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On the convergence of the Abel–Poisson means of multiple Fourier series

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Abstract: Let $A_\varepsilon(x, f)$ be the Abel–Poisson means of an integrable function $f(x)$ on n -dimensional torus \mathbf{T}^n , $-\pi < x_i \leq \pi$, $i = 1, \dots, n$ ($n \geq 2$) in the Euclidean n -space. The famous Bochner’s theorem asserts that for any function $f \in L^1(\mathbf{T}^n)$ the Abel–Poisson means $A_\varepsilon(x, f)$ are pointwise converge to $f(x)$ a.e., that is,

$$\lim_{\varepsilon \rightarrow 0^+} A_\varepsilon(x, f) = f(x), \quad a.e. x \in \mathbf{T}^n.$$

In this paper we investigate the rate of convergence of Abel–Poisson means at the so-called μ -smoothness point of f .

Key words: Abel–Poisson means, multiple Fourier series, Poisson summation formula

1. Introduction and formulation of main results

Let \mathbb{R}^n denote the n -dimensional Euclidean space,

$$\mathbf{T}^n = \{x \in \mathbb{R}^n \mid -\pi < x_i \leq \pi, \quad i = 1, \dots, n\}$$

be n -dimensional torus, and \mathbb{Z}^n be the integral lattice of \mathbb{R}^n . For any $f(x) \in L^1(\mathbf{T}^n)$, we form the Fourier series of its periodical continuation by

$$f(x) \sim \sum a_m e^{im \cdot x}, \quad (1.1)$$

with Fourier coefficient $a_m = \int_{\mathbf{T}^n} f(x) e^{-im \cdot x} dx$, where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, and $m \cdot x = m_1 x_1 + \dots + m_n x_n$.

For a function $f \in L^1(\mathbf{T}^n)$, define the Abel–Poisson means of $f(x)$ by

$$A_\varepsilon(x, f) = \sum_{m \in \mathbb{Z}^n} e^{-\varepsilon|x|} a_m e^{im \cdot x}. \quad (1.2)$$

It is well known that the Fourier transform of the function $e^{-\varepsilon|x|}$ has the following form

$$P_\varepsilon(y) \equiv P(y; \varepsilon) = (e^{-\varepsilon|\cdot|})^\wedge(y) = \frac{c_n \varepsilon}{(\varepsilon^2 + |y|^2)^{(n+1)/2}}, \quad c_n = \pi^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right). \quad (1.3)$$

The function $P_\varepsilon(y)$ is called the Poisson kernel and has the following properties [12, p. 253].

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Proposition 1.1

$$\begin{aligned} (a) \quad & \int_{\mathbb{R}^n} P_\varepsilon(y) dy = 1, \text{ for all } \varepsilon > 0. \\ (b) \quad & P_1(y) \leq A(1 + |y|)^{-(n+\delta)}, \text{ for some } \delta > 0. \end{aligned} \quad (1.4)$$

For $A_\varepsilon(x, f)$, the Abel–Poisson means of $f(x)$, defined by (1.2), we have the following integral representation via Poisson kernel;

Lemma 1.2 *Let the function $f \in L^1(\mathbf{T}^n)$ be periodically continued to \mathbb{R}^n and $P_\varepsilon(y)$ be the Poisson kernel, defined by (1.3). Then*

$$A_\varepsilon(x, f) = \int_{\mathbb{R}^n} f(x-t)P_\varepsilon(t)dt. \quad (1.5)$$

The formula (1.5) is well known and its proof is based on the following Poisson summation formula:

$$\sum_{m \in \mathbb{Z}^n} \Phi(\varepsilon m) e^{im \cdot t} = (2\pi)^n \sum_{m \in \mathbb{Z}^n} P_\varepsilon(t + 2\pi m), \text{ where } \Phi(x) = e^{-|x|}. \quad (1.6)$$

Many works with different perspectives on the summation of multiple Fourier series and integrals have been studied in great detail in the papers [6, 7, 11, 13] (see also Stein, E.M. and Weiss, G. [12] and Weiss, F. [14, 15]). The purpose of this paper is as follows: First we introduce the notion of μ –smoothness point of a function f , which is also a Lebesgue point of f . Then we estimate the error of approximation of $f(x)$ by its Abel–Poisson means $A_\varepsilon(x, f)$ as $\varepsilon \rightarrow 0^+$ at the μ –smoothness point of function f . Note that the analogous problem for the Gauss–Weierstrass means of the relevant higher dimensional Fourier series has been studied in [2]. Some aspects of the rate of convergence, in the case of truncated hypersingular integrals generated by the Poisson and metaharmonic semigroups have been studied in [4], and for truncated hypersingular integrals generated by the Gauss–Weierstrass semigroups have been studied in [3]. Also, the nice papers by Golubov, B.I. [9, 10], by Aliev I.A. [1] and by Bayrakci S., Shafiev M.F., Aliev I.A [5] should be mentioned.

Definition 1.3 (cf.[1-3]) *Let $\rho \in (0, 1)$ be fixed parameter and the function $\mu(r)$, $(0 \leq r \leq \rho)$ be continuous on $[0, \rho]$, be positive on $(0, \rho]$, and $\mu(0) = 0$. We say that a function $\varphi \in L^1_{loc}(\mathbb{R}^n)$ has the μ –smoothness property at a point $x^0 \in \mathbb{R}^n$ if*

$$D_\mu(x^0) \equiv \sup_{0 < r \leq \rho} \frac{1}{r^n \mu(r)} \int_{|x| \leq r} |\varphi(x^0 - x) - \varphi(x^0)| dx < \infty. \quad (1.7)$$

In the sequel it will be assumed that $\mu(r) \geq \alpha r$ for an $\alpha > 0$, and $\mu(r) = \mu(\rho)$ for $\rho \leq r < \infty$.

Remark 1.4 *It is clear that every μ –smoothness point x^0 is also a Lebesgue point of f :*

$$\frac{1}{r^n} \int_{|x| \leq r} |f(x^0 - x) - f(x^0)| dx \stackrel{(1.7)}{\leq} \mu(r) D_\mu(x^0) \rightarrow 0, \text{ as } r \rightarrow 0^+.$$

Now, we state the main results of the paper.

Theorem 1.5 Let a periodical function $f \in L^1(\mathbf{T}^n)$ has the μ -smoothness property at $x^0 \in \mathbf{T}^n$, and suppose that $A_\varepsilon(x^0, f)$ is the Abel-Poisson means of f defined by (1.5). Then

$$|A_\varepsilon(x^0, f) - f(x^0)| \leq cD_\mu(x^0) \left(\varepsilon + \int_0^{\rho/\varepsilon} \frac{\mu(\varepsilon r)}{1+r^2} dr \right), \quad \varepsilon \rightarrow 0^+, \tag{1.8}$$

where c does not depend on ε .

Corollary 1.6 Let the function μ be continuous on $[0, \rho]$, positive on $(0, \rho]$ and $\mu(0) = 0$. Also, suppose $\mu(t) \geq \alpha t$, $(0 \leq t \leq \rho, \alpha > 0)$ and be extended on $[\rho, \infty)$ as $\mu(t) = \mu(\rho)$. In addition, assume that there exists a locally bounded function $w(t) > 0$ such that

$$\mu(\varepsilon t) \leq \mu(\varepsilon)w(t), \quad (0 < \varepsilon < \rho, 0 < t < \rho/\varepsilon), \text{ and } \int_0^\infty \frac{w(t)}{1+t^2} dt < \infty. \tag{1.9}$$

If $x^0 \in \mathbf{T}^n$ is a μ -smoothness point of $f \in L^1(\mathbf{T}^n)$, then the following estimate holds

$$|A_\varepsilon(x^0, f) - f(x^0)| = O(\mu(\varepsilon)), \quad \varepsilon \rightarrow 0^+. \tag{1.10}$$

Corollary 1.7 (a) Let $0 < \gamma < 1, 0 < \rho < 1$, and $\mu(\varepsilon) = \varepsilon^\gamma, 0 < \varepsilon \leq \rho$. If $x^0 \in \mathbf{T}^n$ is μ -smoothness point of $f \in L^1(\mathbf{T}^n)$, then following estimate holds

$$|A_\varepsilon(x^0, f) - f(x^0)| = O(\varepsilon^\gamma), \quad \varepsilon \rightarrow 0^+. \tag{1.11}$$

(b) Let $0 < \gamma < 1, 0 < \rho < 1, 0 < \beta < \infty$, and $\mu(\varepsilon) = \varepsilon^\gamma (\log \frac{1}{\varepsilon})^\beta, 0 < \varepsilon \leq \rho$. If $x^0 \in \mathbf{T}^n$ is μ -smoothness point of $f \in L^1(\mathbf{T}^n)$, then following estimate holds

$$|A_\varepsilon(x^0, f) - f(x^0)| = O(\varepsilon^\gamma (\log \frac{1}{\varepsilon})^\beta), \quad \varepsilon \rightarrow 0^+. \tag{1.12}$$

2. Proofs of the main results

We will use of some techniques from [2] and [4].

Proof of Theorem 1.5: Let $x^0 \in \mathbf{T}^n$ be a μ -smoothness point of f . By (1.4) and (1.5), we have

$$\begin{aligned} |A_\varepsilon(x^0, f) - f(x^0)| &= \left| \int_{\mathbb{R}^n} P_\varepsilon(x) (f(x^0 - x) - f(x^0)) dx \right| \\ &\leq \int_{|x| \leq \rho} |P_\varepsilon(x)| |f(x^0 - x) - f(x^0)| dx + \int_{|x| > \rho} |P_\varepsilon(x)| |f(x^0 - x) - f(x^0)| dx \\ &= I_1(\varepsilon) + I_2(\varepsilon). \end{aligned} \tag{2.1}$$

In order to estimate $I_1(\varepsilon)$ we introduce the function

$$\Psi(x, x^0) = \begin{cases} |f(x^0 - x) - f(x^0)|, & \text{if } |x| \leq \rho, \\ 0, & \text{if } |x| > \rho. \end{cases}$$

Hence, using this function, we have

$$I_1(\varepsilon) = \int_{\mathbb{R}^n} P_\varepsilon(x) \Psi(x, x^0) dx.$$

In the last integral, by change of variables $x = r\theta$, where $r = |x|$, and remembering that the function P_ε is radial, we obtain that

$$I_1(\varepsilon) = \int_0^\infty r^{n-1} |P_\varepsilon(r)| \left(\int_{|\theta|=1} \Psi(r\theta, x^0) d\theta \right) dr.$$

Here, $d\theta$ is the Euclidean area element of the unit sphere $S_{n-1} = \{\theta \in \mathbb{R}^n : |\theta| = 1\}$. If we let

$$g(r) = \int_{|\theta|=1} \Psi(r\theta, x^0) d\theta,$$

and

$$h(r) = \int_0^r g(t) t^{n-1} dt$$

then, we have

$$I_1(\varepsilon) = \int_0^\infty r^{n-1} |P_\varepsilon(r)| g(r) dr = \int_0^\infty |P_\varepsilon(r)| dh(r).$$

Since $P_\varepsilon(r) > 0$ and differentiable on $[0, \infty)$, we get, by integrating by parts,

$$I_1(\varepsilon) = \int_0^\infty P_\varepsilon(r) dh(r) = P_\varepsilon(r) h(r) \Big|_0^\infty - \int_0^\infty P_\varepsilon'(r) h(r) dr. \quad (2.2)$$

Because that $h(0) = 0$, $h(r) = h(\rho)$ (for $r \geq \rho$), and $P_\varepsilon(r) \rightarrow 0$ as $r \rightarrow \infty$

$$P_\varepsilon(r) h(r) \Big|_0^\infty = 0.$$

Now, observing that

$$\begin{aligned} h(r) &= \int_0^r g(t) t^{n-1} dt = \int_{|x| \leq r} \Psi(x, x^0) dx = \int_{|x| \leq r} |f(x^0 - x) - f(x^0)| dx \\ &= r^n \mu(r) \frac{1}{r^n \mu(r)} \int_{|x| \leq r} |f(x^0 - x) - f(x^0)| dx \leq r^n \mu(r) D_\mu(x^0) \end{aligned} \quad (2.3)$$

where $D_\mu(x^0)$ is defined as in (1.7), and using the fact $h(r) = h(\rho) = \text{constant}$ for $r \geq \rho$, we have

$$\begin{aligned} I_1(\varepsilon) &\leq \int_0^\infty h(r)|P'_\varepsilon(r)|dr = \int_0^\rho h(r)|P'_\varepsilon(r)|dr + \int_\rho^\infty h(r)|P'_\varepsilon(r)|dr \\ &\leq D_\mu(x^0) \int_0^\rho r^n \mu(r)|P'_\varepsilon(r)|dr + h(\rho) \int_\rho^\infty |P'_\varepsilon(r)|dr. \end{aligned} \tag{2.4}$$

By (1.3),

$$P'_\varepsilon(r) = c_1 \varepsilon r (\varepsilon^2 + r^2)^{-(n+3)/2}.$$

Thus, from (2.3) and (2.4), we obtain

$$\begin{aligned} I_1(\varepsilon) &\leq c_1 D_\mu(x^0) \int_0^\rho r^{n+1} \mu(r) \varepsilon \frac{1}{(\varepsilon^2 + r^2)^{(n+3)/2}} dr \\ &\quad + c_1 h(\rho) \varepsilon \int_\rho^\infty \frac{r}{(\varepsilon^2 + r^2)^{(n+3)/2}} dr. \end{aligned}$$

Since $n \geq 1$, the latter integral in the above expression is finite, hence

$$I_1(\varepsilon) \leq c_1 D_\mu(x^0) \varepsilon \int_0^\rho r^{n+1} \mu(r) (\varepsilon^2 + r^2)^{-(n+3)/2} dr + c_2 \varepsilon.$$

Finally, denoting $c_3 = \max \left\{ c_1; \frac{c_2}{D_\mu(x^0)} \right\}$, we obtain

$$\begin{aligned} I_1(\varepsilon) &\leq c_3 D_\mu(x^0) \left(\varepsilon + \varepsilon \int_0^\rho r^{n+1} \mu(r) (\varepsilon^2 + r^2)^{-(n+3)/2} dr \right) \\ &= c_3 D_\mu(x^0) \left(\varepsilon + \int_0^{\rho/\varepsilon} r^{n+1} \mu(\varepsilon r) (1 + r^2)^{-(n+3)/2} dr \right) \\ &\leq c_4 D_\mu(x^0) \left(\varepsilon + \int_0^{\rho/\varepsilon} \frac{\mu(\varepsilon r)}{1 + r^2} dr \right). \end{aligned} \tag{2.5}$$

Let us now estimate the term $I_2(\varepsilon)$:

$$I_2(\varepsilon) \leq |f(x^0)| \int_{|x|>\rho} |P_\varepsilon(x)| dx + \int_{|x|>\rho} |P_\varepsilon(x)| |f(x^0 - x)| dx. \tag{2.6}$$

For the first integral,

$$\begin{aligned} |f(x^0)| \int_{|x|>\rho} |P_\varepsilon(x)| dx &= c_5 \int_\rho^\infty r^{n-1} |P_\varepsilon(r)| dr \\ &= c_6 \int_\rho^\infty r^{n-1} \frac{\varepsilon}{(\varepsilon^2 + r^2)^{(n+1)/2}} dr \leq c_7 \varepsilon \int_\rho^\infty \frac{1}{r^2} dr = c_8 \varepsilon. \end{aligned} \tag{2.7}$$

For the second integral, by applying the Hölder’s inequality, we get

$$\begin{aligned} \int_{|x|>\rho} |P_\varepsilon(x)| |f(x^0 - x)| dx &\leq \left(\int_{|x|>\rho} |f(x^0 - x)|^p dx \right)^{1/p} \left(\int_{|x|>\rho} |P_\varepsilon(x)|^q dx \right)^{1/q} \\ &\leq \|f\|_p \varepsilon \left(\int_{|x|>\rho} \frac{dx}{|x|^{(n+1)q}} \right)^{1/q} = c_9 \varepsilon. \end{aligned} \tag{2.8}$$

Hence, by (2.7) and (2.8) we obtain

$$I_2(\varepsilon) \leq c_{10} \varepsilon. \tag{2.9}$$

Finally, using (2.5) and (2.9) in (2.1), we have

$$|A_\varepsilon(x^0, f) - f(x^0)| \leq I_1(\varepsilon) + I_2(\varepsilon) \leq cD_\mu(x^0) \left(\varepsilon + \int_0^{\rho/\varepsilon} \frac{\mu(\varepsilon r)}{1+r^2} dr \right),$$

where the coefficient c does not depend on $\varepsilon > 0$. This clearly proves the theorem.

Proof of Corollary 1.6:

By (1.8) and the condition $\mu(\varepsilon t) \leq \mu(\varepsilon) w(t)$,

$$|A_\varepsilon(x^0, f) - f(x^0)| \leq cD_\mu(x^0) \left(\varepsilon + \mu(\varepsilon) \int_0^{\rho/\varepsilon} \frac{w(t)}{1+t^2} dt \right).$$

Since

$$\int_0^\infty \frac{w(t)}{1+t^2} dt < \infty \quad \text{and} \quad \mu(\varepsilon) \geq \alpha\varepsilon, \quad (0 < \varepsilon < \rho < 1),$$

we obtain

$$|A_\varepsilon(x^0, f) - f(x^0)| \leq c_1 \mu(\varepsilon), \quad (0 < \varepsilon < \rho < 1),$$

as desired.

Proof of Corollary 1.7:

a) It is easy to verify that the function

$$\mu(t) = \begin{cases} t^\gamma, & \text{if } 0 \leq t \leq \rho < 1, \\ \rho^\gamma, & \text{if } t \geq \rho, \end{cases}$$

satisfies all the conditions of Corollary 1.6 with the function $w(t) = t^\gamma$, $(0 < t < \infty)$.

b) If we let

$$\mu(t) = \begin{cases} 0, & \text{if } t = 0, \\ t^\gamma (\ln(1/t))^\beta, & \text{if } 0 < t \leq \rho < 1, \\ \rho^\gamma (\ln(1/\rho))^\beta, & \text{if } t \geq \rho, \end{cases}$$

and

$$w(t) = t^\gamma \left(1 + \frac{|\ln t|}{|\ln \rho|}\right)^\beta, \quad 0 < t < \infty,$$

then

$$\begin{aligned} \mu(\varepsilon t) &= (\varepsilon t)^\gamma \left(\ln \frac{1}{\varepsilon t}\right)^\beta = \varepsilon^\gamma \left(\ln \frac{1}{\varepsilon}\right)^\beta t^\gamma \left(1 + \frac{\ln t^{-1}}{\ln \varepsilon^{-1}}\right)^\beta \\ &\leq \varepsilon^\gamma \left(\ln \frac{1}{\varepsilon}\right)^\beta t^\gamma \left(1 + \frac{\ln |t|}{|\ln \rho|}\right)^\beta, \quad (0 < \varepsilon < \rho, \quad 0 < t < \rho < 1). \end{aligned}$$

Consequently, for the function

$$w(t) = t^\gamma \left(1 + \frac{|\ln t|}{|\ln \rho|}\right), \quad 0 < t < \rho/\varepsilon,$$

we observe that

$$w(\varepsilon t) \leq \mu(\varepsilon) w(t), \quad (0 < \varepsilon < \rho, \quad 0 < t < \rho/\varepsilon),$$

and therefore

$$\int_0^\infty \frac{w(t)}{1+t^2} dt = c \int_0^\infty \frac{t^\gamma (\ln \rho + |\ln t|)^\beta}{1+t^2} dt < \infty.$$

This proves the Corollary 1.7-(b).

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