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Arf numerical semigroups with multiplicity 11 and 13

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Abstract: Parametrizations are given for Arf numerical semigroups with multiplicity up to 10. In this work, we give parametrizations of Arf numerical semigroups with multiplicity 11 and 13, and combining these results with previous results about the number of Arf numerical semigroups with multiplicity 2, 3, 5, 7, we share some observations about the set of Arf numerical semigroups with prime multiplicity.

Key words: Numerical semigroups, Arf numerical semigroups, multiplicity, conductor, Frobenius number, ratio

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the set of nonnegative integers. The cardinality of a set K will be denoted by $|K|$. A subset $S \subseteq \mathbb{N}_0$ satisfying

$$(i)\ 0 \in S \qquad (ii)\ x, y \in S \Rightarrow x + y \in S \qquad (iii)\ |\mathbb{N}_0 \setminus S| < \infty$$

is called a numerical semigroup. It is well known (see, for instance, [2, 4, 8]) that the condition (iii) above is equivalent to saying that the greatest common divisor $\gcd(S)$ of elements of S is 1.

If A is a subset of $\mathbb{N}_0$, we will denote by $\langle A \rangle$ the submonoid of $\mathbb{N}_0$ generated by A. If $S = \langle A \rangle$, A is called a set of generators for S. If $A = \{a_1, \ldots, a_r\}$, we write $\langle A \rangle = \langle a_1, \ldots, a_r \rangle$. The monoid $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

For every numerical semigroup S there exists a unique minimal set of generators $\{a_1, a_2, \ldots, a_e\}$ with $a_1 < a_2 < \cdots < a_e$; that is, $\{a_1, a_2, \ldots, a_e\}$ is a set of generators for S, but no proper subset of $\{a_1, a_2, \ldots, a_e\}$ generates S. The integers a_1 and e are called the multiplicity and the embedding dimension of S, and they are denoted by $m(S)$ and $e(S)$, respectively. The multiplicity $m(S)$ is the smallest positive element of S. It is known that $e(S) \leq m(S)$ (see, for instance Chapter 1 of [8]). S is said to be a numerical semigroup of maximal embedding dimension if $e(S) = m(S)$.

For a numerical semigroup S, the largest integer that is not in S is called the Frobenius number of S and it is denoted by $\mathsf{f}(S)$; the smallest element of S for which all subsequent natural numbers belong to S is called the conductor of S and it is denoted by $c(S)$. Clearly, $c(S) = \mathsf{f}(S) + 1$. We have $c(\mathbb{N}_0) = 0$ and $c(S) \geq 2$ if and only if $S \neq \mathbb{N}_0$.

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From now on we stick to the above notations and we put $\mathsf{f}(S) = \mathsf{f}$, $c(S) = c$.

If S is a numerical semigroup and $a \in S\backslash\{0\}$, the Apéry set of S with respect to a is the set

$$\mathrm{Ap}(S, a) = \{s \in S : s - a \notin S\}.$$

It is easy to see that $\mathrm{Ap}(S, a) = \{w(0) = 0, w(1), \ldots, w(a-1)\}$, where $w(i)$ is the least element of S such that $w(i) \equiv i \pmod{a}$. It is also easy to see that

$$S = \langle a, w(1), \ldots, w(a-1)\rangle \text{ and } \mathsf{f} = \max(\mathrm{Ap}(S, a)) - a.$$

Thus

$$c = \max(\mathrm{Ap}(S, a)) - a + 1 \text{ and } \max(\mathrm{Ap}(S, a)) = c + a - 1.$$

For general concepts and facts about numerical semigroups, we refer to [4] and [8].

2. Arf numerical semigroups

A numerical semigroup S satisfying the additional condition

$$x, y, z \in S; x \geq y \geq z \Rightarrow x + y - z \in S \tag{2.1}$$

is called an Arf numerical semigroup.

This is the original definition of an Arf numerical semigroup given by C. Arf in [1]. We will refer to the condition (2.1) as the Arf condition. Fifteen conditions equivalent to the Arf condition are given in Theorem 1.3.4 of [2]. If $x, y, z \in S$; $x \geq y \geq z$ and $x \geq c$, the conductor of S, then $x + y - z \geq c$ and thus $x + y - z \in S$. Therefore, to prove that a numerical semigroup with conductor c is an Arf numerical semigroup it is enough to check the Arf condition (2.1) for small elements of S; i.e. for elements $x < c$.

The following lemma by Campillo, Farran and Munuera [3] gives a very useful condition equivalent to the Arf condition.

Lemma 2.1 *[[3], Proposition 2.3] A numerical semigroup S is an Arf numerical semigroup if and only if $2x - y \in S$ for all $x, y \in S$ with $x \geq y$.*

Combining Lemma 2.1 and our remark above about the Arf condition, we see that a numerical semigroup S is an Arf numerical semigroup if and only if $2x - y \in S$ for all $x, y \in S$ with $c > x > y$.

The following lemma of Rosales et al. [9] will be crucial for what follows.

Lemma 2.2 *[[8], Lemma 11] Let S be an Arf numerical semigroup and let s be any element of S. If $s + 1 \in S$, then $s + k \in S$ for all $k \in \mathbb{N}_0$ and thus $c \leq s$.*

It is well known (see, for instance, [7] or [8]) that every Arf numerical semigroup is of maximal embedding dimension, that is, $e(S) = m(S)$. Thus if S is an Arf numerical semigroup with multiplicity $m = m(S)$, then

$$\{m, w(1), \ldots, w(m-1)\}$$

is the minimal set of generators for S, where

$$\mathrm{Ap}(S, m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}.$$

The number $\max(\mathrm{Ap}(S,m))$ is called the major of S and it is denoted by $\mathcal{M}$. The smallest minimal generator that is larger than the multiplicity of S is called the ratio of S and it is denoted by $\mathcal{R}$. Thus $\mathcal{M} = c + m - 1$ and $\mathcal{R} = \min(\mathrm{Ap}(S,m)\backslash\{0\})$.

Let S be a numerical semigroup with multiplicity m and conductor c. Since any multiple of m is an element of S, $c \not\equiv 1 \pmod{m}$. Hence $c \equiv k \pmod{m}$, where $k \in \{0, 2, \ldots, m-1\}$. The following lemma by Garcia-Sánchez et al. [5] shows that if S is an Arf numerical semigroup, then $w(1)$ and $w(m-1)$ are completely determined by the conductor c and the multiplicity m.

Lemma 2.3 *[[5], Lemma 13] Let S be an Arf numerical semigroup with multiplicity m and conductor c where $c \equiv k \pmod{m})$, $k \in \{0, 2, \ldots, m-1\}$. Then*

$$(i)\ w(1) = \begin{cases} c+1 & \text{if} \quad k = 0 \ (i.e,\ c \equiv 0 \pmod{m}), \\ c-k+m+1 & \text{if} \quad k \neq 0 \ (i.e,\ c \not\equiv 0 \pmod{m}), \end{cases}$$

(ii) $w(m-1) = c - k + m - 1.$

The next three propositions are taken from [6]. [[5], Lemma 13]

Proposition 2.4 *[[6], Proposition 2.3] Let S be an Arf numerical semigroup with multiplicity $m > 2$, conductor c and $\mathrm{Ap}(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$. Let $k \in \{1, \ldots, m-1\}$. Then*

(i) $w(m-k) + 2k \in S$ and thus $w(k) \leq w(m-k) + 2k$.

(ii) For any positive integer q with $qk < m$, we have $w(k)+(q-1)k \in S$ and thus $w(qk) \leq w(k)+(q-1)k$.

(iii) For $k < \frac{m}{2}$, we have $w(m-2k) \leq w(m-k) + (m-k)$.

Proposition 2.5 *[[6], Proposition 2.4] Let S be an Arf numerical semigroup with multiplicity $m > 2$, conductor c and $\mathrm{Ap}(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$. Assume that the ratio of S is $\mathcal{R} = w(k)$, where $k \in \{1, \ldots, m-1\}$. Let $q \in \mathbb{N}$ such that $qk < m$. Then*

(i) $w(qk) = w(k) + (q-1)k,$

(ii) $w(qk+1) \geq c+1$ if $qk+1 < m,$

(iii) $w(qk-1) \geq c$ if $qk \neq 1$.

Proposition 2.6 *[[6], Proposition 2.5] Let S be an Arf numerical semigroup with multiplicity $m > 2$, conductor c and $\mathrm{Ap}(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$. Assume that the ratio of S is $\mathcal{R} = w(m-k)$, where $k \in \{1, \ldots, m-1\}$. Then for any $q \in \mathbb{N}$*

(i) $q \geq 2$, $qk < m \Longrightarrow w((q-1)k) = w(m-k) + qk$ and $w(m-2k) = w(m-k) + (m-k),$

(ii) $m < qk < m+k \Longrightarrow w((q-1)k) \in \{w(m-k) + qk - m, w(m-k) + qk\}.$

Parametrizations are given for Arf numerical semigroups with multiplicity up to ten and given conductor (see [5, 6]). One can observe from these parametrizations that the number of Arf numerical semigroups with multiplicity 2, 3, 5, 7 and given conductor depends only on the congruence class of the conductor modulo the multiplicity. We shall see below that the same is true for Arf numerical semigroups with multiplicity 11 and 13. Hence it is natural to ask whether this is true for Arf numerical semigroups with any prime multiplicity.

The ratio $\mathcal{R}$ will play an important part in our discussions. Recall that $\mathcal{R}$ is the least element larger than the multiplicity in the minimal set of generators of the numerical semigroup under consideration. It is easily seen that

$$\mathcal{R} \leq c+1 \text{ if } c \equiv 0 \pmod{m} \text{ and } \mathcal{R} \leq c \text{ if } c \not\equiv 0 \pmod{m}.$$

Each proposition in the next two sections will give the list of all Arf numerical semigroups with multiplicity $m = 11$ or $m = 13$ and conductor c for a congruence class of $c \pmod{m}$. Each semigroup in each list is easily seen to be Arf by applying, for instance, Lemma 2.1. Therefore in the proof of each proposition it is enough to verify that the list there contains all Arf numerical semigroups with the given multiplicity and conductor.

3. Arf numerical semigroups with multiplicity 11

Let S be an Arf numerical semigroup with multiplicity 11 and conductor c. Then $c \equiv 0, 2, 3, 4, 5, 6, 7, 8, 9$ or $10 \pmod{11}$.

The following proposition describes all Arf numerical semigroups S with multiplicity 11 and conductor c if $c \equiv 0 \pmod{11}$, $c > 11$.

Proposition 3.1 *Let S be a numerical semigroup with multiplicity 11 and conductor c, where $c > 11$ and $c \equiv 0 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following 9 semigroups:*

$\langle 11, c-8, c-5, c-2, c+1, c+2, c+4, c+5, c+7, c+8, c+10\rangle$ *; or*

$\langle 11, c-7, c-3, c+1, c+2, c+3, c+5, c+6, c+7, c+9, c+10\rangle$ *; or*

$\langle 11, c-5, c-2, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+10\rangle$ *; or*

$\langle 11, c-5, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10\rangle$ *; or*

$\langle 11, c-4, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10\rangle$ *; or*

$\langle 11, c-4, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10\rangle$ *, or*

$\langle 11, c-3, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10\rangle$ *; or*

$\langle 11, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10\rangle$ *; or*

$\langle 11, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10\rangle$ *.*

Proof Let S be an Arf numerical semigroup with multiplicity 11 and conductor $c \equiv 0 \pmod{11}$, where $c > 11$. Then $w(1) = c+1$ and $w(10) = c+10 = \mathcal{M}$ by Lemma 2.3. Using Proposition 2.4(ii), $c+10 = w(10) \leq w(2)+8$ which implies $w(2) = c+2$. Similarly, $w(5) = c+5$. Now applying Proposition 2.4(i) to $w(2)$ and $w(5)$, we get $w(9) \geq c-2$ and $w(6) \geq c-5$. Proposition 2.4(ii) applied to $w(6)$ gives $w(3) \geq c-8$, and Proposition 2.4(iii) applied to $w(5)$ gives $w(8) \geq c-3$. Now we apply Proposition 2.4(ii) to $w(8)$ and get $w(4) \geq c-7$. Finally, we apply Proposition 2.4(i) to $w(4)$ and get $w(7) \geq c-15$. However, $w(7) = c-15$ is impossible, because otherwise $2(c-11)-(c-15) = c-7 \in S$, and $c-15+11 = c-4 \in S$ which yield $2(c-4)-(c-7) = c-1 \in S$, a contradiction. It follows that

$$\mathcal{R} \in \{w(3) = c-8, w(4) = c-7, w(6) = c-5, w(7) = c-4, w(8) = c-3, w(9) = c-2, w(1) = c+1\}.$$

If $\mathcal{R} = c-8$, then $w(4) = c+4$, $w(7) = c+7$, $w(8) = c+8$ by Proposition 2.5. By the same proposition, we have $w(6) = c-5$, and $w(9) = c-2$. Hence

$$S = \langle 11, c-8, c-5, c-2, c+1, c+2, c+4, c+5, c+7, c+8, c+10\rangle.$$

If $\mathcal{R} = c-7$, then $w(3) = c+3$, $w(7) = c+7$, $w(9) = c+9$ and $w(8) = c-3$ by Proposition 2.5. We also note that $c-5 \notin S$, because otherwise we would have $2(c-3)-(c-5) = c-1 \in S$ which is impossible. So $w(6) = c+6$ and we have

$$S = \langle 11, c-7, c-3, c+1, c+2, c+3, c+5, c+6, c+7, c+9, c+10 \rangle.$$

If $\mathcal{R} = c-5$, then $w(7) = c+7$ by Proposition 2.5. As $w(i) > w(6)$ for each $i \neq 6$, we have $w(3) = c+3$, $w(4) = c+4$. Let us also note that $c-3 \notin S$ since $2(c-3)-(c-5) = c-1$. So $w(8) = c+8$. Finally, we note that $w(9) \in \{c-2, c+9\}$. Thus we have

$$S = \langle 11, c-5, c-2, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+10 \rangle; \text{ or}$$
$$S = \langle 11, c-5, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10 \rangle.$$

If $\mathcal{R} = c-4$, then $w(6) = c+6$, $w(8) = c+8$ by Proposition 2.5. As $w(i) > w(7)$ for each $i \neq 7$, we have $w(3) = c+3$, $w(4) = c+4$. Note also that $w(9) \in \{c-2, c+9\}$. Hence

$$S = \langle 11, c-4, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10 \rangle; \text{ or}$$
$$S = \langle 11, c-4, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10 \rangle.$$

If $\mathcal{R} = c-3$, $w(7) = c+7$, then and $w(9) = c+9$ by Proposition 2.5. We also have $w(3) = c+3$, because $w(8) = c-3 < w(3)$. Similarly, $w(4) = c+4$, $w(6) = c+6$ and we have

$$S = \langle 11, c-3, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10 \rangle.$$

If $\mathcal{R} = c-2$, then $w(8) = c+8$ by Proposition 2.5. We also have $w(3) = c+3$, because $c-2 = w(9) < w(3)$. Similarly, $w(4) = c+4$, $w(6) = c+6$, $w(7) = c+7$ and we have

$$S = \langle 11, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10 \rangle.$$

If $\mathcal{R} = c+1$, then

$$S = \langle 11, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10 \rangle.$$

This completes the proof. □

The proof of each of the remaining propositions in this section is very similar to the above proof. Therefore we omit them.

Proposition 3.2 *Let S be a numerical semigroup with multiplicity 11 and conductor c, where $c > 13$ and $c \equiv 2 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following 5 semigroups:*

$\langle 11, c-6, c-4, c, c+1, c+2, c+3, c+4, c+6, c+8, c+10 \rangle$; *or*

$\langle 11, c-6, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c-5, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c-4, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10 \rangle$; *or*

$\langle 11, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10 \rangle$.

Proposition 3.3 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 14$ and $c \equiv 3 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 3 *semigroups:*

$\langle 11, c-7, c, c+1, c+2, c+3, c+5, c+6, c+7, c+9, c+10 \rangle$; *or*

$\langle 11, c-6, c, c+1, c+2, c+3, c+4, c+6, c+7, c+9, c+10 \rangle$; *or*

$\langle 11, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10 \rangle$.

Proposition 3.4 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 15$ and $c \equiv 4 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 3 *semigroups:*

$\langle 11, c-6, c, c+1, c+2, c+3, c+4, c+6, c+8, c+9, c+10 \rangle$; *or*

$\langle 11, c-2, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10 \rangle$; *or*

$\langle 11, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10 \rangle$.

Proposition 3.5 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 16$ and $c \equiv 5 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 3 *semigroups:*

$\langle 11, c-8, c-2, c, c+1, c+2, c+4, c+5, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c-2, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10 \rangle$.

Proposition 3.6 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 17$ and $c \equiv 6 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 5 *semigroups:*

$\langle 11, c-8, c-4, c-2, c, c+1, c+2, c+4, c+6, c+8, c+10 \rangle$; *or*

$\langle 11, c-4, c-2, c, c+1, c+2, c+3, c+4, c+6, c+8, c+10 \rangle$; *or*

$\langle 11, c-3, c, c+1, c+2, c+3, c+4, c+6, c+7, c+9, c+10 \rangle$; *or*

$\langle 11, c-2, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+9, c+10 \rangle$.

Proposition 3.7 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 18$ and $c \equiv 7 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 3 *semigroups:*

$\langle 11, c-3, c, c+1, c+2, c+3, c+5, c+6, c+7, c+9, c+10 \rangle$; *or*

$\langle 11, c-2, c, c+1, c+2, c+3, c+5, c+6, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c, c+1, c+2, c+3, c+5, c+6, c+7, c+8, c+9, c+10 \rangle$.

Proposition 3.8 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 19$ and $c \equiv 8 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 7 *semigroups:*

$\langle 11, c-6, c-4, c-2, c, c+1, c+2, c+4, c+6, c+8, c+10 \rangle$; *or*

$\langle 11, c-5, c-2, c, c+1, c+2, c+4, c+5, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c-4, c-2, c, c+1, c+2, c+4, c+5, c+6, c+8, c+10 \rangle$; *or*

$\langle 11, c-4, c, c+1, c+2, c+4, c+5, c+6, c+8, c+9, c+10 \rangle$; *or*

$\langle 11, c-3, c, c+1, c+2, c+4, c+5, c+6, c+7, c+9, c+10 \rangle$; *or*

$\langle 11, c-2, c, c+1, c+2, c+4, c+5, c+6, c+7, c+8, c+10 \rangle$; *or*

$\langle 11, c, c+1, c+2, c+4, c+5, c+6, c+7, c+8, c+9, c+10 \rangle$.

Proposition 3.9 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 20$ and $c \equiv 9 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 6 *semigroups:*

$\langle 11, c-6, c-3, c, c+1, c+3, c+4, c+6, c+7, c+9, c+10 \rangle$; *or*
$\langle 11, c-4, c-2, c, c+1, c+3, c+4, c+5, c+6, c+8, c+10 \rangle$; *or*
$\langle 11, c-4, c, c+1, c+3, c+4, c+5, c+6, c+8, c+9, c+10 \rangle$; *or*
$\langle 11, c-3, c, c+1, c+3, c+4, c+5, c+6, c+7, c+9, c+10 \rangle$; *or*
$\langle 11, c-2, c, c+1, c+3, c+4, c+5, c+6, c+7, c+8, c+10 \rangle$; *or*
$\langle 11, c, c+1, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10 \rangle$.

Proposition 3.10 *Let S be a numerical semigroup with multiplicity* 11 *and conductor c, where $c > 21$ and $c \equiv 10 \pmod{11}$. Then S is an Arf semigroup if and only if S is one of the following* 10 *semigroups:*

$\langle 11, c-8, c-6, c-4, c-2, c, c+2, c+4, c+6, c+8, c+10 \rangle$; *or*
$\langle 11, c-6, c-4, c-2, c, c+2, c+3, c+4, c+6, c+8, c+10 \rangle$; *or*
$\langle 11, c-6, c-2, c, c+2, c+3, c+4, c+6, c+7, c+8, c+10 \rangle$; *or*
$\langle 11, c-5, c-2, c, c+2, c+3, c+4, c+5, c+7, c+8, c+10 \rangle$; *or*
$\langle 11, c-5, c, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10 \rangle$; *or*
$\langle 11, c-4, c-2, c, c+2, c+3, c+4, c+5, c+6, c+8, c+10 \rangle$; *or*
$\langle 11, c-4, c, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10 \rangle$; *or*
$\langle 11, c-3, c, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10 \rangle$; *or*
$\langle 11, c-2, c, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10 \rangle$; *or*
$\langle 11, c, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10 \rangle$.

For any rational number x, the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$. The number of Arf numerical semigroups with multiplicity m and conductor c is denoted by $N_{ARF}(m,c)$.

Corollary 3.11 *Let c be a positive integer such that $\lfloor \frac{c}{11} \rfloor > 1$. The number of Arf numerical semigroups with multiplicity* 11 *and conductor c is*

$$N_{ARF}(11,c) = \begin{cases} 9 & \textit{if} \quad c \equiv 0 \pmod{11}, \\ 5 & \textit{if} \quad c \equiv 2 \textit{ or } 6 \pmod{11}, \\ 3 & \textit{if} \quad c \equiv 3,4,5 \textit{ or } 7 \pmod{11}, \\ 7 & \textit{if} \quad c \equiv 8 \pmod{11}, \\ 6 & \textit{if} \quad c \equiv 9 \pmod{11}, \\ 10 & \textit{if} \quad c \equiv 10 \pmod{11}. \end{cases}$$

4. Arf numerical semigroups with multiplicity 13

Let S be an Arf numerical semigroup with multiplicity 13 and conductor c. Then $c \equiv 0,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11$ or $12 \pmod{13}$.

The following proposition describes all Arf numerical semigroups S with multiplicity 13 and conductor c if $c \equiv 0 \pmod{13}$, $c > 13$. We will sketch the proof of this proposition and skip the proofs of the remaining propositions which are very similar to the proof below.

Proposition 4.1 *Let S be a numerical semigroup with multiplicity* 13 *and conductor c, where $c > 13$ and $c \equiv 0 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following* 12 *semigroups:*

$\langle 13, c-8, c-3, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+9, c+11, c+12\rangle$ *; or*

$\langle 13, c-6, c-4, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10, c+12\rangle$ *; or*

$\langle 13, c-6, c-3, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+11, c+12\rangle$ *; or*

$\langle 13, c-6, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+12\rangle$ *; or*

$\langle 13, c-6, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+11, c+12\rangle$ *; or*

$\langle 13, c-5, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+12\rangle$ *; or*

$\langle 13, c-5, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+11, c+12\rangle$ *; or*

$\langle 13, c-4, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+12\rangle$ *; or*

$\langle 13, c-4, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+11, c+12\rangle$ *, or*

$\langle 13, c-3, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+11, c+12\rangle$ *; or*

$\langle 13, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+12\rangle$ *; or*

$\langle 13, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+11, c+12\rangle$.

Proof As in the proof of Proposition 3.1, $w(1) = c+1$, $w(12) = c+12$, $w(2) = c+2$, $w(3) = c+3$, $w(4) = c+4$, $w(6) = c+6$, and $w(11) \geq c-2$, $w(10) \geq c-3$, $w(9) \geq c-4$, $w(8) \geq c-5$, $w(7) \geq c-6$, $w(5) \geq c-8$. It follows that $\mathcal{R}$ belongs to the set

$$\{w(5) = c-8, w(7) = c-6, w(8) = c-5, w(9) = c-4, w(10) = c-3, w(11) = c-2, w(1) = c+1\}.$$

If $\mathcal{R} = w(5) = c-8$, then

$$S = \langle 13, c-8, c-3, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+9, c+11, c+12\rangle.$$

If $\mathcal{R} = w(7) = c-6$, then

$$\langle 13, c-6, c-4, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10, c+12\rangle;\ \text{or}$$
$$\langle 13, c-6, c-3, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+11, c+12\rangle;\ \text{or}$$
$$\langle 13, c-6, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+12\rangle;\ \text{or}$$
$$\langle 13, c-6, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+11, c+12\rangle.$$

If $\mathcal{R} = w(8) = c-5$, then

$$\langle 13, c-5, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+12\rangle;\ \text{or}$$
$$\langle 13, c-5, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+11, c+12\rangle.$$

If $\mathcal{R} = w(9) = c-4$, then

$$\langle 13, c-4, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+12\rangle;\ \text{or}$$
$$\langle 13, c-4, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+11, c+12\rangle.$$

If $\mathcal{R} = w(10) = c-3$, then

$$\langle 13, c-3, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+11, c+12 \rangle .$$

If $\mathcal{R} = w(11) = c-2$, then

$$\langle 13, c-2, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+12 \rangle .$$

If $\mathcal{R} = w(1) = c+1$, then

$$S = \langle 13, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+11, c+12 \rangle .$$

□

Proposition 4.2 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 15$ and $c \equiv 2 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 7 semigroups:*

$\langle 13, c-10, c-5, c, c+1, c+2, c+4, c+5, c+6, c+7, c+9, c+10, c+12 \rangle$ *; or*

$\langle 13, c-7, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10, c+12 \rangle$ *; or*

$\langle 13, c-6, c-4, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10, c+12 \rangle$ *, or*

$\langle 13, c-6, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+12 \rangle$ *, or*

$\langle 13, c-5, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+12 \rangle$ *; or*

$\langle 11, c-4, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+12 \rangle$ *; or*

$\langle 13, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+12 \rangle$ *.*

Proposition 4.3 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 16$ and $c \equiv 3 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 4 semigroups:*

$\langle 13, c-8, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+9, c+11, c+12 \rangle$ *; or*

$\langle 13, c-7, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+11, c+12 \rangle$ *; or*

$\langle 13, c-6, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+11, c+12 \rangle$ *; or*

$\langle 13, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+11, c+12 \rangle$ *.*

Proposition 4.4 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 17$ and $c \equiv 4 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 6 semigroups:*

$\langle 13, c-8, c-6, c-2, c, c+1, c+2, c+3, c+4, c+6, c+8, c+10, c+12 \rangle$ *; or*

$\langle 13, c-8, c-2, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+10, c+12 \rangle$ *; or*

$\langle 13, c-8, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$ *; or*

$\langle 13, c-6, c-2, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10, c+12 \rangle$ *; or*

$\langle 13, c-2, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+12 \rangle$ *; or*

$\langle 13, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$ *.*

Proposition 4.5 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 18$ and $c \equiv 5 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 4 semigroups:*

$\langle 13, c-10, c-2, c, c+1, c+2, c+4, c+5, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c-8, c-2, c, c+1, c+2, c+3, c+4, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c-2, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c, c+1, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+11, c+12 \rangle$.

Proposition 4.6 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 19$ and $c \equiv 6 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 6 semigroups:*

$\langle 13, c-9, c-3, c, c+1, c+2, c+3, c+5, c+6, c+8, c+9, c+11, c+12 \rangle$, *or*

$\langle 11, c-8, c-4, c-2, c, c+1, c+2, c+3, c+4, c+6, c+8, c+10, c+12 \rangle$; *or*

$\langle 13, c-4, c-2, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+10, c+12 \rangle$; *or*

$\langle 13, c-3, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+11, c+12 \rangle$; *or*

$\langle 13, c-2, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c, c+1, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+11, c+12 \rangle$.

Proposition 4.7 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 20$ and $c \equiv 7 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 3 semigroups :*

$\langle 13, c-3, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+11, c+12 \rangle$; *or*

$\langle 13, c-2, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c, c+1, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10, c+11, c+12 \rangle$.

Proposition 4.8 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 21$ and $c \equiv 8 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 8 semigroups:*

$\langle 13, c-10, c-6, c-4, c-2, c, c+1, c+2, c+4, c+6, c+8, c+10, c+12 \rangle$; *or*

$\langle 13, c-6, c-4, c-2, c, c+1, c+2, c+3, c+4, c+6, c+8, c+10, c+12 \rangle$; *or*

$\langle 13, c-5, c-2, c, c+1, c+2, c+3, c+4, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c-4, c-2, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+10, c+12 \rangle$; *or*

$\langle 13, c-4, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$; *or*

$\langle 13, c-3, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+9, c+11, c+12 \rangle$; *or*

$\langle 13, c-2, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c, c+1, c+2, c+3, c+4, c+6, c+7, c+8, c+9, c+10, c+11, c+12 \rangle$.

Proposition 4.9 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 22$ and $c \equiv 9 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 6 semigroups:*

$\langle 13, c-6, c-3, c, c+1, c+2, c+3, c+5, c+6, c+8, c+9, c+11, c+12 \rangle$; *or*

$\langle 13, c-4, c-2, c, c+1, c+2, c+3, c+5, c+6, c+7, c+8, c+10, c+12 \rangle$; *or*

$\langle 13, c-4, c, c+1, c+2, c+3, c+5, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$; *or*

$\langle 13, c-3, c, c+1, c+2, c+3, c+5, c+6, c+7, c+8, c+9, c+11, c+12 \rangle$; *or*

$\langle 13, c-2, c, c+1, c+2, c+3, c+5, c+6, c+7, c+8, c+9, c+10, c+12 \rangle$; *or*

$\langle 13, c, c+1, c+2, c+3, c+5, c+6, c+7, c+8, c+9, c+10, c+11, c+12 \rangle$.

Proposition 4.10 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 23$ and $c \equiv 10 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 10 semigroups:*

$\langle 13, c-8, c-6, c-4, c-2, c, c+1, c+2, c+4, c+6, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-6, c-4, c-2, c, c+1, c+2, c+4, c+5, c+6, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-6, c-2, c, c+1, c+2, c+4, c+5, c+6, c+8, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c-5, c-2, c, c+1, c+2, c+4, c+5, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c-5, c, c+1, c+2, c+4, c+5, c+6, c+7, c+9, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-4, c-2, c, c+1, c+2, c+4, c+5, c+6, c+7, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-4, c, c+1, c+2, c+4, c+5, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-3, c, c+1, c+2, c+4, c+5, c+6, c+7, c+8, c+9, c+11, c+12 \rangle$; *or*
$\langle 13, c-2, c, c+1, c+2, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c, c+1, c+2, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+11, c+12 \rangle$.

Proposition 4.11 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 24$ and $c \equiv 11 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 9 semigroups:*

$\langle 13, c-8, c-5, c-2, c, c+1, c+3, c+4, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c-7, c-3, c, c+1, c+3, c+4, c+5, c+7, c+8, c+9, c+11, c+12 \rangle$; *or*
$\langle 13, c-5, c-2, c, c+1, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c-5, c, c+1, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-4, c-2, c, c+1, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-4, c, c+1, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-3, c, c+1, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+11, c+12 \rangle$; *or*
$\langle 13, c-2, c, c+1, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c, c+1, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+11, c+12 \rangle$.

Proposition 4.12 *Let S be a numerical semigroup with multiplicity 13 and conductor c, where $c > 25$ and $c \equiv 12 \pmod{13}$. Then S is an Arf semigroup if and only if S is one of the following 17 semigroups:*

$\langle 13, c-10, c-8, c-6, c-4, c-2, c, c+2, c+4, c+6, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-9, c-6, c-3, c, c+2, c+3, c+5, c+6, c+8, c+9, c+11, c+12 \rangle$; *or*
$\langle 13, c-8, c-6, c-4, c-2, c, c+2, c+3, c+4, c+6, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-8, c-4, c-2, c, c+2, c+3, c+4, c+6, c+7, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-8, c-4, c, c+2, c+3, c+4, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-7, c-2, c, c+2, c+3, c+4, c+5, c+7, c+8, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c-6, c-4, c-2, c, c+2, c+3, c+4, c+5, c+6, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-6, c-3, c, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+11, c+12 \rangle$; *or*
$\langle 13, c-6, c-2, c, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c-6, c, c+2, c+3, c+4, c+5, c+6, c+8, c+9, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-5, c-2, c, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+12 \rangle$; *or*
$\langle 13, c-5, c, c+2, c+3, c+4, c+5, c+6, c+7, c+9, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-4, c-2, c, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+12 \rangle$; *or*
$\langle 13, c-4, c, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+10, c+11, c+12 \rangle$; *or*
$\langle 13, c-3, c, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+11, c+12 \rangle$; *or*

$\langle 13, c-2, c, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+12\rangle$ *; or*
$\langle 13, c, c+2, c+3, c+4, c+5, c+6, c+7, c+8, c+9, c+10, c+11, c+12\rangle$.

Corollary 4.13 *Let c be a positive integer such that $\lfloor \frac{c}{13} \rfloor > 1$. The number of Arf numerical semigroups with multiplicity* 13 *and conductor c is*

$$N_{ARF}(13,c) = \begin{cases} 12 & \textit{if} \quad c \equiv 0 \pmod{13}, \\ 7 & \textit{if} \quad c \equiv 2 \pmod{13}, \\ 4 & \textit{if} \quad c \equiv 3 \textit{ or } 5 \pmod{13}, \\ 6 & \textit{if} \quad c \equiv 4 \textit{ or } 6 \textit{ or } 9 \pmod{13}, \\ 3 & \textit{if} \quad c \equiv 7 \pmod{13}, \\ 8 & \textit{if} \quad c \equiv 8 \pmod{13}, \\ 10 & \textit{if} \quad c \equiv 10 \pmod{13}, \\ 9 & \textit{if} \quad c \equiv 11 \pmod{13}, \\ 17 & \textit{if} \quad c \equiv 12 \pmod{13}. \end{cases}$$

5. Conclusion and open questions

For any prime p and any positive integer c, let $\mathscr{S}_{ARF}(p,c)$ denote the set of all Arf numerical semigroups with multiplicity p and conductor c. Our observations about Arf numerical semigroups with prime multiplicity $p \leq 13$ suggest that the following questions have affirmative answers.

Question 1

Let $S \in \mathscr{S}_{ARF}(p,c)$ where $c > 2p$. Is it always true that

$$\mathrm{Ap}(S,p) \subset (c-p, c+p) \quad ?$$

Question 2

$$\mathscr{S}_{ARF}(p, c+p) = \{(p+S) \cup \{0\} : S \in \mathscr{S}_{ARF}(p,c)\}$$

and thus

$$N_{ARF}(p, c+p) = N_{ARF}(p,c) \quad ?$$

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