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# The integer-antimagic spectra of a disjoint union of Hamiltonian graphs

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**Abstract:** Let $A$ be a nontrivial abelian group. A simple graph $G=(V,E)$ is $A$-antimagic, if there exists an edge labeling $f: E(G) \to A\backslash\{0\}$ such that the induced vertex labeling $f^+(v)=\sum_{uv\in E(G)} f(uv)$ is a one-to-one map. The integer-antimagic spectrum of a graph $G$ is the set $\text{IAM}(G)=\{k: G \text{ is } \mathbb{Z}_k\text{-antimagic and } k\geq 2\}$. In this paper, we determine the integer-antimagic spectra for a disjoint union of Hamiltonian graphs.

**Key words:** Disjoint union, Hamiltonian graphs, graph labeling, integer-antimagic labeling

## 1. Introduction

A labeling of a graph is defined to be an assignment of values to the vertices and/or edges of the graph. Graph labeling is a very diverse and active field of study. A dynamic survey [2] maintained by Gallian contains 2922 references to research papers and books on the topic.

Let $G$ be a simple graph. For any nontrivial abelian group $A$ (written additively), let $A^*=A\backslash\{0\}$, where $0$ is the additive identity of $A$ (sometimes denoted by $0_A$). Let function $f: E(G)\to A^*$ be an edge labeling of $G$. Any such labeling induces a map $f^+: V(G)\to A$, defined by $f^+(v)=\sum_{uv\in E(G)} f(uv)$. If there exists such an edge labeling $f$ whose induced map $f^+$ on $V(G)$ is one-to-one, we say that $f$ is an $A$-antimagic labeling and that $G$ is an $A-\textit{antimagic}$ graph. The integer-antimagic spectrum of a graph $G$ is the set $\text{IAM}(G)=\{k: G \text{ is } \mathbb{Z}_k\text{-antimagic and } k\geq 2\}$. Let $f: E(G)\to\mathbb{Z}^+$ be an edge labeling of $G$ and $f^+$ be its induced vertex labeling. We will denote the range of $f^+$ by $\mathcal{R}_f(G)$.

The concept of the $A$-antimagicness property for a graph $G$ (introduced independently in [1, 3]) naturally arises as a variation of the $A$-magic labeling problem (where the induced vertex labeling is a constant map). There is a large body of research on $A$-magic graphs within the mathematical literature. As for $A$-antimagic graphs (which is the focus of our paper), cycles, paths, various classes of trees, dumbbells, graphs with a chord, Hamiltonian graphs, multicyclic graphs, complete bipartite graphs, complete bipartite graphs with a deleted edge, tadpoles and lollipop graphs were investigated in [1, 3–6, 9–11].

Now, we include some known results which will be used in the rest of the paper. In particular, the results from the theorems in this section are used in the constructions of new $\mathbb{Z}_k$-antimagic labelings.

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A trivial lower bound for the least element of $\mathrm{IAM}(G)$ is the order of $G$. However, this is not always achieved, as seen in the following result from [1].

**Lemma 1.1 ([1])** *A graph of order $4m+2$, for all $m \in \mathbb{Z}^+$, is not $\mathbb{Z}_{4m+2}$-antimagic.*

Motivation for our current work is found in the following conjecture. The analogous conjecture for connected simple graphs was given in [6].

**Conjecture 1.2** *Let $G$ be a simple graph. If $t$ is the least positive integer such that $G$ is $\mathbb{Z}_t$-antimagic, then $\mathrm{IAM}(G) = \{k : k \geq t\}$.*

A result of Jones and Zhang [3] finds the minimum element of $\mathrm{IAM}(G)$ for all connected graphs on three or more vertices. In their paper, a $\mathbb{Z}_n$-antimagic labeling of a graph on $n$ vertices is referred to as a nowhere-zero modular edge-graceful labeling. This is a variation of a graceful labeling (originally called a $\beta$-valuation) which was introduced by Rosa [7] in 1967. The result is as follows, where the terminology has been adapted to better suit this paper.

**Theorem 1.3 ([3])** *If $G$ is a connected simple graph of order $n \geq 3$, then $min\{t : t \in \mathrm{IAM}(G)\} \in \{n, n+1, n+2\}$. Furthermore,*

- *$min\{t : t \in \mathrm{IAM}(G)\} = n$ if and only if $n \not\equiv 2 \pmod 4$, $G \neq K_3$, and $G$ is not a star of even order,*
- *$min\{t : t \in \mathrm{IAM}(G)\} = n+1$ if and only if $G = K_3$ or $n \equiv 2 \pmod 4$ and $G$ is not a star of even order, and*
- *$min\{t : t \in \mathrm{IAM}(G)\} = n+2$ if and only if $G$ is a star of even order.*

If $a$ and $b$ are integers with $a \leq b$, let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$. Let $(v_0, v_1, \ldots, v_{n-1})$ denote the $n$-cycle with edges $v_i v_{i+1}$ for $i \in [0, n-2]$ and $v_0 v_{n-1}$. Consider the cycle $C_n = (v_0, v_1, \ldots, v_{n-1})$. Define the x-alternating cycle labeling of $C_n$, starting with the edge $v_i v_{i+1}$ to be the function $g_x : E(C_n) \to \{x, -x\}$, such that $g_x(v_i v_{i+1}) = x$, and $g_x$ alternates between $-x$ and $x$ where $x \in A$ for some additive group $A$.

In [1, 6, 8**?** –11], Conjecture 1.2 was shown to be true for various classes of graphs. We will make use of the following result in our main construction.

**Theorem 1.4 ([1])** *$C_{4m+r}$, for all $m \in \mathbb{N}$, is $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+r$, if $r = 0, 1, 3$. $C_{4m+2}$ for all $m \in \mathbb{N}$ are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+3$.*

In [8], the integer-antimagic spectra of a disjoint union of cycles was established. We will use this theorem in the proof of Theorem 4.1.

**Theorem 1.5 ([8])** *Let $G$ be a disjoint union of cycles, where $|V(G)| = n$. Then,*

$$\mathrm{IAM}(G) = \begin{cases} [4, \infty) & \text{if } G = K_3 \\ [n, \infty) & \text{if } n \equiv 0, 1, 3 \pmod 4 \text{ and } G \neq K_3 \\ [n+1, \infty) & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

The purpose of this paper is to provide additional evidence for Conjecture 1.2 by verifying it for a large family of graphs (namely, all disjoint unions of Hamiltonian graphs). This is accomplished by explicit construction of the labelings. Here, a disjoint union of Hamiltonian graphs is a graph whose components are each Hamiltonian.

## 2. Graphs with an odd path

Let $[v_0, v_1, \ldots, v_{n-1}]$ denote the $n$-path with edges $v_iv_{i+1}$ for $i \in [0, n-2]$. Consider the path $P = [v_0, v_1, \ldots, v_{n-1}]$. We will use $P[v_0, v_{n-1}]$ to denote the path with end vertices $v_0$ and $v_{n-1}$. Define the x-alternating path labeling of $P$, starting with the edge $v_iv_{i+1}$ to be the function $t_x : E(P) \to \{x, -x\}$, such that $t_x(v_iv_{i+1}) = x$ and $t_x$ alternates between $-x$ and $x$ where $x \in A$ for some additive group $A$.

If we are given a $\mathbb{Z}_k$-antimagic labeling $f$ of a path $P$, then $f^+(v) = (f + t_x)^+(v)$ where $x \in \mathbb{Z}_k \backslash \{0\}$, that is, adding an integer $x$ and its additive inverse $-x$ to the alternating edges of the path does not change the labels of the interior vertices of the path. Moreover, applying $x$-alternating path labeling to the path $P = [v_0, v_1, \ldots, v_{2n}]$, starting with the edge $v_0v_1$, for a special case when $x = f^+(v_{2n}) - f^+(v_0)$, replaces the labels of the end vertices $v_0$ and $v_{2n}$ of the path.

**Observation 2.1** *For an odd integer $n \geq 2$, suppose $f : E(P) \to \mathbb{Z}_k \backslash \{0\}$ is a $\mathbb{Z}_k$-antimagic labeling of a path $P = [v_0, v_1, \ldots, v_{n-1}]$ and let $t_x$ be the $x$-alternating path labeling of the path $P$ starting with the edge $v_0v_1$ where $x = f^+(v_{n-1}) - f^+(v_0)$. Then, $h : E(P) \to \mathbb{Z}_k$ is a labeling of the path with $\mathcal{R}_f(P) = \mathcal{R}_h(P)$ where $h$ is defined as $h = f + t_x$.*

Note that for the graph $G'$ obtained from a given antimagic graph $G$ by adding an edge between two nonadjacent vertices in $G$, it is always possible to find an edge labeling of $G'$ such that the range of the induced vertex labeling is the same as the vertex labeling of $G$. However, it could be the case that $(f + t_x)(uv) = 0$ for some edges $uv$ and choices of $x \in \mathbb{Z}_k \backslash \{0\}$. By the following lemma, if it exists, we will be able to find such an integer $x$, such that $(f + t_x)(uv) \neq 0$ for all edges $uv \in E(P)$.

**Lemma 2.2** *Let $f$ be a $\mathbb{Z}_k$-antimagic labeling of a graph $G$ and suppose $G' = G \cup \{uv\}$ where $u, v \in V(G)$ and $uv \notin E(G)$. Suppose $uv$ lies on an odd order path $P[y, z]$ in $G'$ such that $f^+(y) - f^+(z) \neq \pm f(e)$ for each edge $e \in P$. Then, there is a $\mathbb{Z}_k$-antimagic labeling $h$ of $G'$ such that $\mathcal{R}_f(G) = \mathcal{R}_h(G')$.*

**Proof** Let $f : E(G) \to \mathbb{Z}_k \backslash \{0\}$ be a $\mathbb{Z}_k$-antimagic labeling of $G$ and $P = [y = v_0, v_1, \ldots, v_{2n} = z]$ be a path in $G'$ including the edge $uv$. We define $h : E(G') \to \mathbb{Z}_k \backslash \{0\}$ by

$$h(e) = f(e) + w(e),$$

where addition is in $\mathbb{Z}_k$ and

$$w(e) = \begin{cases} t_x(e) & \text{if } e \in P \\ 0 & \text{otherwise.} \end{cases}$$

Here, $x = f^+(v_{2n}) - f^+(v_0)$ and $t_x$ is the $x$-alternating path labeling of the path $P$ starting with the edge $v_0v_1$. Clearly, $t_x^+(v_i) = 0$ for all $1 \leq i \leq 2n-1$, and $t_x^+(v_0) = f^+(v_{2n}) - f^+(v_0)$ and $t_x^+(v_{2n}) = f^+(v_0) - f^+(v_{2n})$. So, $f^+(v_i) = h^+(v_i)$ for all $1 \leq i \leq 2n-1$, and $f^+(v_0) = h^+(v_{2n})$ and $f^+(v_{2n}) = h^+(v_0)$. Also, since

$f^+(v_{2n}) - f^+(v_0) \neq \pm f(e)$ for each edge $e \in P$, $h(e) \neq 0$ for each $e \in G'$. Thus, $h$ is the desired $\mathbb{Z}_k$-antimagic labeling of $G'$ with $\mathcal{R}_f(G) = \mathcal{R}_h(G')$. □

*Example 1.* In order to show how the alternating path labelings are used in the proof of Lemma 2.2, an example is given in Figure 1. In 1(a), a $\mathbb{Z}_k$-antimagic labeling of a graph $G$ for $k \geq 7$ is illustrated. In 1(b), (c), and (d), the overlaying of different path labelings (onto the original labeling of $G$) are illustrated, which in turn give $\mathbb{Z}_k$-antimagic labelings of $G \cup \{uv\}$. Furthermore, these new $\mathbb{Z}_k$-antimagic labelings maintain the same range of vertex labels. ◊

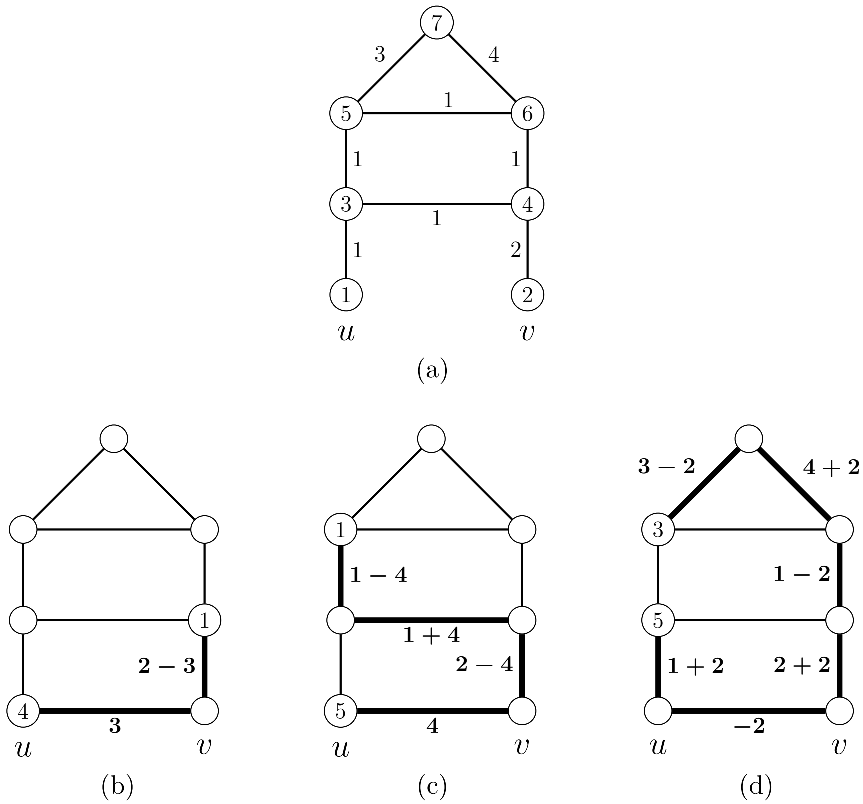


**Figure 1**. Three possible odd order paths including edge $uv$ and their labelings.

## 3. Graphs with a cycle

Let $(v_0, v_1, \ldots, v_{n-1})$ denote the $n$-cycle with edges $v_iv_{i+1}$ for $i \in [0, n-2]$ and $v_0v_{n-1}$. Consider the cycle $C_n = (v_0, v_1, \ldots, v_{n-1})$. Define the x-alternating cycle labeling of $C_n$, starting with the edge $v_iv_{i+1}$ to be the function $g_x : E(C_n) \to \{x, -x\}$, such that $g_x(v_iv_{i+1}) = x$ and $g_x$ alternates between $-x$ and $x$ where $x \in A$ for some additive group $A$.

If we are given a $\mathbb{Z}_k$-antimagic labeling $f$ of an even cycle $C$, then $f^+(v) = (f + g_x)^+(v)$ where $x \in \mathbb{Z}_k \backslash \{0\}$, that is, adding an integer $x$ and its additive inverse $-x$ to the alternating edges of the cycle does not change the vertex labels of the cycle.

**Observation 3.1** *Suppose $f : E(C) \to \mathbb{Z}_k \backslash \{0\}$ is a $\mathbb{Z}_k$-antimagic labeling of an even cycle $C$ and let $g_x$ be the*

*$x$-alternating cycle labeling of the cycle $C$. Then, $h : E(C) \to \mathbb{Z}_k$ is a labeling of the cycle with $\mathcal{R}_f(C) = \mathcal{R}_h(C)$ where $h$ is defined as $h = f + g_x$.*

From a given $\mathbb{Z}_k$-antimagic graph $G$ with two nonadjacent vertices $a$ and $b$, let us construct $G'$ by adding an edge between $a$ and $b$. Notice that if the new edge $ab$ lies on an even cycle in $G'$, then by the above labeling of even cycles, we can label $ab$ preserving the vertex labels of $G$. However, it could be the case that $(f + g_x)(uv) = 0$ for some edge $uv$ and choices of $x \in \mathbb{Z}_k \backslash \{0\}$. The following result, given in [5], shows the existence of such an integer $x$ such that $(f + g_x)(uv) \neq 0$ for all edges $uv \in E(C)$.

**Lemma 3.2 ([5])** *Let $f$ be a $\mathbb{Z}_k$-antimagic labeling of a graph $G$ and let $G' = G \cup \{uv\}$ where $u, v \in V(G)$ and $uv \notin E(G)$. Suppose $uv$ lies on a non-Hamiltonian even cycle $C_m$ in $G'$, then there is a $\mathbb{Z}_k$-antimagic labeling $h$ of $G'$ such that $\mathcal{R}_f(G) = \mathcal{R}_h(G')$.*

Suppose that $G$ is a graph, and $C$ is the cycle $(v_0, v_1, ..., v_{n-1})$ in $G$. A chord of a cycle $C$ is an edge not in $E(C)$ whose endpoints lie in the vertex set $V(C)$. If $C$ has at least one chord, then it is called a chorded cycle. We define $C(l)$ to be the graph obtained from $C$ by adding the chord $v_iv_j$, where $l = min\{|i-j|, n-|i-j|\}$ which is called the length of the chord. Note that the length of any chord in a cycle $C$ is at least 2 and at most $\lfloor \frac{n}{2} \rfloor$.

We call the chorded cycle $C(l)$ a cycle with a chord of length $l$, as well. Note that $C(l)$ is the union of two cycles which share exactly one edge – the chord. We call the shorter of the two cycles the minor subcycle of $C(l)$, denoted by $C^-(l)$, and the longer of the two cycles the major subcycle of $C(l)$, denoted by $C^+(l)$.

In [1], Conjecture 1.2 was shown to be true for cycles with a chord.

**Theorem 3.3 ([4])** *Let $n$ be an integer and let $l \in [2, \lfloor \frac{n}{2} \rfloor]$ be an integer. Then, $\mathrm{IAM}(C_n(l)) = \{k : k \geq n\}$ if $n \equiv 0, 1, 3 \pmod 4$ and $\mathrm{IAM}(C_n(l)) = \{k : k \geq n+1\}$ if $n \equiv 2 \pmod 4$.*

The following lemma provides a tool for handling even cycles with even chords.

**Lemma 3.4** *Let $f$ be a $\mathbb{Z}_k$-antimagic labeling of an even cycle $C = (v_0, v_1, \ldots, v_{n-1})$. Then, for every even integer $l \in [2, \frac{n}{2}]$, there is at least one integer $i \in \mathbb{Z}_n$ such that $f(v_iv_{i+1}) \neq f^+(v_{i+1}) - f^+(v_{i+l})$ or $f(v_iv_{i+1}) \neq f^+(v_i) - f^+(v_{i-l+1})$.*

**Proof** We will prove the lemma by contradiction. By the definition of $f^+$, $f^+(v_i) = f(e_{i-1}) + f(e_i)$ where $e_i = v_iv_{i+1}$ for all $i \in \mathbb{Z}_n$.

First suppose that for all $i \in \mathbb{Z}_n$, $f(e_i) = f^+(v_{i+1}) - f^+(v_{i+l})$. So, we have $f^+(v_i) = f(e_{i-1}) + f(e_i) = f^+(v_i) - f^+(v_{i+l-1}) + f^+(v_{i+1}) - f^+(v_{i+l})$ or simply

$$f^+(v_{i+1}) = f^+(v_{i+l-1}) + f^+(v_{i+l}) \tag{3.1}$$

for all $i \in \mathbb{Z}_n$. Now suppose that for all $i \in \mathbb{Z}_n$, $f(e_i) = f^+(v_{i+1}) - f^+(v_{i+l}) = f^+(v_i) - f^+(v_{i-l+1})$. Since $f^+(v_i) = f(e_{i-1}) + f(e_i)$, we have $f^+(v_i) = f(e_{i-1}) + f(e_i) = f^+(v_i) - f^+(v_{i+l-1}) + f^+(v_i) - f^+(v_{i-l+1})$ or simply

$$f^+(v_i) = f^+(v_{i+l-1}) + f^+(v_{i-l+1}) \tag{3.2}$$

for all $i \in \mathbb{Z}_n$.

From equation (3.2), one can obtain $f^+(v_{i+1}) = f^+(v_{i+l}) + f^+(v_{i-l+2})$, and if we substitute this in equation (3.1), then we have

$$f^+(v_{i+l-1}) = f^+(v_{i-l+2}) \tag{3.3}$$

for all $i \in \mathbb{Z}_n$. It is a contradiction since otherwise, we would have $i + l - 1 \equiv i - l + 2 \pmod{n}$. However, $2l \not\equiv 3 \pmod{n}$ since $l$ and $n$ are both even. □

Now, we extend the result of Theorem 3.3 by including the range condition for the vertex labeling. This is extensively used in our main construction.

**Theorem 3.5** *Let $f$ be a $\mathbb{Z}_k$-antimagic labeling of a cycle $C$ and $l \geq 2$ be an integer. Then, there exists $h : E(C(l)) \to \mathbb{Z}_k \backslash \{0\}$ which is an antimagic labeling of $C(l)$ such that $\mathcal{R}_f(C) = \mathcal{R}_h(C(l))$.*

**Proof** Assume that $c$ is a chord in $C = (v_0, v_1, \ldots, v_{n-1})$ of length $l \in [2, \lfloor \frac{n}{2} \rfloor]$. If $l$ is odd, then the minor subcycle $C^-(l)$ is an even cycle in $C(l)$ with $|V(C^-(l))| < n - 1$. So applying Lemma 3.2 gives the result for odd length chord $l$. Similarly, if $n$ is odd, then $|V(C^+(l))|$ and $|V(C^-(l))|$ have different parities, that is, there exists at least one even cycle in $C(l)$ which includes the chord $c$. Thus, by Lemma 3.2, we have the desired labeling of $C(l)$. So we may assume $n$ and $l$ are even integers.

By Lemma 3.4, there is always an integer $i \in \mathbb{Z}_n$ such that at least one of $f^+(v_{i+1}) - f^+(v_{i+l})$ and $f^+(v_i) - f^+(v_{i-l+1})$ is different from $f(v_i v_{i+1})$. Let $C^{(i)}(l)$ be the chorded cycle with the chord $v_i v_{i+l}$. For $0 \leq i \leq n-1$, the 3-paths $P_1^{(i)} = [v_{i+1}, v_i, v_{i+l}]$ and $P_2^{(i)} = [v_i, v_{i+1}, v_{i-l+1}]$ belong to $C^{(i)}(l)$ and $C^{(i-l+1)}(l)$, respectively, including the chord. For the sake of brevity, we will denote the path $P_j^i$ as $P^*$ where $i$ is the integer satisfying conditions of Lemma 3.4 and $j = 1$ if $f(v_i v_{i+l}) \neq f^+(v_{i+1}) - f^+(v_{i+l})$ otherwise $j = 2$. Also, let $c = v_i v_{i+l}$ if $f(v_i v_{i+1}) \neq f^+(v_{i+1}) - f^+(v_{i+l})$ otherwise $c = v_{i+1} v_{i-l+1}$. The chord $c$ lies on an odd order path $P^*$ in either $C^{(i)}(l)$ or $C^{(i-l+1)}(l)$ satisfying the conditions of Lemma 2.2. It is obvious that $C^{(i)}(l) \cong C^{(j)}(l)$ for all $i, j \in [0, n-1]$. Thus, by Lemma 2.2, we have the desired labeling of $C$ with the chord $c$. □

*Example 2*. In order to show how alternating path labelings are used in the proof of Theorem 3.5, examples are given in Figure 2. These figures show that the odd order path we use is not unique, although Theorem 3.5 guarantees that such a path always exists. In 2(a), a $\mathbb{Z}_k$-antimagic labeling of $C_8$ for $k \geq 8$ is illustrated. In 2(b) and 2(c), the overlaying of $P_3$ and $P_5$ labelings, respectively, (onto the original labelings of $C_8$) are illustrated. This, in turn, gives $\mathbb{Z}_k$-antimagic labelings of $C_8 \cup \{uv\}$. Furthermore, the new $\mathbb{Z}_k$-antimagic labelings maintain the same range of vertex labels. ◊

Also, we will use the following integer-antimagic labeling of a cycle with more than one chord, as found in [5].

**Lemma 3.6** **([5])** *Let $m$ be an integer and let $l_1, l_2 \in [2, \lfloor \frac{m}{2} \rfloor]$, where $l_1$ and $l_2$ have the same parity when $m$ is even. Also let $f$ be a $\mathbb{Z}_k$-antimagic labeling of a graph $G$ and let $G' = G \cup \{c_1, c_2\}$. If the edges $c_1$ and $c_2$ are two different chords of lengths $l_1$ and $l_2$, respectively, of a cycle $C_m$ in $G'$, then there is a $\mathbb{Z}_k$-antimagic labeling $h$ of $G'$ such that $\mathcal{R}_f(G) = \mathcal{R}_h(G')$.*

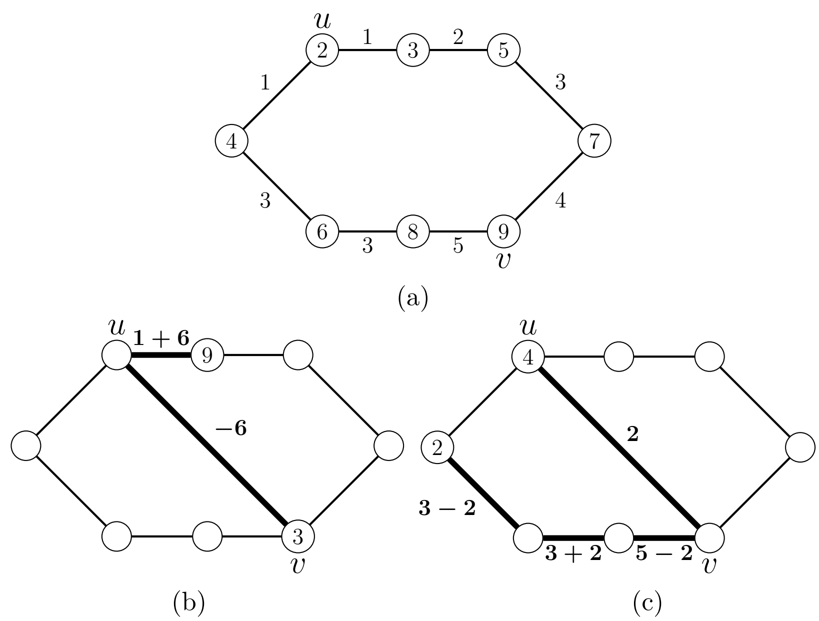


**Figure 2**. Two different odd order paths used to preserve $\mathbb{Z}_k$-antimagicness for $k \geq 8$, when adding a chord to $C_8$.

## 4. Main result

Now we prove our main result, which gives a complete characterization of the integer-antimagic spectra for a disjoint union of Hamiltonian graphs.

**Theorem 4.1** *Let $G$ be a disjoint union of Hamiltonian graphs, where $|V(G)| = n$. Then,*

$$\mathrm{IAM}(G) = \begin{cases} [4, \infty) & \text{if } G = K_3 \\ [n, \infty) & \text{if } n \equiv 0,1,3 \pmod 4 \text{ and } G \neq K_3 \\ [n+1, \infty) & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

**Proof** The result is obvious if $G = K_3$, so we may assume that the order of $G$ is at least 4. Since the integer-antimagic spectra of Hamiltonian graphs have been determined [5], we may assume that $G$ contains at least two connected components. Let $H_1, H_2, \ldots, H_{m-1}$ and $H_m$ be disjoint Hamiltonian graphs possessing the spanning cycles $C_{n_1}, C_{n_2}, \ldots$ and $C_{n_m}$, respectively. Also, let $G$ be the disjoint union of the $H_i$'s, i.e. $G = \bigcup_{i=1}^{m} H_i$. Note that $G$ has order $n = \sum_{i=1}^{m} n_i$. We can think of each element of $E(H_i) \backslash E(C_{n_i})$ as a chord of $C_{n_i}$. We will construct $G$ by adding chords to the Hamilton cycles $C_{n_i}$, for each $1 \leq i \leq m$. By Theorem 1.5, there exists a $\mathbb{Z}_k$-antimagic labeling $f$ of $\bigcup_{i=1}^{m} C_{n_i}$ for $k \geq n$ when $n \equiv 0,1,3 \pmod 4$ and $k \geq n+1$ when $n \equiv 2 \pmod 4$. First, we will keep adding chords to each cycle $C_{n_i}$ for $1 \leq i \leq m$, updating the edge labeling as we go, until we get all the edges of $H_i$ in the resulting graph. Here, we separate the edge adding and labeling procedure into two cases depending on the parity of $n_i$.

For each $i \in \{1, \ldots, m\}$ where $n_i$ is odd, the lengths of the major and minor subcycles have different parities; that is, there exists at least one even cycle in that component which includes the chord. Thus, by Lemma 3.2, we have a $\mathbb{Z}_k$-antimagic labeling of that component which preserves the vertex labels induced by

$f$. Repeat this process for each chord in the component.

For each $i \in \{1, \dots, m\}$ where $n_i$ is even, we will separate into two cases: when the number of even length chords in $H_i$ is even, and when the number of even length chords in $H_i$ is odd. In both cases, we will construct $H_i$ by adding all even length chords and then remaining odd length chords, in turn. If the number of even length chords is even, then we can pair up these even length chords and add them to $C_{n_i}$ as pairs. Thus, by Lemma 3.6, this edge addition does not change the vertex labels induced by $f$. If the number of even length chords is odd, then we first add a single chord of even length $l$ to $C_{n_i}$. By Theorem 3.5, the updated labeling of $C_{n_i}(l)$ induces the same vertex labels as $f$. Again, we can pair up the remaining even length chords and keep adding them to $C_{n_i}(l)$ as pairs until we have all the even length chords in the component $H_i$. Lastly, we add the odd length chords in $H_i$. As we mentioned before, if the length of a chord is odd, then the corresponding minor subcycle is an even cycle. Thus, by Lemma 3.2, regardless of how many odd length chords are added, the resulting graph is always $\mathbb{Z}_k$-antimagic.

□

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