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Gradient estimates of a nonlinear elliptic equation for the V -Laplacian on noncompact Riemannian manifolds

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Abstract: In this paper, we consider gradient estimates for positive solutions to the following equation

$$\Delta_V u + au^p \log u = 0$$

on complete noncompact Riemannian manifold with k -dimensional Bakry-Émery Ricci curvature bounded from below. Using the Bochner formula and the Cauchy inequality, we obtain upper bounds of $|\nabla u|$ with respect to the lower bound of the Bakry-Émery Ricci curvature.

Key words: Gradient estimates, V -Laplacian, Riemannian manifolds, Bakry-Émery Ricci curvature

1. Introduction

It is an interesting problem to consider gradient estimates for equations on Riemannian manifolds. Li and Yau [8] derived some parabolic gradient estimates for positive solutions to the following equation

$$u_t = \Delta u \tag{1.1}$$

on Riemannian manifold with Ricci curvature bounded from below. Hamilton [5] obtained some elliptic type gradient estimates for positive solutions to (1.1). Nowadays, gradient estimate of Li-Yau type or of Hamilton type was extended to other equations on Riemannian manifolds with various curvature conditions. Ma [9] got some gradient estimates for positive solutions to the following equation

$$\Delta u + au \log u = 0 \tag{1.2}$$

on complete noncompact Riemannian manifold with Ricci curvature bounded below. Dung [3] got a sharp gradient estimates for the following equation

$$u_t = \Delta u + au \log u. \tag{1.3}$$

Yang [12], Guo-Ishida [4], and Chen [2] discussed gradient estimates for positive solutions to the following equation

$$u_t = \Delta u + au \log u + bu. \tag{1.4}$$

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Wang [10] studied gradient estimates for the following equation

$$u_t(x, t) = \Delta u(x, t) + au(x, t) \log u(x, t) + b(x, t)u(x, t). \tag{1.5}$$

Huang-Ma [7] and Wu [11] studied gradient estimates for the following equation

$$u_t = \Delta_f u + au \log u + bu \tag{1.6}$$

on complete noncompact Riemannian manifold with k -dimensional Bakry-Émery Ricci curvature bounded from below. Chen and Qiu [1] obtained some gradient estimates for positive solutions to the following equation

$$u_t = \Delta_V u + au \log u. \tag{1.7}$$

Zhao [13] discussed gradient estimates for positive solutions to the following equation

$$\Delta_V(u^p) + bu = 0. \tag{1.8}$$

Huang and Li [6] studied gradient estimates for positive solutions to the following equation

$$\Delta_V u + au \log u = 0. \tag{1.9}$$

Motivated by [6, 9, 11, 13], we study gradient estimates for positive solutions to the following equation

$$\Delta_V u + au^p \log u = 0 \tag{1.10}$$

on complete noncompact Riemannian manifold with k -dimensional Bakry-Émery Ricci curvature bounded from below.

2. Preliminaries and notations

Let (M^n, g) be a complete n -dimensional Riemannian manifold. The V -Laplacian Δ_V is defined by

$$\Delta_V = \Delta + \langle V, \nabla \cdot \rangle,$$

where V is a smooth vector field on (M^n, g) , Δ is the Laplacian operator, ∇ is the gradient operator. If $V = \nabla f$, then Δ_V is called the f -Laplacian. Therefore the V -Laplacian Δ_V could be viewed as a generalization of the f -Laplacian Δ_f . For $k \geq n$, we can define the Bakry-Émery Ricci curvature as follows:

$$\begin{aligned} \text{Ric}_V &= \text{Ric} - \frac{1}{2} \mathfrak{L}_V g, \\ \text{Ric}_V^k &= \text{Ric}_V - \frac{1}{k-n} V^* \otimes V^*, \end{aligned} \tag{2.1}$$

where Ric is the Ricci curvature of (M^n, g) , \mathfrak{L} denotes the Lie derivative and V^* is the dual 1-form of V . Throughout this paper, we denote by $B_q(R)$ the geodesic ball centered at q with radius R and use the convention that $k = n$ if and only if $V \equiv 0$. Usually, Ric_V^k is called the k -dimensional Bakry-Émery Ricci curvature and Ric_V can seen the ∞ -dimensional Bakry-Émery Ricci curvature. For Ric_V^k , there is an important formula which can be read as

$$\frac{1}{2} \Delta_V |\nabla u|^2 = |\text{Hess}u|^2 + \langle \nabla u, \nabla \Delta_V u \rangle + \text{Ric}_V(\nabla u, \nabla u), \tag{2.2}$$

where $\text{Hess}u$ is the Hessian of u . Formula (2.1) is usually called the Bochner formula for Ric_V^k .

3. Main results and their proof

Theorem 3.1 *Let (M^n, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_V^k(B_q(2R)) \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10) on $B_q(2R)$, $p \neq \frac{k+2}{k}$, $\epsilon = 1 - \frac{kp}{k+2}$ and $h = u^\epsilon$. Then on $B_q(2R)$, the following inequality holds:*

$$\begin{aligned} \frac{1}{2} \Delta_V |\nabla h|^2 &\geq \frac{kp}{kp-k-2} \left(\frac{p}{kp-k-2} - 1 \right) \frac{|\nabla h|^4}{h^2} + \frac{kp}{kp-k-2} \frac{\langle \nabla h, \nabla |\nabla h|^2 \rangle}{h} \\ &\quad - ah^{\frac{(p-1)(2+k)}{2+k-kp}} |\nabla h|^2 - A |\nabla h|^2. \end{aligned} \tag{3.1}$$

Proof Direct calculation shows that

$$\Delta_V h = \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} + \epsilon u^{\epsilon-1} \Delta_V u = \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - ah^{\frac{\epsilon+p-1}{\epsilon}} \log h. \tag{3.2}$$

Therefore

$$\begin{aligned} \langle \nabla h, \nabla \Delta_V h \rangle &= \frac{\epsilon - 1}{\epsilon} \langle \nabla h, \nabla \frac{|\nabla h|^2}{h} \rangle - a \langle \nabla h, \nabla (h^{\frac{\epsilon+p-1}{\epsilon}} \log h) \rangle \\ &= \frac{\epsilon - 1}{\epsilon h} \langle \nabla h, \nabla |\nabla h|^2 \rangle - \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^4}{h^2} - \frac{a(\epsilon + p - 1)}{\epsilon} h^{\frac{p-1}{\epsilon}} |\nabla h|^2 \log h - ah^{\frac{p-1}{\epsilon}} |\nabla h|^2. \end{aligned} \tag{3.3}$$

By (2.1), (2.2), (3.2) and (3.3), we have

$$\begin{aligned} \frac{1}{2} \Delta_V |\nabla h|^2 &\geq \frac{1}{k} (\Delta_V h)^2 + \langle \nabla h, \nabla \Delta_V h \rangle + \text{Ric}_V^k(\nabla h, \nabla h) \\ &= \frac{1}{k} \left[\frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - ah^{\frac{\epsilon+p-1}{\epsilon}} \log h \right]^2 + \frac{\epsilon - 1}{\epsilon h} \langle \nabla h, \nabla |\nabla h|^2 \rangle - \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^4}{h^2} \\ &\quad - \frac{a(\epsilon + p - 1)}{\epsilon} h^{\frac{p-1}{\epsilon}} |\nabla h|^2 \log h - ah^{\frac{p-1}{\epsilon}} |\nabla h|^2 + \text{Ric}_V^k(\nabla h, \nabla h) \\ &= \left[\frac{(\epsilon - 1)^2}{k\epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right] \frac{|\nabla h|^4}{h^2} - \left[\frac{2(\epsilon - 1)}{k\epsilon} + \frac{\epsilon + p - 1}{\epsilon} \right] ah^{\frac{p-1}{\epsilon}} |\nabla h|^2 \log h \\ &\quad + \frac{a^2}{k} (h^{\frac{\epsilon+p-1}{\epsilon}} \log h)^2 + \frac{\epsilon - 1}{\epsilon h} \langle \nabla h, \nabla |\nabla h|^2 \rangle - ah^{\frac{p-1}{\epsilon}} |\nabla h|^2 + \text{Ric}_V^k(\nabla h, \nabla h). \end{aligned} \tag{3.4}$$

Since $\epsilon = 1 - \frac{kp}{k+2}$, we have $\frac{\epsilon-1}{\epsilon} = \frac{kp}{kp-k-2}$ and

$$\frac{2(\epsilon - 1)}{k\epsilon} + \frac{\epsilon + p - 1}{\epsilon} = 0, \quad \frac{(\epsilon - 1)^2}{k\epsilon^2} - \frac{\epsilon - 1}{\epsilon} = \frac{kp}{kp-k-2} \left(\frac{p}{kp-k-2} - 1 \right).$$

Therefore, by (3.4) and the condition $\text{Ric}_V^k(B_q(2R)) \geq -A$, we conclude that (3.1) is true. The proof of Theorem 3.1 is complete. \square

Theorem 3.2 Let (M^n, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_V^k(B_q(2R)) \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10) on $B_q(2R)$, $1 \leq p < \frac{k+2}{k-1}$ and $p \neq \frac{k+2}{k}$, then the following inequalities hold on $B_q(R)$:

(1) If $a > 0$, then

$$|\nabla u| \leq M \sqrt{\beta(M^{p-1}a + A) + \frac{C}{R^2}(1 + \sqrt{(k-1)A} \coth(\sqrt{\frac{A}{k-1}}R))}; \tag{3.5}$$

(2) If $a < 0$, then

$$|\nabla u| \leq M \sqrt{\beta \max\{0, m^{p-1}a + A\} + \frac{C}{R^2}(1 + \sqrt{(k-1)A} \coth(\sqrt{\frac{A}{k-1}}R))}, \tag{3.6}$$

where $M = \sup_{x \in B_q(2R)} u(x)$, $m = \inf_{x \in B_q(2R)} u(x)$, $\beta = \frac{2(k+2)^2}{kp(p+k+2-kp)}$ and C is a constant depending only on k and p .

Proof As in [1] and [6], we define a cut-off function $\psi \in C^2[0, +\infty)$ by

$$\psi(t) = \begin{cases} 1, & t \in [0, 1] \\ 0, & t \in [2, +\infty) \end{cases}$$

satisfying $\psi(t) \in [0, 1]$, and

$$\psi'(t) \leq 0, \quad \psi''(t) \geq -C_1, \quad \frac{|\psi'(t)|^2}{\psi(t)} \leq C_1.$$

Let

$$\varphi = \psi\left(\frac{r(x)}{R}\right), \quad G = \varphi|\nabla h|^2.$$

According to [1] and [6], we have

$$\frac{|\nabla \varphi|^2}{\varphi} \leq \frac{C_1}{R^2} \tag{3.7}$$

and

$$-\Delta_V \varphi \leq \frac{\sqrt{C_1(k-1)A} \coth(\sqrt{\frac{A}{k-1}}R)}{R^2} + \frac{C_1}{R^2}. \tag{3.8}$$

Assume that G achieves its maximum at the point $x_0 \in B_q(2R)$ and assume that $G(x_0) > 0$ (otherwise the proof is trivial). Then at the point x_0 , it holds that

$$\nabla G = 0, \quad \Delta_V G \leq 0, \quad \nabla|\nabla h|^2 = -\frac{|\nabla h|^2}{\varphi} \nabla \varphi.$$

Direct calculation shows that

$$\Delta_V G = \varphi \Delta_V |\nabla h|^2 + \frac{\Delta_V \varphi}{\varphi} G - \frac{2|\nabla \varphi|^2}{\varphi^2} G$$

$$\begin{aligned} &\geq 2\varphi\left[\frac{kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right)\frac{|\nabla h|^4}{h^2}+\frac{kp}{kp-k-2}\frac{\langle\nabla h,\nabla|\nabla h|^2\rangle}{h}\right. \\ &\quad \left.-ah^{\frac{(p-1)(2+k)}{2+k-kp}}|\nabla h|^2-A|\nabla h|^2\right]+\frac{\Delta_V\varphi}{\varphi}G-\frac{2|\nabla\varphi|^2}{\varphi^2}G \\ &=\frac{2kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right)\frac{G^2}{h^2\varphi}-\frac{2kp}{kp-k-2}\frac{G}{\varphi}\frac{\langle\nabla h,\nabla\varphi\rangle}{h} \\ &\quad -2ah^{\frac{(p-1)(2+k)}{2+k-kp}}G-2AG+\frac{\Delta_V\varphi}{\varphi}G-\frac{2|\nabla\varphi|^2}{\varphi^2}G. \end{aligned}$$

Therefore at the point x_0 , it holds that

$$\frac{2kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right)\frac{G}{h^2}\leq\frac{2kp}{kp-k-2}\frac{\langle\nabla h,\nabla\varphi\rangle}{h}+2ah^{\frac{(p-1)(2+k)}{2+k-kp}}\varphi+2A\varphi-\Delta_V\varphi+\frac{2|\nabla\varphi|^2}{\varphi}. \tag{3.9}$$

If $1\leq p<\frac{k+2}{k}$, then $\frac{2kp}{kp-k-2}<0$. Therefore, by the Cauchy inequality we have

$$\frac{2kp}{kp-k-2}\frac{\langle\nabla h,\nabla\varphi\rangle}{h}\leq-\frac{2kp}{kp-k-2}\left[\frac{kp-k-2}{2(kp-k-2-p)}\frac{|\nabla\varphi|^2}{\varphi}+\frac{kp-k-2-p}{2(kp-k-2)}\frac{G}{h^2}\right]. \tag{3.10}$$

According to (3.9) and (3.10), we get

$$\frac{kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right)\frac{G}{h^2}\leq\frac{-kp}{kp-k-2-p}\frac{|\nabla\varphi|^2}{\varphi}+2au^{p-1}\varphi+2A\varphi-\Delta_V\varphi+\frac{2|\nabla\varphi|^2}{\varphi}. \tag{3.11}$$

If $\frac{k+2}{k}<p<\frac{k+2}{k-1}$, then $\frac{p}{kp-k-2}>1$, $p+k+2-kp>0$ and

$$\frac{2kp}{kp-k-2}\frac{\langle\nabla h,\nabla\varphi\rangle}{h}\leq\frac{2kp}{kp-k-2}\left[\frac{kp-k-2}{2(p+k+2-kp)}\frac{|\nabla\varphi|^2}{\varphi}+\frac{p+k+2-kp}{2(kp-k-2)}\frac{G}{h^2}\right]. \tag{3.12}$$

According to (3.9) and (3.12), we get

$$\frac{kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right)\frac{G}{h^2}\leq\frac{kp}{p+k+2-kp}\frac{|\nabla\varphi|^2}{\varphi}+2au^{p-1}\varphi+2A\varphi-\Delta_V\varphi+\frac{2|\nabla\varphi|^2}{\varphi}. \tag{3.13}$$

If $a>0$, by (3.7), (3.8), (3.11) and (3.13) we conclude that there exists a constant C depending only on k and p such that

$$\frac{kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right)G(x)\leq h^2(x_0)[2M^{p-1}a+2A+\frac{C}{R^2}(1+\sqrt{(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R))] \tag{3.14}$$

holds for $x\in B_p(2R)$.

If $x\in B_p(R)$, then $\varphi=1$ and $G(x)=|\nabla h|^2(x)=\varepsilon^2u^{2\varepsilon-2}(x)|\nabla u|^2(x)$. Therefore, by (3.14) we get

$$|\nabla u|^2(x)\leq\frac{(k+2)^2}{kp(p+k+2-kp)}u^{2-2\varepsilon}(x)u^{2\varepsilon}(x_0)H\leq\frac{(k+2)^2}{kp(p+k+2-kp)}M^2H, \tag{3.15}$$

where

$$H = 2M^{p-1}a + 2A + \frac{C}{R^2}(1 + \sqrt{(k-1)A} \coth(\sqrt{\frac{A}{k-1}}R)). \tag{3.16}$$

By (3.15) and (3.16), we conclude that (3.5) is true. On the other hand, if $a < 0$ then $2am^{p-1}\varphi + 2A\varphi \leq 2\max\{0, am^{p-1} + A\}$. Similar to the case of $a > 0$, we can get (3.11) and (3.13). Therefore, we conclude that (3.6) is true. The proof of Theorem 3.2 is complete. \square

Theorem 3.3 *Let (M^n, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_V^k(B_q(2R)) \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10) on $B_q(2R)$, then the following inequalities hold on $B_q(R)$:*

(1) *If $a > 0$, $0 < p \leq 1$ and $\inf_{x \in B_q(2R)} u(x) \neq 0$, then*

$$|\nabla u| \leq M \sqrt{\beta(am^{p-1} + A) + \frac{C}{R^2}(1 + \sqrt{(k-1)A} \coth(\sqrt{\frac{A}{k-1}}R))}; \tag{3.17}$$

(2) *If $a < 0$ and $0 < p \leq 1$, then*

$$|\nabla u| \leq M \sqrt{\beta \max\{0, aM^{p-1} + A\} + \frac{C}{R^2}(1 + \sqrt{(k-1)A} \coth(\sqrt{\frac{A}{k-1}}R))}, \tag{3.18}$$

where $M = \sup_{x \in B_q(2R)} u(x)$, $m = \inf_{x \in B_q(2R)} u(x)$, $\beta = \frac{2(k+2)^2}{kp(p+k+2-kp)}$ and C is a constant depending only on k and p .

Proof As in the proof of Theorem 3.2, we can arrive at (3.9). If $0 < p \leq 1$, then $\frac{2kp}{kp-k-2} < 0$. Therefore, by the Cauchy inequality we can get (3.11). If $a > 0$, $0 < p \leq 1$ and $\inf_{x \in B_q(2R)} u(x) \neq 0$, then

$$ah^{\frac{(p-1)(2+k)}{2+k-kp}} \varphi = au^{p-1}\varphi \leq am^{p-1}. \tag{3.19}$$

According to (3.11), (3.19) and the methods in the proof of Theorem 3.2, we conclude that (3.17) is true. If $a < 0$ and $0 < p \leq 1$, then

$$ah^{\frac{(p-1)(2+k)}{2+k-kp}} \varphi + A\varphi = au^{p-1}\varphi + A\varphi \leq \max\{0, aM^{p-1} + A\}. \tag{3.20}$$

By (3.11), (3.20) and the methods in the proof of Theorem 3.2, we conclude that (3.18) is true. The proof of Theorem 3.3 is complete. \square

Letting $R \rightarrow +\infty$, we obtain the following global estimates on complete noncompact Riemannian manifolds:

Theorem 3.4 *Let (M^n, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_V^k \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10), $1 \leq p < \frac{k+2}{k-1}$ and $p \neq \frac{k+2}{k}$, then the following inequalities hold*

(1) *If $a > 0$, then*

$$|\nabla u| \leq M \sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)}(M^{p-1}a + A)};$$

(2) If $a < 0$, then

$$|\nabla u| \leq M \sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)} \max\{0, m^{p-1}a + A\}},$$

where $M = \sup_{x \in M^n} u(x)$, $m = \inf_{x \in M^n} u(x)$.

Theorem 3.5 Let (M^n, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_V^k \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10), then the following inequalities hold

(1) If $a > 0$, $0 < p \leq 1$ and $\inf_{x \in M^n} u(x) \neq 0$, then

$$|\nabla u| \leq M \sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)} (am^{p-1} + A)};$$

(2) If $a < 0$ and $0 < p \leq 1$, then

$$|\nabla u| \leq M \sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)} \max\{0, aM^{p-1} + A\}},$$

where $M = \sup_{x \in M^n} u(x)$, $m = \inf_{x \in M^n} u(x)$.

In particular, we have

Theorem 3.6 Let (M^n, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}_V^k \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a bounded nonconstant positive solution to (1.10), $M = \sup_{x \in M^n} u(x)$ and $m = \inf_{x \in M^n} u(x)$. If $a < 0$ and $0 < p \leq 1$, then $M^{1-p} > -\frac{a}{A}$. If $a < 0$, $1 \leq p < \frac{k+2}{k-1}$ and $p \neq \frac{k+2}{k}$, then $m^{p-1} < -\frac{A}{a}$.

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References

- [1] Chen Q, Qiu H. Gradient estimates and Harnack inequalities of a nonlinear parabolic equation for the V -Laplacian. *Annals of Global Analysis and Geometry* 2016; 50:47-64.
- [2] Chen L, Chen W. Gradient estimates for a nonlinear parabolic equation on complete non-compact Riemannian manifolds, *Annals of Global Analysis and Geometry* 2009; 35:397-404.
- [3] Dung HT, Dung NT. Sharp gradient estimates for a heat equation in Riemannian manifolds. *Proceedings of the American Mathematical Society* 2019; 147: 5329-5338.
- [4] Guo H, Ishida M. Harnack estimates for nonlinear backward heat equations in geometric flows. *Journal of Functional Analysis* 2014; 267: 2638-2662.

- [5] Hamilton R. A matrix Harnack estimates for the heat equation. *Communications in Analysis and Geometry* 1993; 1: 113-126.
- [6] Huang G, Li Z. Liouville type theorems of nonlinear elliptic equation for the V -Laplacian. *Analysis and Mathematical Physics* 2018; 8: 123-134.
- [7] Huang G, Ma B. Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. *Archiv der Mathematik* 2010; 94: 265-275.
- [8] Li P, Yau S. On the parabolic kernel of the Schrodinger operator. *Acta Mathematica* 1986; 156: 153-201.
- [9] Ma L. Gradient estimates for a simple elliptic equation on non-compact Riemannian manifolds. *Journal of Functional Analysis* 2006; 241: 374-382.
- [10] Wang W. Upper bounds of Hessian matrix and gradient estimates of positive solutions to the nonlinear parabolic equation along Ricci flow. *Nonlinear Analysis* 2022; 214: 1-43.
- [11] Wu J. Elliptic gradient estimates for a weighted heat equation and applications. *Mathematische Zeitschrift* 2015; 280: 451-468.
- [12] Yang Y. Gradient estimates for a nonlinear parabolic equation on Riemannian manifold. *Proceedings of the American Mathematical Society* 2008; 136: 4095-4102.
- [13] Zhao G. Gradient estimates of a nonlinear elliptic equation for the V -Laplacian. *Archiv der Mathematik* 2020; 114: 457-469.