h(z)|}{1 + |h(z)|}$$

which gives that

$$F(p(z)) \le F(h(z)),$$

i.e.

$$p(z) \prec_F h(z), \ z \in U.$$

Proof. Without loss of generality, we can assume that p, φ, and h satisfy the conditions of the theorem on the closed disc $\overline{U}$.

Let the function $\psi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ be defined as

$$\psi(r, s) = r + s \cdot \varphi(r), \ r, s \in \mathbb{C}. \tag{2.4}$$

For $r = p(z)$, $s = zp'(z)$, relation (2.4) becomes

$$\psi(p(z), zp'(z)) = p(z) + zp'(z) \cdot \varphi[p(z)], \ z \in U. \tag{2.5}$$

Using (2.5) in (2.1), we have

$$\frac{|\psi(p(z), zp'(z))|}{1+|\psi(p(z), zp'(z))|} \leq \frac{|h(z)|}{1+|h(z)|},\ z \in U, \tag{2.6}$$

and relation (2.2) becomes

$$\psi(p(z), zp'(z)) \prec_F h(z),\ z \in U. \tag{2.7}$$

For $z = z_0$, using Remark 1.7, relation (2.7) is equivalent to

$$\psi(p(z_0), z_0 p'(z_0)) \in h(U). \tag{2.8}$$

In order to prove that (2.1) or (2.7) implies that p is subordinate to function h, Lemma A must be applied. For that, we assume that the functions p, φ, h satisfy the conditions in Lemma A in the unit disc $\overline{U}$. If p is not subordinate to h, by Lemma A, there exist points $z_0 = r_0 e^{i\theta_0} \in U$, $\zeta_0 \in \partial U \setminus E(q)$ and $m \geq 1$, that satisfy

$$p(z_0) = h(\zeta_0),\ z_0 p'(z_0) = m\zeta_0 h'(\zeta_0),\ |\zeta_0| = 1.$$

Since h is analytic in $\overline{U}$, the function

$$\mathcal{L}(z,t) = h(z) + tzh'(z)\varphi[p(z_0)] = a_1(t) + a_2(t)z^2 + \ldots \tag{2.9}$$

is analytic in U, for all $t \geq 0$, and is continuously differentiable on $[0, \infty)$ for all $z \in \overline{U}$. Differentiating (2.9), with respect to z, we obtain

$$\frac{\partial \mathcal{L}(z,t)}{\partial z} = a_1(t) + 2a_2(t)z + \ldots \tag{2.10}$$

$$= h'(z) + th'(z)\varphi[p(z_0)] + tzh''(z) \cdot \varphi[p(z_0)].$$

For $z = 0$, we have

$$a_1(t) = h'(0)(1 + t\varphi[p(z_0)]) \neq 0 \text{ for } t \geq 0,$$

since h is a convex function in U, hence univalent, and $\lim_{t\to\infty} |a_1(t)| = \infty$.

Differentiating (2.9) with respect to t, we obtain

$$\frac{\partial L(z,t)}{\partial t} = z \cdot h'(z) \cdot \varphi[p(z_0)],\ z \in \overline{U}. \tag{2.11}$$

A simple calculation combined with (k) and (kk) yields

$$\text{Re}\left[\frac{z \cdot \partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] = \text{Re}\,\frac{1}{\varphi(p(z_0))} + t \cdot \text{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > 0, \tag{2.12}$$

for $z \in \overline{U}$ and $t \geq 0$.

Using Lemma B, relation (2.12) gives that $L(z,t)$ is a subordination chain, which by definition implies

$$L(z,s) \prec L(z,t),\ \text{when } 0 \leq s \leq t. \tag{2.13}$$

For $z = 0$, we have

$$L(z,0) \prec L(z,t),\ t \geq 0. \tag{2.14}$$

On the other hand, if $L(z,0) = h(z)$, the relation (2.14) becomes

$$h(z) \prec L(z,t) \tag{2.15}$$

and we deduce that $L(z,t) \notin h(z)$, for $z \in \overline{U}$ and $t \geq 0$.

For $z = \zeta_0$, $|\zeta_0| = 1$ and $t = m \geq 0$, we have

$$L(\zeta_0, m) \notin h(z). \tag{2.16}$$

We have supposed that $p \not\prec_F h$, then using Lemma A, we have that

$$p(z_0) = h(\zeta_0) \text{ and } z_0 p'(z_0) = m\zeta_0 h'(\zeta_0).$$

We calculate

$$\begin{aligned} \psi(p(z_0), z_0 p'(z_0)) &= p(z_0) + z_0 p'(z_0) \cdot \varphi[p(z_0)] \\ &= h(\zeta_0) + m\zeta_0 h'(\zeta_0) \cdot \varphi[h(\zeta_0)] \\ &= L(\zeta_0, m). \end{aligned}$$

Using (2.15), we obtain that $\psi(p(z_0), z_0 p'(z_0)) \notin h(U)$. Since this contradicts (2.8), we conclude that

$$p(z) \prec_F h(z), \ z \in U.$$

Remark 2.2 *If* $h(z) = \frac{1+z}{1-z}$ *is a convex function with* $\operatorname{Re} h(z) > 0$, *then we obtain the following criterion for univalence:*

Corollary 2.3 *Let* p *and* $h(z) = \frac{1+z}{1-z}$ *be analytic in* U, *with* $p(0) = h(0) = 1$, *let* $\varphi : D \subset \mathbb{C} \to \mathbb{C}$ *be analytic in the domain* D *containing* $p(U)$, *and* $F : \mathbb{C} \to [0,1]$,

$$F(z) = \frac{|z|}{1+|z|}, \ z \in U.$$

If

(ℓ) $\operatorname{Re} \varphi[p(z)] > 0, \ z \in U$, *and*

($\ell\ell$) h *is convex in* U, *then*

$$\frac{|p(z) + zp'(z) \cdot \varphi[p(z)]|}{1 + |p(z) + zp'(z) \cdot \varphi[p(z)]|} \leq \frac{\left|\frac{1+z}{1-z}\right|}{1 + \left|\frac{1+z}{1-z}\right|}$$

that is

$$F(p(z) + zp'(z) \cdot \varphi[p(z)]) \leq F\left(\frac{1+z}{1-z}\right),$$

i.e.

$$p(z) + zp'(z) \cdot \varphi[p(z)] \prec_F \frac{1+z}{1-z}, \ z \in U,$$

implies

$$\frac{|p(z)|}{1+|p(z)|} \leq \frac{\left|\frac{1+z}{1-z}\right|}{1+\left|\frac{1+z}{1-z}\right|}$$

which gives that

$$F(p(z)) \leq F\left(\frac{1+z}{1-z}\right)$$

i.e.

$$p(z) \prec_F \frac{1+z}{1-z}, \;\; z \in U$$

and $p \in \mathcal{P}$.

Proof. From the proof of Theorem 2.1, we get

$$p(z) \prec_F h(z) = \frac{1+z}{1-z}, \;\; z \in U. \tag{2.17}$$

We now show that function h is convex in U which is equivalent to

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > 0.$$

We evaluate:

$$h'(z) = \frac{2}{(1-z)^2}; \quad h''(z) = \frac{4}{(1-z)^3}.$$

We now have

$$\frac{zh''(z)}{h'(z)} + 1 = \frac{\frac{4z}{(1-z)^3}}{\frac{2}{(1-z)^2}} + 1 = \frac{2z}{1-z} + 1 = \frac{1+z}{1-z}.$$

In order to compute $\mathrm{Re}\frac{1+z}{1-z}$, we consider $z = \rho(\cos\alpha + i\sin\alpha) \in U$, $0 \leq \rho < 1$. Then we obtain

$$\mathrm{Re}\frac{1+z}{1-z} = \mathrm{Re}\frac{1+\rho\cos\alpha + \rho i \sin\alpha}{1-\rho\cos\alpha - \rho i\sin\alpha} =$$

$$\mathrm{Re}\frac{(1+\rho\cos\alpha+\rho i\sin)(1-\rho\cos\alpha+\rho i\sin)}{(1-\rho\cos\alpha)^2+\rho^2\sin^2\alpha} =$$

$$\mathrm{Re}\frac{1-\rho^2\cos^2\alpha-\rho^2\sin^2\alpha}{(1-\rho\cos\alpha)^2+\rho^2\sin^2\alpha} = \mathrm{Re}\frac{1-\rho^2}{(1-\rho\cos\alpha)^2+\rho^2\sin^2\alpha} > 0,$$

since $\rho < 1$.

Hence, $\mathrm{Re}\frac{zh''(z)}{h'(z)} + 1 = \mathrm{Re}\frac{1+z}{1-z} > 0$, $z \in U$, which gives that function h is convex in U.

Since h is a convex function, $h(U)$ is a convex domain and

$$\mathrm{Re}\, h(z) = \mathrm{Re}\,\frac{1+z}{1-z} > 0, \;\; z \in U,$$

the differential subordination (2.17) is equivalent to

$$\operatorname{Re} p(z) > \operatorname{Re} \frac{1+z}{1-z} > 0,$$

hence $p \in \mathcal{P}$.

Remark 2.4 *If $h(z) = \frac{1}{1-z}$, $z \in U$, is a convex function with*

$$\operatorname{Re} \frac{1}{1-z} = \frac{1}{2}, \ z \in U,$$

then we obtain the following univalence criterion:

Corollary 2.5 *Let p and $h(z) = \frac{1}{1-z}$ be analytic in U, with $p(0) = h(0) = 1$, let $\varphi : D \subset \mathbb{C} \to \mathbb{C}$ be analytic in a domain D containing $p(U)$ and*

$$F : \mathbb{C} \to [0,1], \ F(z) = \frac{|z|}{1+|z|}, \ z \in U.$$

If

(m) $\operatorname{Re} \varphi[p(z)] > 0, \ z \in U$, *and*

(mm) $h(z) = \frac{1}{1-z}$ *convex in* U, *then*

$$\frac{|p(z) + zp'(z) \cdot \varphi[p(z)]|}{1 + |p(z) + zp'(z) \cdot \varphi[p(z)]|} \le \frac{\left|\frac{1}{1-z}\right|}{1 + \left|\frac{1}{1-z}\right|}, \ z \in U,$$

that is

$$F(p(z) + zp'(z) \cdot \varphi[p(z)]) \le F\left(\frac{1}{1-z}\right), \ z \in U,$$

i.e.

$$p(z) + zp'(z) \cdot \varphi[p(z)] \prec_F \frac{1}{1-z}, \ z \in U,$$

implies

$$\frac{|p(z)|}{1+|p(z)|} \le \frac{\left|\frac{1}{1-z}\right|}{1 + \left|\frac{1}{1-z}\right|}, \ z \in U,$$

which gives that

$$F(p(z)) \le F\left(\frac{1}{1-z}\right), \ z \in U,$$

i.e.

$$p(z) \prec_F h(z) = \frac{1}{1-z}, \ z \in U$$

and $p \in \mathcal{P}$.

Proof. From the proof of Theorem 2.1, we have that

$$p(z) \prec_F h(z) = \frac{1}{1-z}, \ z \in U. \tag{2.18}$$

We next prove that function h is convex in U which is equivalent to

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > 0.$$

We have

$$h'(z) = \frac{1}{(1-z)^2}; \quad h''(z) = \frac{2}{(1-z)^3}.$$

We obtain

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{\frac{2z}{(1-z)^3}}{\frac{1}{(1-z)^2}} + 1 = \frac{2z}{1-z} + 1 = \frac{1+z}{1-z}.$$

We have seen in the proof of Corollary 2.3 that $\mathrm{Re}\frac{1+z}{1-z} > 0$; hence, function $h(z) = \frac{1}{1-z}$ is convex in U.

Since h is a convex function, $h(U)$ is a convex domain and

$$\mathrm{Re}\, h(z) = \mathrm{Re}\, \frac{1}{1-z} = \frac{1}{2}, \ z \in U,$$

differential subordination (2.18) is equivalent to

$$\mathrm{Re}\, p(z) > \mathrm{Re}\, \frac{1}{1-z} = \frac{1}{2}, \ z \in U;$$

hence, $p \in \mathcal{P}$.

Example 2.6 *Let $p(z) = 1 + \frac{1}{2}z$, $z \in U$, let $h(z) = \frac{1}{1-z}$ be a convex function in U with $p(0) = h(0) = 1$. Let $\varphi : D \subset \mathbb{C} \to \mathbb{C}, \varphi(w) = 1 + w$ and $F : \mathbb{C} \to [0,1]$, $F(z) = \frac{1}{1+|z|}$, where $z = \rho\cos\alpha + i\rho\sin\alpha$, $z \in U$, $0 \le \rho < 1$.*

We have $p(z) + zp'(z) \cdot \varphi[p(z)] = 1 + \frac{3}{2}z + \frac{1}{4}z^2$ and using Corollary 2.5, we can write:

If

(m)

$$\mathrm{Re}\varphi[p(z)] = \mathrm{Re}[1 + p(z)] = \mathrm{Re}\left[2 + \frac{1}{2}z\right] = \mathrm{Re}\left(2 + \frac{1}{2}\rho\cos\alpha + i\frac{1}{2}\rho\sin\alpha\right)$$

$$= 2 + \frac{1}{2}\rho\cos\alpha = \frac{3}{2} + \frac{1+\rho\cos\alpha}{2} > \frac{3}{2} > 0, \quad z \in U;$$

(mm)

$$h(z) = \frac{1}{1-z}, \quad \mathrm{Re}\left[\frac{zh''(z)}{h'(z)} + 1\right] > 0, \ \ z \in U,$$

then

$$F\left(1+\frac{3}{2}z+\frac{1}{4}z^2\right)\leq F\left(\frac{1}{1-z}\right),$$

equivalently written

$$\frac{1}{1+\left|1+\frac{3}{2}z+\frac{1}{4}z^2\right|}\leq\frac{1}{1+\left|\frac{1}{1-z}\right|},$$

which is the fuzzy differential subordination

$$1+\frac{3}{2}z+\frac{1}{4}z^2\prec_F\frac{1}{1-z},\quad z\in U,$$

implies

$$F\left(p\left(z\right)\right)\leq F\left(h\left(z\right)\right),$$

equivalently written

$$\frac{1}{1+\left|1+\frac{1}{2}z\right|}\leq\frac{1}{1+\left|\frac{1}{1-z}\right|},\quad z\in U,$$

which is the fuzzy differential subordination

$$1+\frac{1}{2}z\prec_F\frac{1}{1-z},\quad z\in U. \tag{2.19}$$

Since $h\left(z\right)=\frac{1}{1-z}$ *is convex in* U*, fuzzy differential subordination (2.19) is equivalent to:*

$$\mathrm{Re}\left(1+\frac{1}{2}z\right)>\mathrm{Re}\frac{1}{1-z},\quad z\in U.$$

Since $\mathrm{Re}\frac{1}{1-z}=\frac{1}{2}$*, we can write* $\mathrm{Re}\left(1+\frac{1}{2}z\right)>\frac{1}{2}$*, which means that* $p\left(z\right)=1+\frac{1}{2}z\in\mathcal{P}$*.*

Indeed, since $p\left(0\right)=h\left(0\right)=1$*, we get that* $p\left(U\right)=\left\{t\in\mathbb{C}:\mathrm{Re}\ t>\frac{3}{2}\right\}$ *and* $h\left(U\right)=\left\{v\in\mathbb{C}:\mathrm{Re}\ v>\frac{1}{2}\right\}$*; hence,* $p\left(U\right)\subset h\left(U\right)$*.*

*Using Definition 1.6 and Remark 1.7*d*, we get that*

$$F\left(p\left(z\right)\right)\leq F\left(h\left(z\right)\right),$$

which is the fuzzy differential subordination

$$1+\frac{1}{2}z\prec_F\frac{1}{1-z}. \tag{2.20}$$

Since $h\left(z\right)=\frac{1}{1-z}$ *is convex in* U*, fuzzy differential subordination (2.20) is equivalent to*

$$\mathrm{Re}\left(1+\frac{1}{2}z\right)>\mathrm{Re}\frac{1}{1-z}=\frac{1}{2};$$

hence, $p\left(z\right)=1+\frac{1}{2}z\in\mathcal{P}$*.*

3. Conclusion

The connection between fuzzy sets theory and geometric theory of analytic functions is clearly strong and durable and adapting the notions of the theory of differential subordination to the fuzzy sets theory obviously works and has outcomes that are of interest for the researchers in the field of complex analysis who want to extend their focus as it can be seen from the few papers cited in the present paper which are, of course, just a selection not all the work done on this topic. The original results of this paper are just the starting point for a new direction concerning the study of the univalence of analytic functions. The results stated here prove that stating such conditions is possible using the theory of fuzzy differential subordination. The next step is to associate to the study of univalence with different types of operators just as they were associated with the general investigations done concerning fuzzy differential subordinations. Then this study could be continued using the dual notion of fuzzy differential superordination, and connections between the two theories might be established through sandwich-type theorems familiar to the field of geometric theory of analytic functions.

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