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MUSTAFA AYDIN

NAZIM I. MAHMUDOV

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Study of the φ-generalized type k-fractional integrals or derivatives and some of their properties

Mustafa AYDIN[1,2,*], **Nazım I. MAHMUDOV**[2]
[1]Department of Mathematics, Faculty of Science and Letters, İstanbul Technical University, İstanbul, Turkey
[2]Department of Mathematics, Faculty of Arts & Sciences, Eastern Mediterranean University, Famagusta, North Cyprus

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Abstract: A novel fractional integral in the sense of Riemann-Liouville integral and two new fractional derivatives in the sense of Riemann-Liouville derivative and Caputo derivative with respect to another function and two parameters are introduced. Some significant properties of them are presented like semigroup property, inverse property, etc. The solution of the Cauchy-type problem for the nonhomogenous linear differential equation with the φ-generalized Caputo k-fractional derivative is given by using the method of successive approximation.

Key words: φ-generalized Riemann-Liouville k-fractional integral, φ-generalized Riemann-Liouville k-fractional derivative, φ-generalized Caputo k-fractional derivative, Cauchy type problem

1. Introduction

It was understood that n was one of the nonnegative integers when one talked about the derivative of order n or n-fold integrals because the former was in need of knowing instantaneous rates of change, areas under or between curves, the slopes of curves, and accumulation of quantities. These needs produced the well-known traditional calculus. Unlike traditional calculus, although fractional calculus at that time was a production of only innocent curiosity which is in Leibnitz's letter to L' Hospital in 1695, it has been widely improved along with the extension of the needs in the recent decades. Many researchers not only in the past like Euler, Fourier, Abel, Liouville, Riemann, Grünwald, Hadamard, Weyl, Erdélyi-Kober, Caputo have tried to understand and define fractional derivatives and integrals[17][16][20][27][21][22][4][19], but ones also in the present make an attempt to define a new derivative or integral of fractional order depending generalizing available concepts like gamma function and appearing new ones and needs. For instance, Katugampola [12] introduced a novel fractional operator generalizing the well-known Hadamard fractional and the Riemann-Liouville derivatives to an individual form. Romero [23] et al. presented a novel fractional derivative named by k-Riemann-Liouville fractional derivative by utilizing the k-gamma function and relationships with the k-Riemann-Liouville integral and some features employing Laplace and Fourier transforms. Sarikaya [26] et al. gave a new version of fractional integral called (k, s)-Riemann-Liouville fractional integral generalizing the Riemann-Liouville fractional integral and presented some features for this one as well as new integral inequalities employing the novel version of fractional integral. Subsequently, Azam [3] et al. developed the generalized k-fractional derivative in the sense of Riemann-Liouville

*Correspondence: mustafaaydin@itu.edu.tr

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and generalized Caputo type k-fractional derivative which are the generalized forms of some existing fractional derivatives. Almeida [1] studied a Caputo type fractional derivative with respect to another function and investigated some features, like the inverse law and the semigroup law, Fermat's and Taylor's theorems, etc.

Fractional calculus has a prevailing usage in the scientific world. Nowadays, it has been employed in the areas of mathematical physics, statistical mechanics, electrochemistry, electric conductance of biological systems, astrophysics, computed tomography, control theory, the mathematical modeling of viscoelastic material, thermodynamics, the modeling of diffusion, biophysics, electric conductance of biological systems, fractional order models of neurons, hydrology, geological surveying, signal and image possessing, engineering, finance, etc. Almeida [1] et al. took a Population Growth Model into consideration and demonstrated that the process utilizing a Caputo FD with respect to different functions (kernels) can be more accurately modeled. With the help of the generalized fractional derivatives, mathematically the variant of post-Newtonian mechanics and the relativistic-covariant generalization of the traditional equations in the gravitational field are studied by Kobelev [14].

In addition to the fact that each of all counted papers above is a source of inspiration, the pioneer works [26][3][1] motivated us to define a new fractional integral and two novel fractional derivatives which can cover most of existing fractional integrals and derivatives. We investigated some of their features and discovered relations with each other and lastly solved the Cauchy type problem which naturally comes to light. Before starting the main contributions, we remind a few concepts.

By Diaz and Pariguan [6], the k-gamma and the k-beta functions are defined as follows

$$\Gamma_k\left(\omega\right)=\int_0^\infty e^{-\frac{y^k}{k}}y^{\omega-1}dy, \tag{1.1}$$

and

$$B_k\left(\omega,\varpi\right)=\frac{1}{k}\int_0^1 y^{\frac{\omega}{k}-1}\left(1-y\right)^{\frac{\varpi}{k}-1}dy, \tag{1.2}$$

where $Re\left(\omega\right)>0$ and $Re\left(\varpi\right)>0$, respectively. Their relations with the well-known gamma and beta functions and themselves are given by

$$\Gamma\left(\omega\right)=\lim_{k\to 1}\Gamma_k\left(\omega\right),\qquad \Gamma_k\left(\omega\right)=k^{\frac{\omega}{k}-1}\Gamma\left(\frac{\omega}{k}\right),\qquad \Gamma_k\left(k\right)=1,\qquad \Gamma_k\left(\omega+k\right)=\omega\Gamma_k\left(\omega\right),$$

and

$$B_k\left(\omega,\varpi\right)=\frac{\Gamma_k\left(\omega\right)\Gamma_k\left(\varpi\right)}{\Gamma_k\left(\omega+\varpi\right)}=\frac{1}{k}B\left(\frac{\omega}{k},\frac{\varpi}{k}\right).$$

Define two norms $\left\|.\right\|_C : C\left[a,b\right]\to\mathbb{R}$ and $\left\|.\right\|_{C_\varphi^{[n]}} : C^n\left[a,b\right]\to\mathbb{R}$ by

$$\left\|f\right\|_C:=\max_{x\in[a,b]}\left|f\left(x\right)\right|,\qquad \left\|f\right\|_{C_\varphi^{[n]}}:=\sum_{m=0}^{n}\left\|f_\varphi^{[m]}\right\|_C,$$

where $|.|$ is an arbitrary norm of $\mathbb{R}$, $n\in\mathbb{N}$.

Gösta Mittag-Leffler [18] defined Mittag-Leffler function $E_\eta\left(\varphi\right)$ by

$$E_\eta\left(\varphi\right)=\sum_{k=0}^{\infty}\frac{\varphi^k}{\Gamma\left(\eta k+1\right)},\quad \eta\in\mathbb{C},\quad Re\left(\eta\right)>0.$$

Subsequently, Wiman [28] introduced a generalized Mittag-Leffler function $E_{\eta,\mu}(\varphi)$, given by

$$E_{\eta,\mu}(\varphi)=\sum_{k=0}^{\infty}\frac{\varphi^{k}}{\Gamma(\eta k+\mu)},\quad \eta,\mu\in\mathbb{C},\quad Re(\eta)>0.$$

2. Main contributions

2.1. The φ-generalized R-L k-fractional integral and derivative

In this section, we introduce both the φ-generalized Riemann Liouville k-fractional integral (φ-GRL k-FI) of order $\alpha>0$ and the φ-generalized Riemann Liouville k-fractional derivative (φ-GRL k-FD) of order $\alpha>0$. We examine some properties and relations between them. Now, let us start with the definition of (φ-GRL k-FI).

Definition 2.1 *Let f be a continuous function on the real interval $[a,b]$ and let $\varphi\in C^{1}[a,b]$ be an increasing function with $\varphi'(x)\neq 0,\ \ \forall x\in[a,b]$. Then the φ-generalized Riemann Liouville k-fractional integral of $\alpha>0$ is given by*

$$\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)=\frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_{k}(\alpha)}\int_{a}^{x}\left(\varphi^{s}(x)-\varphi^{s}(t)\right)^{\frac{\alpha}{k}-1}\varphi'(t)\,\varphi^{s-1}(t)\,f(t)\,dt,$$

where $k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. For the sake of simplicity, we denote φ-GRL k-FI using the differential concept by

$$\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)=\frac{s^{-\frac{\alpha}{k}}}{k\Gamma_{k}(\alpha)}\int_{a}^{x}\left(\varphi^{s}(x)-\varphi^{s}(t)\right)^{\frac{\alpha}{k}-1}f(t)\,d\varphi^{s}(t).$$

Based on the choices of k,s,φ, we obtain different definitions of fractional integrals, e.g., φ-GRL k-FI reduces to (k,s)-Riemann-Liouville fractional integral[26] provided that $\varphi(x)=x$, and for $k\to 1,\ s=1$, it coincides with the φ-Riemann-Liouville fractional integrals [13][25][2][1]. φ-GRL k-FI with $\varphi(x)=x,\ k\to 1,$ $s=1$ corresponds to the well-known Riemann-Liouville fractional integrals. Selecting $\varphi(x)=x,\ k\to 1,$ $s\to 0^{+}$ turns φ-GRL k-FI into the Hadamard fractional integral [10], etc.

The following theorem expresses the semigroup and commutative property of φ-GRL k-FI.

Theorem 2.2 *Let f be a continuous function on the real interval $[a,b]$ and let $\varphi\in C^{1}[a,b]$ be an increasing function with $\varphi'(x)\neq 0,\ \ \forall x\in[a,b]$. Then, $\forall\alpha,\beta>0$*

$${}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\alpha,\varphi}_{k,s}\left[{}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\beta,\varphi}_{k,s}f(x)\right]={}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\alpha+\beta,\varphi}_{k,s}f(x)={}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\beta,\varphi}_{k,s}\left[{}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\alpha,\varphi}_{k,s}f(x)\right].$$

Proof Assume that given conditions are satisfied. By using Fubini's theorem, consider

$$\begin{aligned}{}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\alpha,\varphi}_{k,s}\left[{}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\beta,\varphi}_{k,s}f(x)\right]&=\frac{s^{-\frac{\alpha}{k}}}{k\Gamma_{k}(\alpha)}\int_{a}^{x}\left(\varphi^{s}(x)-\varphi^{s}(y)\right)^{\frac{\alpha}{k}-1}\left[\frac{s^{-\frac{\beta}{k}}}{k\Gamma_{k}(\beta)}\int_{a}^{y}\left(\varphi^{s}(y)-\varphi^{s}(t)\right)^{\frac{\beta}{k}-1}d\varphi^{s}(t)\,f(t)\right]d\varphi^{s}(y)\\&=\frac{s^{-\frac{\alpha+\beta}{k}}}{k^{2}\Gamma_{k}(\alpha)\Gamma_{k}(\beta)}\int_{a}^{x}f(t)\left[\int_{t}^{x}\left(\varphi^{s}(x)-\varphi^{s}(y)\right)^{\frac{\alpha}{k}-1}\left(\varphi^{s}(y)-\varphi^{s}(t)\right)^{\frac{\beta}{k}-1}d\varphi^{s}(y)\right]d\varphi^{s}(t)\end{aligned}$$

By substituting $z=\frac{\varphi^s(y)-\varphi^s(t)}{\varphi^s(x)-\varphi^s(t)}$, we obtain $z=0,\ z=1,\ [\varphi^s(x)-\varphi^s(t)]\,z=d\varphi^s(y)$, and

$$\begin{aligned}
{}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}\left[{}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\beta,\varphi}f(x)\right] &= \frac{s^{-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x(\varphi^s(x)-\varphi^s(t))^{\frac{\alpha+\beta}{k}-1}f(t)\left[\int_0^1(1-z)^{\frac{\alpha}{k}-1}z^{\frac{\beta}{k}-1}dz\right]d\varphi^s(t)\\
&= \frac{s^{-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x(\varphi^s(x)-\varphi^s(t))^{\frac{\alpha+\beta}{k}-1}f(t)\,d\varphi^s(t)\,B_k\left(\frac{\alpha}{k},\frac{\beta}{k}\right)\\
&= \frac{s^{-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x(\varphi^s(x)-\varphi^s(t))^{\frac{\alpha+\beta}{k}-1}f(t)\,d\varphi^s(t)\,k\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}\\
&={}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha+\beta,\varphi}f(x).
\end{aligned}$$

By changing places of α and β, commutativity of φ-GRL k-FI can be easily followed. □

The following corollary says that φ-GRL k-FI is linear.

Corollary 2.3 *Let g and h be a continuous function on the real interval $[a,b]$ and let $\varphi\in C^1[a,b]$ be an increasing function with $\varphi'(x)\neq 0,\quad\forall x\in[a,b],\ \alpha\in\mathbb{R}^+,\ \mu\in\mathbb{R}$. Then*

$${}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}[g(x)+\mu h(x)]={}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}g(x)+\mu\,{}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}h(x).$$

Lemma 2.4 *Let an increasing function $\varphi\in C^1[a,b]$ have the property of $\varphi'(x)\neq 0,\quad\forall x\in[a,b]$ and let $\alpha,\beta,k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then we have*

$${}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}(\varphi^s(x)-\varphi^s(a))^{\frac{\beta}{k}-1}=\frac{\Gamma_k(\beta)}{s^{\frac{\alpha}{k}}\Gamma_k(\alpha+\beta)}(\varphi^s(x)-\varphi^s(a))^{\frac{\alpha+\beta}{k}-1}.$$

Proof In the light of the definition of ${}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}$

$${}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}(\varphi^s(x)-\varphi^s(a))^{\frac{\beta}{k}-1}=\frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x(\varphi^s(x)-\varphi^s(t))^{\frac{\alpha}{k}-1}(\varphi^s(t)-\varphi^s(a))^{\frac{\beta}{k}-1}d\varphi^s(t)$$

By substituting $z=\frac{\varphi^s(t)-\varphi^s(a)}{\varphi^s(x)-\varphi^s(a)}$, we obtain $z=0,\ z=1,\ [\varphi^s(x)-\varphi^s(a)]\,z=d\varphi^s(t)$, and

$$\begin{aligned}
{}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}(\varphi^s(x)-\varphi^s(a))^{\frac{\beta}{k}-1} &= \frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1(\varphi^s(x)-\varphi^s(a))^{\frac{\alpha+\beta}{k}-1}(1-z)^{\frac{\alpha}{k}-1}z^{\frac{\beta}{k}-1}dz\\
&= \frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(\varphi^s(x)-\varphi^s(a))^{\frac{\alpha+\beta}{k}-1}B_k\left(\frac{\alpha}{k},\frac{\beta}{k}\right)
\end{aligned}$$

which provides the desired result. □

Definition 2.5 *Let f be a continuous function on $[0,\infty)$ and let $\varphi\in C^1[0,\infty)$ be an increasing function with $\varphi'(x)\neq 0,\quad\forall x\in[0,\infty)$. $s,\alpha\in\mathbb{R}^+$, and $n,k\in\mathbb{N}$ with $n=[\alpha]+1$. Then the φ-generalized Riemann Liouville k-fractional derivative of $\alpha>0$ is given by*

$$\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}f\right)(x)=\frac{s^{\frac{\alpha-nk+k}{k}}}{k\Gamma_k(nk-\alpha)}\left(\varphi^{1-s}(x)\frac{1}{\varphi'(x)}\frac{d}{dx}\right)^n\int_a^x(\varphi^s(x)-\varphi^s(t))^{\frac{nk-\alpha}{k}-1}\varphi'(t)\,\varphi^{s-1}(t)\,f(t)\,dt,$$

where $\forall 0<a<x$. *For the sake of simplicity and making calculations easy, we denote* φ*-GRL* k*-FD using the differential concept by*

$$\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}f\right)(x)=\frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\int_a^x\left(\varphi^s(x)-\varphi^s(t)\right)^{\frac{nk-\alpha}{k}-1}f(t)\,d\varphi^s(t).$$

It can be expressed as follows

$$\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}f\right)(x)=\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk-\alpha,\varphi}f\right)(x).$$

Based on the choices of k,s,φ, we can reach to many of fractional derivatives, e.g., φ-GRL k-FD reduces to the generalized k-fractional derivative [3] provided that $\varphi(x)=x$, and for $k\to1$, $s=1$, it coincides with the φ-Riemann-Liouville fractional derivative [13][25][2][1]. φ-GRL k-FD with $\varphi(x)=x$, $k\to1$, $s=1$ corresponds to the well-known Riemann-Liouville fractional derivative. Depending on selecting suitable choices of φ,s,k from φ-GRL k-FD, one can easily obtain the generalized fractional derivative [12], the k-Riemann-Liouville fractional derivative [23], the k-Weyl fractional derivative [24], the k-Hadamard fractional derivative [7] as well as classical Riemann-Liouville fractional derivative, Weyl fractional derivative, Hadamard fractional derivative, etc. One can find more details in the references [13][25][2][1].

Now, we discuss the inverse property of the φ-GRL k-FD.

Theorem 2.6 *Let* f *be a continuous function on* $[0,\infty)$ *and let* $\varphi\in C^1[0,\infty)$ *be an increasing function with* $\varphi'(x)\neq0$, $\forall x\in[0,\infty)$. $s,\alpha\in\mathbb{R}^+$, *and* $n,k\in\mathbb{N}$ *with* $n=[\alpha]+1$. *Then* $\forall 0<a<x$,

$${}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}f\right)(x)=\frac{1}{k^n}f(x).$$

Proof With the help of both their definitions, we get

$$\begin{aligned}&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}f\right)(x)\\&=\frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\int_a^x\left(\varphi^s(x)-\varphi^s(y)\right)^{\frac{nk-\alpha}{k}-1}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}f\right)(y)\,d\varphi^s(y)\\&=\frac{s^{\frac{\alpha}{k}-n-\frac{\alpha}{k}}}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\int_a^x\left(\varphi^s(x)-\varphi^s(y)\right)^{\frac{nk-\alpha}{k}-1}\left[\int_a^y\left(\varphi^s(y)-\varphi^s(t)\right)^{\frac{\alpha}{k}-1}f(t)\,d\varphi^s(t)\right]d\varphi^s(y)\end{aligned}$$

By utilizing Fubini's theorem, we have

$$\begin{aligned}&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}f\right)(x)\\&=\frac{s^{\frac{\alpha}{k}-n-\frac{\alpha}{k}}}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\int_a^x f(t)\left[\int_t^x\left(\varphi^s(x)-\varphi^s(y)\right)^{\frac{nk-\alpha}{k}-1}\left(\varphi^s(y)-\varphi^s(t)\right)^{\frac{\alpha}{k}-1}d\varphi^s(t)\right]d\varphi^s(y)\end{aligned}$$

By substituting $z=\frac{\varphi^s(y)-\varphi^s(t)}{\varphi^s(x)-\varphi^s(t)}$, we obtain $z=0$, $z=1$, $[\varphi^s(x)-\varphi^s(t)]\,z=d\varphi^s(y)$, and

$$\begin{aligned}&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}f\right)(x)\\&=\frac{s^{-n}}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\int_a^x\left(\varphi^s(x)-\varphi^s(t)\right)^{n-1}f(t)\int_0^1(1-z)^{\frac{nk-\alpha}{k}-1}z^{\frac{\alpha}{k}-1}dz\,d\varphi^s(t)\end{aligned}$$

In the light of the definition and properties of beta function,

$$
\begin{aligned}
&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}f\right)(x)\\
&=\frac{s^{-n}}{k^n\Gamma_k(n)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\int_a^x\left(\varphi^s(x)-\varphi^s(t)\right)^{n-1}f(t)\,d\varphi^s(t)\\
&=\left(\varphi^{1-s}(x)\frac{1}{\varphi'(x)}\right)^n\frac{s^{1-n}}{k^n\Gamma_k(n)}\frac{d^n}{dx^n}\int_a^x\left(\varphi^s(x)-\varphi^s(t)\right)^{n-1}\varphi'(t)\,\varphi^{s-1}(t)\,f(t)\,d(t)\,.
\end{aligned}
$$

By applying the derivative of an integral of a two-variable function by n-times to the above equality, we get the required result. □

Lemma 2.7 *Let f be a continuous function on $[0,\infty)$ and let $\varphi\in C^1[0,\infty)$ be an increasing function with $\varphi'(x)\neq 0,\quad\forall x\in[0,\infty)$. $s,\alpha,\beta\in\mathbb{R}^+$, and $n,k\in\mathbb{N}$ with $n=[\alpha]+1$. Then $\forall 0<a<x$,*

$$
{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\beta,\varphi}f\right)(x)=\frac{1}{k^n}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha-\beta,\varphi}f\right)(x)\,.
$$

In the following theorem, semigroup property of φ-GRL k-FD is demonstrated.

Theorem 2.8 *Let $\varphi\in C^1[0,\infty)$ be an increasing function with $\varphi'(x)\neq 0,\quad\forall x\in[0,\infty)$. For continuous f on $[0,\infty)$, $s,\alpha,\beta\in\mathbb{R}^+$, and $n,k,m\in\mathbb{N}$ with $n=[\alpha]+1$, $m=[\beta]+1$ such that $\alpha+\beta<nk$. Then $\forall 0<a<x$,*

$$
{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\varphi}f\right)(x)=\frac{1}{k^n}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha+\beta,\varphi}f\right)(x)\,.
$$

Proof From the inverse and semigroup properties of φ-GRL k-FD and φ-GRL k-FI, respectively, we obtain

$$
\begin{aligned}
{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\varphi}f\right)(x)&=\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n{}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk-\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\varphi}f\right)(x)\\
&=\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n{}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk-\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\varphi}\right)\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\beta,\varphi}\right)\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{-\beta,\varphi}f\right)(x)\\
&=\frac{1}{k^n}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk-\alpha,\varphi}\right)\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{-\beta,\varphi}f\right)(x)\\
&=\frac{1}{k^n}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk-(\alpha+\beta),\varphi}f\right)(x)
\end{aligned}
$$

which is the desired result. □

Here is the commutativity and linearity of φ-GRL k-FD.

Corollary 2.9 *Let $\varphi\in C^1[0,\infty)$ be an increasing function with $\varphi'(x)\neq 0,\quad\forall x\in[0,\infty)$. For continuous f on $[0,\infty)$, $s,\alpha,\beta\in\mathbb{R}^+$, and $n,k,m\in\mathbb{N}$ with $n=[\alpha]+1$, $m=[\beta]+1$ such that $\alpha+\beta<nk$, then $\forall 0<a<x$,*

$$
{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\varphi}f\right)(x)={}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\varphi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\varphi}f\right)(x)\,.
$$

Corollary 2.10 *Let $\varphi \in C^1[0,\infty)$ be an increasing function with $\varphi'(x) \neq 0, \quad \forall x \in [0,\infty)$. For continuous g,h on $[0,\infty)$, $s,\alpha \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$ and $n,k \in \mathbb{N}$ with $n = [\alpha]+1$, then $\forall 0 < a < x$,*

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left[g(x) + \mu h(x)\right] = {}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s} g(x) + \mu {}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s} h(x). $$

Lemma 2.11 *Let $\varphi \in C^1[0,\infty)$ be an increasing function with $\varphi'(x) \neq 0, \quad \forall x \in [0,\infty)$, and let $s,\alpha,\gamma \in \mathbb{R}^+$, $n,k \in \mathbb{N}$ with $n = [\alpha]+1$. Then $\forall 0 < a < x$,*

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left(\varphi^s(x) - \varphi^s(a)\right)^{\frac{\gamma}{k}} = \frac{s^{\frac{\alpha-nk}{k}}\Gamma_k(k+\gamma)}{k\Gamma_k(nk+k+\gamma-\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n \left(\varphi^s(x) - \varphi^s(a)\right)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}}. $$

Proof Because of its definition, we have

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left(\varphi^s(x) - \varphi^s(a)\right)^{\frac{\gamma}{k}} = \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n \int_a^x \left(\varphi^s(x) - \varphi^s(t)\right)^{\frac{nk-\alpha}{k}-1}\left(\varphi^s(t) - \varphi^s(a)\right)^{\frac{\gamma}{k}} d\varphi^s(t) $$

By substituting $z = \frac{\varphi^s(t)-\varphi^s(a)}{\varphi^s(x)-\varphi^s(a)}$, we obtain $z=0$, $z=1$, $[\varphi^s(x) - \varphi^s(a)]z = d\varphi^s(t)$, and

$$ \begin{aligned} {}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left(\varphi^s(x) - \varphi^s(a)\right)^{\frac{\gamma}{k}} &= \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n \int_0^1 \left(\varphi^s(x) - \varphi^s(a)\right)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}}(1-z)^{\frac{nk-\alpha}{k}-1} z^{\frac{\gamma}{k}} dz \\ &= \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n \left(\varphi^s(x) - \varphi^s(a)\right)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}} B_k\left(\frac{nk-\alpha}{k}, \frac{\gamma}{k}+1\right) \end{aligned} $$

which grants the desired result. □

2.2. The φ-generalized Caputo k-fractional derivative

In this section, we introduce the φ-generalized Caputo k-fractional derivative(φ-GC k-FD) of order $\alpha > 0$. We will discuss relations of φ-GC k-FD with φ-GRL k-FD and φ-GRL k-FI as well as some simple properties.

Here is the definition of φ-GC k-FD.

Definition 2.12 *Let $f,\varphi \in C^n[0,\infty)$ be two functions such that φ is increasing and $\varphi'(x)$, $x \in [0,\infty)$ and let $s,\alpha \in \mathbb{R}^+$, $n,k \in \mathbb{N}$ such that $n := [\alpha]+1$ and $k(n-1) < \alpha < nk$. Then $\forall 0 < a < x$, the φ-generalized Caputo k-fractional derivative(φ-GC k-FD) of order $\alpha > 0$ is*

$$ \left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s} f\right)(x) = \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\int_a^x \left(\varphi^s(x) - \varphi^s(t)\right)^{\frac{nk-\alpha}{k}-1}\left[\left(\varphi^{1-s}(t)\frac{d}{d\varphi(t)}\right)^n f(t)\right] d\varphi^s(t). $$

It can be expressed as follows

$$ \left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s} f\right)(x) = {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{nk-\alpha,\varphi}_{k,s}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n f(x). $$

Again, we want to emphasize that we can obtain different kinds of fractional derivatives apart from the above-mentioned ones depending on selecting the choices of k, s, φ. For instance, φ-GC k-FD reduces to the a generalized Caputo type k-fractional derivative [3] when $\varphi(x)=x$. On choosing $k\to 1$, $s=1$, it coincides with the φ-Caputo fractional derivative [13][25][2][1]. φ-GC k-FD with $\varphi(x)=x$, $k\to 1$, $s=1$ corresponds to the well-known Caputo fractional derivative. With the appropriate selections of φ, s, k, one can derive a k-Caputo fractional derivative [5], the k-Caputo Hadamard fractional derivative, the Caputo modification of the Hadamard fractional derivative [8], the Caputo type Weyl fractional derivative in addition to the Caputo–Hadamard fractional derivative [8][11], the Caputo–Erdélyi–Kober fractional derivative [15]. One can find more details in the references [13][25][2][1].

Lemma 2.13 *Let $\varphi\in C^n[0,\infty)$ be a function with $\varphi'\neq 0$, $x\in[0,\infty)$ and let $\alpha,\beta,s\in\mathbb{R}^+$, $n,k\in\mathbb{N}$. Then $0<a<x$,*

$$ {}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left(\varphi^s(x)-\varphi^s(a)\right)^{\frac{\beta}{k}-1}=s^{\frac{\alpha-nk}{k}}\frac{\Gamma_k(\beta-nk)\,\Gamma\left(\frac{\beta}{k}\right)}{\Gamma_k(\beta-\alpha)\,\Gamma\left(\frac{\beta}{k}-n\right)}\left(\varphi^s(x)-\varphi^s(a)\right)^{\frac{\beta-\alpha}{k}-1}. $$

Proof It is easy to calculate the following equality

$$ \left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n\left(\varphi^s(x)-\varphi^s(a)\right)^{\frac{\beta}{k}-1}=\frac{\Gamma\left(\frac{\beta}{k}\right)}{\Gamma\left(\frac{\beta}{k}-n\right)}\left(\varphi^s(x)-\varphi^s(a)\right)^{\frac{\beta}{k}-n-1}. $$

By using the given definition of φ-GC k-FD

$$ {}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\varphi}\left(\varphi^s(x)-\varphi^s(a)\right)^{\frac{\beta}{k}-1}=\frac{s^{\frac{\alpha-nk}{k}}\Gamma\left(\frac{\beta}{k}\right)}{\Gamma_k(nk-\alpha)\,\Gamma\left(\frac{\beta}{k}-n\right)}\int_a^x\left(\varphi^s(x)-\varphi^s(a)\right)^{\frac{\beta-\alpha}{k}-1}\left(\varphi^s(t)-\varphi^s(a)\right)^{\frac{\beta-\alpha}{k}-1}d\varphi^s(t). $$

The desired thing is obtained from substituting $y=\frac{\varphi^s(t)-\varphi^s(a)}{\varphi^s(x)-\varphi^s(a)}$, and using k-Beta function and its properties. □

Theorem 2.14 *Let $f,\varphi\in C^n[0,\infty)$ be two functions such that φ is increasing and $\varphi'(x)$, $x\in[0,\infty)$ and let $s,\alpha\in\mathbb{R}^+$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$,*

$$ {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\varphi}f\right)(x)=\frac{1}{k^n}\left(f(x)-\sum_{m=0}^{n-1}\frac{1}{m!}\left(\varphi^s(x)-\varphi^s(a)\right)^m f_\varphi^{(m)}(a)\right) $$

where $f_\varphi^{(m)}(x)=\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^m f(x)$.

Proof

$$ \begin{aligned} {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\varphi}f\right)(x) &= {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}\,{}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk-\alpha,\varphi}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n f(x)\\ &={}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk,\varphi}\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^n f(x)\\ &=\frac{s^{-n}}{k\Gamma_k(nk)}\int_a^x\left(\varphi^s(x)-\varphi^s(t)\right)^{n-1}\left[\left(\varphi^{1-s}(t)\frac{d}{d\varphi(t)}\right)^n f(t)\right]d\varphi^s(t)=:\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{0,\varphi}f\right)(x) \end{aligned} $$

By keeping relations between k-gamma and well-known gamma functions and applying n-times integration by parts, we reach the craved result. □

Corollary 2.15 *Let $f,\varphi \in C^{n}[0,\infty)$ be two functions such that φ is increasing and $\varphi'(x)$, $x\in[0,\infty)$ and let $s,\alpha\in\mathbb{R}^{+}$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$,*

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\beta,\varphi}_{k,s}\left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}f\right)(x)=\left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha-\beta,\varphi}_{k,s}f\right)(x). $$

Corollary 2.16 *Let $g,h,\varphi \in C^{n}[0,\infty)$ be two functions such that φ is increasing and $\varphi'(x)$, $x\in[0,\infty)$ and let $c_1,c_2\in\mathbb{R}$, $s,\alpha\in\mathbb{R}^{+}$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$,*

$$ {}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left[c_1 g(x)+c_2 h(x)\right]=c_1\,{}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}g(x)+c_2\,{}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}h(x). $$

Corollary 2.17 *Let $f,\varphi \in C^{n}[0,\infty)$ be two functions such that φ is increasing and $\varphi'(x)$, $x\in[0,\infty)$ and let $s,\alpha\in\mathbb{R}^{+}$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$,*

$$ \left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}f\right)(x)={}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left(f(x)-\sum_{m=0}^{n-1}\frac{1}{m!}\left(\varphi^{s}(x)-\varphi^{s}(a)\right)^{m}f^{(m)}_{\varphi}(a)\right). $$

Theorem 2.18 *$\varphi\in C^{1}[a,b], a>0$ is increasing with $\varphi'(x)\neq 0$, $x\in[a,b]$ and let $s,\alpha\in\mathbb{R}^{+}$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. If $f\in C^{1}[a,b]$, then $\forall 0<a<x$,*

$$ {}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)=\frac{1}{k^{n}}f(x). $$

Proof By using corollary 2.17, we have

$$ {}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)={}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left(\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)-\sum_{m=0}^{n-1}\frac{1}{m!}\left(\varphi^{s}(x)-\varphi^{s}(a)\right)^{m}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)^{(m)}_{\varphi}(a)\right) $$

Considering that

$$ \begin{aligned}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)^{(m)}_{\varphi}(x)&=\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^{m}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)\\&=\left(\varphi^{1-s}(x)\frac{d}{d\varphi(x)}\right)^{m}\frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(\varphi^{s}(x)-\varphi^{s}(t)\right)^{\frac{\alpha}{k}-1}f(t)\,d\varphi^{s}(t).\end{aligned} $$

One can easily infer the following inequality from the above equation

$$ \left|\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)^{(m)}_{\varphi}(x)\right|\leq\frac{s^{m-\frac{\alpha}{k}}}{k^{\frac{\alpha}{k}}\Gamma_k\left(\frac{\alpha}{k}-m+1\right)}\left(\varphi^{s}(x)-\varphi^{s}(a)\right)^{\frac{\alpha}{k}-m}\|f\|_{C}, $$

and so $\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)^{(m)}_{\varphi}(a)=0$ for all $k=0,1,...,n-1$. Therefore,

$$ {}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)={}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}f\right)(x)=\frac{1}{k^{n}}f(x). $$

This completes the proof. □

Theorem 2.19 *The φ generalized Caputo k-fractional derivatives of order $\alpha > 0$ are bounded operators, i.e let $f, \varphi \in C^n[a,b], a > 0$ be two functions such that φ is increasing and $\varphi'(x)$, $x \in [a,b]$ and let $s, \alpha \in \mathbb{R}^+$, $n, k \in \mathbb{N}$ such that $n := [\alpha] + 1$ and $k(n-1) < \alpha < nk$.*

$$\left\| \left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s} f \right)(x) \right\|_C \leq M \|f\|_{C^{[n]}_{\varphi}}$$

where

$$M = \frac{s^{\frac{\alpha-nk}{k}}}{\Gamma_k(nk+k-\alpha)} \left[\frac{\varphi^{1-s}(b)}{\min_{x\in[a,b]} |\varphi'(x)|} \right]^n (\varphi^s(x) - \varphi^s(a))^{\frac{nk-\alpha}{k}}.$$

Proof For all $0 < a < x$,

$$\begin{aligned} \left| \left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s} f \right)(x) \right| &\leq \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)} \int_a^x (\varphi^s(x) - \varphi^s(t))^{\frac{nk-\alpha}{k}-1} d\varphi^s(t) \left[\frac{\varphi^{1-s}(b)}{\min_{x\in[a,b]} |\varphi'(x)|} \right]^n \|f\|_{C^{[n]}_{\varphi}} \\ &= \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)} \left[\frac{\varphi^{1-s}(b)}{\min_{x\in[a,b]} |\varphi'(x)|} \right]^n \frac{(\varphi^s(x) - \varphi^s(a))^{\frac{nk-\alpha}{k}}}{\frac{nk-\alpha}{k}} \|f\|_{C^{[n]}_{\varphi}} \\ &= \frac{s^{\frac{\alpha-nk}{k}}}{\frac{nk-\alpha}{k}\Gamma_k(nk-\alpha)} \left[\frac{\varphi^{1-s}(b)}{\min_{x\in[a,b]} |\varphi'(x)|} \right]^n (\varphi^s(x) - \varphi^s(a))^{\frac{nk-\alpha}{k}} \|f\|_{C^{[n]}_{\varphi}} \\ &= \frac{s^{\frac{\alpha-nk}{k}}}{\Gamma_k(nk+k-\alpha)} \left[\frac{\varphi^{1-s}(b)}{\min_{x\in[a,b]} |\varphi'(x)|} \right]^n (\varphi^s(x) - \varphi^s(a))^{\frac{nk-\alpha}{k}} \|f\|_{C^{[n]}_{\varphi}} \end{aligned}$$

which is the desired result. □

The solution of nonhomogenous linear differential equation with the φ-generalized Caputo k-fractional derivative

2.3. Applications

In this section, we look for a solution to the Cauchy-type problem for nonhomogeneous linear φ- generalized Caputo k- fractional differential equation. A solution of the Cauchy-type problem for φ-GRL k-FDEs in the same form can be examined in the similar manner.

Theorem 2.20 *For two functions $y, \varphi \in C[0,\infty)$ such that φ is increasing and $\varphi'(x)$, $x \in [0,\infty)$ and $s \in \mathbb{R}^+$, $0 < \alpha < 1$, $k \in \mathbb{N}$ such that $0 < \alpha < k$ and $\lambda, c \in \mathbb{R}$. The following fractional initial value problem*

$${}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s} y(x) - \lambda y(x) = f(x), \tag{2.1}$$

$$y(a) = c, \tag{2.2}$$

is of the solution

$$y(x) = cE_{\frac{\alpha}{k}}\left(\varphi^{\alpha,\varphi}_{k,s}(x,a)\right) + \frac{s^{-\frac{\alpha}{k}}}{k^{\frac{\alpha}{k}-1}} \int_a^x (\varphi^s(x) - \varphi^s(t))^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\varphi^{\alpha,\varphi}_{k,s}(x,t)\right) f(t)\, d\varphi^s(t), \tag{2.3}$$

where $\varphi^{\alpha,\varphi}_{k,s}(x,y) := k^{1-\frac{\alpha}{k}} \lambda \left(\frac{\varphi^s(x) - \varphi^s(y)}{s} \right)^{\frac{\alpha}{k}}$.

Proof By applying ${}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s}$ to both sides of 2.1 and using Theorem 2.6 and Corollary 2.17, we get

$$y(x) = y(a) + k\lambda {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} y(x) + k {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} f(x).$$

To solve this integral equation, we use the method of successive approximation. According to this method, we set:

$$y_0(x) = y(a),$$
$$y_m(x) = y_0(x) + k\lambda {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} y_{m-1}(x) + k {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} f(x)$$

where $m \geq 1$. For $m = 1$, we have

$$y_1(x) = y_0(x) + k\lambda {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} y_0(x) + k {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} f(x).$$

By rewriting and regulating

$$y_1(x) = y(a) + k\lambda y(a) \frac{s^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} (\varphi^s(x) - \varphi^s(t))^{\frac{\alpha}{k}} + k {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} f(x).$$

Similarly we find for $y_2(x)$ that

$$y_2(x) = y_0(x) + k\lambda \left[y(a) + k\lambda y(a) \frac{s^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} (\varphi^s(x) - \varphi^s(t))^{\frac{\alpha}{k}} + k {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} f(x) \right] + k {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\varphi}_{k,s} f(x).$$

With the help of Lemma 2.4, one can easily reach to

$$y_2(x) = y(a) \sum_{j=1}^{3} \frac{k^{j-1}\lambda^{j-1} s^{-(j-1)\frac{\alpha}{k}}}{\Gamma_k((j-1)\alpha+k)} (\varphi^s(x) - \varphi^s(t))^{\frac{(j-1)\alpha}{k}} + \sum_{j=1}^{2} k^j \lambda^{j-1} {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{j\alpha,\varphi}_{k,s} f(x).$$

By keeping on this process, we derive the following equation for $y_m(x)$, $m \geq 1$

$$y_m(x) = y(a) \sum_{j=1}^{m+1} \frac{k^{j-1}\lambda^{j-1} s^{-(j-1)\frac{\alpha}{k}}}{\Gamma_k((j-1)\alpha+k)} (\varphi^s(x) - \varphi^s(t))^{\frac{(j-1)\alpha}{k}} + \sum_{j=1}^{m} k^j \lambda^{j-1} {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{j\alpha,\varphi}_{k,s} f(x).$$

Taking the limit while m tends to ∞, we get the following explicit pattern of $y(x)$ to the solution of 2.1 and 2.2:

$$y(x) = y(a) \sum_{j=1}^{\infty} \frac{k^{j-1}\lambda^{j-1} s^{-(j-1)\frac{\alpha}{k}}}{\Gamma_k((j-1)\alpha+k)} (\varphi^s(x) - \varphi^s(t))^{\frac{(j-1)\alpha}{k}} + \sum_{j=1}^{\infty} k^j \lambda^{j-1} {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{j\alpha,\varphi}_{k,s} f(x).$$

By replacing the index of summation j by $j-1$, we have

$$y(x) = y(a) \sum_{j=0}^{\infty} \frac{k^{j}\lambda^{j} s^{-j\frac{\alpha}{k}}}{\Gamma_k(j\alpha+k)} (\varphi^s(x) - \varphi^s(t))^{\frac{j\alpha}{k}} + \sum_{j=0}^{\infty} k^{j+1} \lambda^{j} {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{(j+1)\alpha,\varphi}_{k,s} f(x).$$

which provides us the required result by keeping in mind the given definition of φ-GRL k-FI and k-Gamma and k-Beta functions and their features. □

Corollary 2.21 *For $0 < \alpha < 1$, the following special case of the fractional initial value problem (2.1)-(2.2):*

$$\begin{aligned} {}^{\mathfrak{C}}_{0^+}\mathfrak{D}^{\alpha,x^2}_{2,1} y(x) - 2^{\frac{\alpha}{2}-1} y(x) &= 1, \\ y(0) &= 1, \end{aligned}$$

is of the solution obtained from the equation 2.4

$$y(x) = E_{\frac{\alpha}{2}}\left(\left(x^2\right)^{\frac{\alpha}{2}}\right) + 2^{\frac{\alpha}{2}-1} \int_0^x \left(x^2 - t^2\right)^{\frac{\alpha}{2}-1} E_{\frac{\alpha}{2},\frac{\alpha}{2}}\left(\left(x^2 - t^2\right)^{\frac{\alpha}{2}}\right) dt^2, \tag{2.4}$$

which is equal to

$$y(x) = E_{\frac{\alpha}{2}}\left(x^{\alpha}\right) + 2^{1-\frac{\alpha}{2}} x^{\alpha} E_{\frac{\alpha}{2}\frac{\alpha}{2}+1}\left(x^{\alpha}\right).$$

3. Conclusion

In recent times, lots of new types of fractional integral and derivatives were introduced and served to many real-world problems. Therefore, our aims are to combine these conceptions into a united one and improve a theory for FDEs with a unified novel derivative. In this study, we presented quite comprehensive φ-generalized Riemann-Liouville k-fractional integral and φ-generalized Riemann-Liouville and Caputo k-fractional derivatives which can be reduced to most of the well-known fractional integrals and derivatives depending on the choices of k, s, φ. Some fundamental features are discussed to build the theory's basement.

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