

Oscillation criteria for third-order neutral differential equations with unbounded neutral coefficients and distributed deviating arguments

Yibing SUN^{1,*}, Yige ZHAO¹, Qiangqiang XIE²

¹School of Mathematical Sciences, University of Jinan, Jinan, China

²Department of Natural Resources, Inspur Software Technology Co., Ltd., Jinan, China

Received: 15.01.2022

Accepted/Published Online: 23.02.2022

Final Version: 11.03.2022

Abstract: This paper focuses on the oscillation criteria for the third-order neutral differential equations with unbounded neutral coefficients and distributed deviating arguments. Using comparison principles, new sufficient conditions improve some known existing results substantially due to less constraints on the considered equation. At last, two examples are established to illustrate the given theorems.

Key words: Oscillation, third-order, neutral, unbounded neutral coefficients, distributed deviating arguments

1. Introduction

Differential equations have played important roles in several practical applications, such as population growth, economics, engineering and neural networks. The study of oscillation criteria for first-, second-, third-, and high-order differential equations are significant for practical reasons and the development of qualitative theory; see [3, 9, 25, 37]. The problem of establishing sufficient conditions for the oscillatory and asymptotic behavior of third-order neutral functional differential equations and dynamic equations on time scales has attracted much research attention during the last few decades, and we refer the reader to the papers [1, 2, 4–8, 12, 14–16, 20, 22, 24, 26, 27, 32, 35, 36]. Chatzarakis et al. [14] considered the following differential equation

$$z'''(t) + q(t)x^\lambda(g(t)) = 0, \quad (1.1)$$

and explored some new oscillation criteria for solutions of (1.1), where $z(t) = x(t) + p(t)x(\eta(t))$, and $p(t) \geq 1$ is the unbounded neutral coefficient. The obtained results essentially improved the related results in the literature.

Recently, much research activity has been focused on the oscillation of various of classes of third-order neutral differential and dynamic equations with distributed deviating arguments; see for example [10, 11, 13, 17, 23]. In what follows, the following differential equation

$$(b(t)(y''(t))^\alpha)' + \int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi = 0, \quad (1.2)$$

and its special cases have been well studied, where $y(t) = x(t) + p(t)x(\tau(t))$ or $y(t) = x(t) + \int_a^b p(t, \xi)x(\tau(t, \xi))d\xi$. Sun and Zhao [31], Tian et al. [33], Xiang [38] and Zhang et al. [39] studied the oscillation of (1.2) under the

*Correspondence: sun_yibing@126.com

2010 AMS Mathematics Subject Classification: 34K11

assumption $0 \leq p(t) \leq P < 1$ or $0 \leq \int_a^b p(t, \xi) d\xi \leq P < 1$, while Jiang and Li [18] investigated the oscillatory and asymptotic behavior of (1.2) by assuming that $0 \leq p(t) \leq p_0 < \infty$. We can see that the neutral coefficient $p(t)$ in such papers are bounded. To the best of our knowledge, there are few results dealing with the oscillation of third-order differential equations with distributed deviating arguments and unbounded neutral coefficients. In this case, Sun and Zhao [28–30] and Tunç [34] gave some oscillation criteria for third-order differential equations with distributed deviating arguments and assumed $p(t) \geq 1$. On the other hand, some additional conditions are also needed in certain papers, such as Jiang et al. [17], Sun and Zhao [28] and Tian et al. [33] needed $\alpha \geq 1$, which is a ratio of odd positive integers.

In this paper, we will consider (1.2), where $y(t) = x(t) + m(t)x(\tau(t))$, $t \geq t_0 > 0$, $c < d$, and α is the ratio of odd positive integers. In this paper, the following conditions are assumed to hold:

$$(D1) \quad b(t), m(t) \in C([t_0, \infty), \mathbb{R}), b(t) > 0, \int_{t_0}^{\infty} b^{-\frac{1}{\alpha}}(t) dt = \infty, m(t) \geq 1 \text{ and } m(t) \neq 1;$$

$$(D2) \quad q(t, \xi) \in C([t_0, \infty) \times [c, d], [0, \infty)) \text{ and } q(t, \xi) \neq 0 \text{ for large } t;$$

$$(D3) \quad \tau(t) \in C([t_0, \infty), \mathbb{R}) \leq t \text{ is a strictly increasing function, } \lim_{t \rightarrow \infty} \tau(t) = \infty, \text{ and } \sigma(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R}) \text{ is nonincreasing for } \xi \text{ satisfying } \liminf_{t \rightarrow \infty} \sigma(t, \xi) = \infty \text{ for } \xi \in [c, d];$$

$$(D4) \quad f(x) \in C(\mathbb{R}, \mathbb{R}) \text{ and there exists } K > 0 \text{ such that } f(x)/x^\alpha \geq K \text{ for any variable } x \neq 0.$$

A function $x(t)$ is a solution of (1.2), if it satisfies (1.2) on the interval $[T_x, \infty)$ and has the properties $y(t)$, $y'(t)$ and $b(t)(y''(t))^\alpha \in C^1([T_x, \infty), \mathbb{R})$, respectively. We consider only those solutions to (1.2) here, which satisfy the condition $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq T_x$ and assume that (1.2) possesses such solutions. A solution of (1.2) is said to be oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise it is called nonoscillatory. Equation (1.2) is said to be oscillatory if all its solutions are oscillatory.

The objective of this paper is to obtain some new oscillation criteria for solutions of (1.2) by using comparison principles. Moreover, our proposed results improve a number of related results in the literature due to less restrictive assumptions on the coefficients of (1.2), and can easily be extended to more general third-order differential equations with unbounded neutral coefficients and distributed deviating arguments. In what follows, all functional inequalities are assumed to hold eventually.

2. Some lemmas

We begin this section with some lemmas, which are essential in establishing our main results. For simplicity, the following notations are used:

$$\begin{aligned} \bar{\sigma}(t) &= \sigma(t, c), \quad \tilde{\sigma}(t) = \sigma(t, d), \\ \delta_1(t, t_1) &= \int_{t_1}^t b^{-\frac{1}{\alpha}}(s) ds \quad \text{for } t \geq t_1 > t_0, \\ \delta_2(t, t_2) &= \int_{t_2}^t \delta_1(s, t_1) ds \quad \text{for } t \geq t_2 > t_1. \end{aligned}$$

Furthermore, we assume that

$$m_1(t) = \frac{1}{m(\tau^{-1}(t))} \left(1 - \frac{1}{m(\tau^{-1}(\tau^{-1}(t)))} \right) \geq 0, \tag{2.1}$$

$$m_2(t) = \frac{1}{m(\tau^{-1}(t))} \left(1 - \frac{\delta_2(\tau^{-1}(\tau^{-1}(t)), t_2)}{m(\tau^{-1}(\tau^{-1}(t)))\delta_2(\tau^{-1}(t), t_2)} \right) \geq 0, \tag{2.2}$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. Let

$$Q_1(t) = K \int_c^d q(t, \xi) m_1^\alpha(\sigma(t, \xi)) d\xi,$$

$$Q_2(t) = K \int_c^d q(t, \xi) m_2^\alpha(\sigma(t, \xi)) d\xi.$$

Lemma 2.1 *Assume that conditions (D1)-(D4) hold, and $x(t)$ is an eventually positive solution of (1.2). Then for sufficiently large t , $y(t)$ satisfies either*

(I) $y(t) > 0, y'(t) > 0, y''(t) > 0$ and $(b(t)(y''(t))^\alpha)' \leq 0;$

or

(II) $y(t) > 0, y'(t) < 0, y''(t) > 0$ and $(b(t)(y''(t))^\alpha)' \leq 0.$

The proof of this lemma is similar to that of some existing papers, such as Elabbasy and Moaaz ([10], Lemma 2.1), Sun and Zhao ([29], Lemma 2.1), and Tian et al. ([33], Lemma 2.1). So it is omitted.

Lemma 2.2 *Assume that conditions (D1)-(D4) and (2.2) hold, and $x(t)$ is an eventually positive solution of (1.2) with $y(t)$ satisfying case (I) of Lemma 2.1. Then for sufficiently large t , we get*

$$(b(t)(y''(t))^\alpha)' + Q_2(t)y^\alpha(\tau^{-1}(\tilde{\sigma}(t))) \leq 0. \tag{2.3}$$

Proof Since $x(t)$ is an eventually positive solution of (1.2), there exists $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(\sigma(t, \xi)) > 0, \xi \in [c, d]$ for $t \geq t_1$. It is easy to yield $y(t) > x(t) > 0$ eventually, and

$$y(\tau^{-1}(t)) = x(\tau^{-1}(t)) + m(\tau^{-1}(t))x(t), \tag{2.4}$$

based on the definition of $y(t)$. Following from (2.4), we have

$$\begin{aligned} x(t) &= \frac{1}{m(\tau^{-1}(t))} \left(y(\tau^{-1}(t)) - x(\tau^{-1}(t)) \right) \\ &= \frac{y(\tau^{-1}(t))}{m(\tau^{-1}(t))} - \frac{y(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))m(\tau^{-1}(\tau^{-1}(t)))} \\ &\geq \frac{y(\tau^{-1}(t))}{m(\tau^{-1}(t))} - \frac{y(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))m(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{2.5}$$

From (1.2), (D2) and (D4), we get

$$(b(t)(y''(t))^\alpha)' \leq 0,$$

and $b(t)(y''(t))^\alpha$ is nonincreasing for $t \geq t_1$, from which we have

$$y'(t) = y'(t_1) + \int_{t_1}^t \frac{(b(s)(y''(s))^\alpha)^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}}(s)} ds \geq b^{\frac{1}{\alpha}}(t)y''(t)\delta_1(t, t_1), \tag{2.6}$$

since $y(t)$ satisfies the case (I) of Lemma 2.1. We can see from (2.6) that

$$\left(\frac{y'(t)}{\delta_1(t, t_1)}\right)' \leq 0 \text{ for } t \geq t_2 > t_1,$$

which yields that $y'(t)/\delta_1(t, t_1)$ is nonincreasing for $t \geq t_2$. Furthermore, we obtain

$$\begin{aligned} y(t) &= y(t_2) + \int_{t_2}^t \frac{y'(s)}{\delta_1(s, t_1)} \delta_1(s, t_1) ds \\ &\geq \frac{\delta_2(t, t_2)}{\delta_1(t, t_1)} y'(t) \text{ for } t \geq t_2, \end{aligned} \tag{2.7}$$

which leads to

$$\left(\frac{y(t)}{\delta_2(t, t_2)}\right)' \leq 0 \text{ for } t \geq t_3 > t_2.$$

Then we have

$$y(\tau^{-1}(\tau^{-1}(t))) \leq \frac{\delta_2(\tau^{-1}(\tau^{-1}(t)), t_2)y(\tau^{-1}(t))}{\delta_2(\tau^{-1}(t), t_2)} \tag{2.8}$$

for $t \geq t_3$, in views of the fact that $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$. Substituting (2.8) into (2.5), we get

$$x(t) \geq m_2(t)y(\tau^{-1}(t)),$$

which yields that

$$\begin{aligned} x(\sigma(t, \xi)) &\geq m_2(\sigma(t, \xi))y(\tau^{-1}(\sigma(t, \xi))) \\ &\geq m_2(\sigma(t, \xi))y(\tau^{-1}(\tilde{\sigma}(t))). \end{aligned} \tag{2.9}$$

Combining (1.2) and (2.9), we arrive at (2.3), due to the monotonicity of $y(t)$ and $\tau(t)$. □

3. Main results

Theorem 3.1 *Assume that conditions (D1)-(D4) hold. Furthermore, assume that there exists a function $g(t) \in C([t_0, \infty), \mathbb{R})$ such that $\sigma(t, \xi) \leq g(t) < \tau(t)$ for $t \geq t_0$. If the two first-order delay differential equations*

$$z'(t) + Q_2(t)\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_2)z(\tau^{-1}(\tilde{\sigma}(t))) = 0 \tag{3.1}$$

and

$$\omega'(t) + Q_1(t)G^\alpha(t)\omega(\tau^{-1}(g(t))) = 0 \tag{3.2}$$

oscillate for large $t \geq t_2 > t_1 \geq t_0$, where

$$G(t) = \int_{\tau^{-1}(\tilde{\sigma}(t))}^{\tau^{-1}(g(t))} \delta_1(\tau^{-1}(g(t)), l) dl,$$

then (1.2) is oscillatory.

Proof Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.2). In general, we assume that $x(t)$ is eventually positive, since $-x(t)$ is also a solution of (1.2), and the proof is similar when $x(t) < 0$ eventually. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t, \xi)) > 0$, $\xi \in [c, d]$ for $t \geq t_1$. $y(t)$ satisfies two possible cases (I) and (II) following from Lemma 2.1.

Firstly, if case (I) holds, then we obtain (2.3), (2.5) and (2.6), proceeding as in the proof of Lemma 2.2. Integrating (2.6) from $t_2 (t_2 > t_1)$ to t , we get

$$y(t) \geq b^{\frac{1}{\alpha}}(t)y''(t)\delta_2(t, t_2). \tag{3.3}$$

Substituting (3.3) into (2.3), we have

$$(b(t)(y''(t))^\alpha)' + Q_2(t)b(\tau^{-1}(\tilde{\sigma}(t)))(y''(\tau^{-1}(\tilde{\sigma}(t))))^\alpha \delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_2) \leq 0$$

with $\tau^{-1}(\tilde{\sigma}(t)) \geq t_2$. Letting $z(t) = b(t)(y''(t))^\alpha$, we can see that $z(t)$ is a positive solution of the following first-order delay differential inequality

$$z'(t) + Q_2(t)\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_2)z(\tau^{-1}(\tilde{\sigma}(t))) \leq 0.$$

We conclude from [21] that (3.1) also has a positive solution, which contradicts the fact that (3.1) oscillates.

Secondly, we assume that case (II) holds. Since $y'(t) < 0$ and $\tau(t) \leq t$, we have

$$y(\tau^{-1}(t)) \geq y(\tau^{-1}(\tau^{-1}(t))). \tag{3.4}$$

Combining (2.5) and (3.4), we get

$$x(t) \geq m_1(t)y(\tau^{-1}(t)),$$

and

$$x(\sigma(t, \xi)) \geq m_1(\sigma(t, \xi))y(\tau^{-1}(\sigma(t, \xi))). \tag{3.5}$$

Substitute (3.5) into (1.2) and use (D4) to obtain

$$(b(t)(y''(t))^\alpha)' + K \int_c^d q(t, \xi)m_1^\alpha(\sigma(t, \xi))y^\alpha(\tau^{-1}(\sigma(t, \xi)))d\xi \leq 0,$$

which indicates that

$$(b(t)(y''(t))^\alpha)' + Q_1(t)y^\alpha(\tau^{-1}(\tilde{\sigma}(t))) \leq 0. \tag{3.6}$$

Since $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$ and $(b(t)(y''(t))^\alpha)' \leq 0$, one can get for $t_2 \leq u \leq v$,

$$y'(v) - y'(u) = \int_u^v \frac{(b(s)(y''(s))^\alpha)^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}}(s)} ds,$$

which yields that

$$-y'(u) \geq b^{\frac{1}{\alpha}}(v)y''(v)\delta_1(v, u).$$

Integrating the above inequality from u to v , we obtain

$$\begin{aligned} y(u) &\geq y(v) + b^{\frac{1}{\alpha}}(v)y''(v) \int_u^v \delta_1(v, l) dl \\ &\geq b^{\frac{1}{\alpha}}(v)y''(v) \int_u^v \delta_1(v, l) dl. \end{aligned} \tag{3.7}$$

From the fact that $\bar{\sigma}(t) \leq g(t)$ and $\tau(t)$ is strictly increasing, we get $\tau^{-1}(\bar{\sigma}(t)) \leq \tau^{-1}(g(t))$. Substituting $u = \tau^{-1}(\bar{\sigma}(t))$ and $v = \tau^{-1}(g(t))$ into (3.7), we have

$$y(\tau^{-1}(\bar{\sigma}(t))) \geq G(t)b^{\frac{1}{\alpha}}(\tau^{-1}(g(t)))y''(\tau^{-1}(g(t))). \tag{3.8}$$

Using (3.6) and (3.8), one can see that

$$(b(t)(y''(t))^\alpha)' + Q_1(t)G^\alpha(t)b(\tau^{-1}(g(t)))(y''(\tau^{-1}(g(t))))^\alpha \leq 0.$$

Letting $\omega(t) = b(t)(y''(t))^\alpha$, we deduce that $\omega(t)$ is a positive solution of the following differential inequality

$$\omega'(t) + Q_1(t)G^\alpha(t)\omega(\tau^{-1}(g(t))) \leq 0.$$

The rest of the proof is similar to that of the former case, and so we omit it here. This completes the proof. \square

Next, we introduce a well-known result. Consider the first-order delay differential equation

$$x'(t) + R(t)x(\tau(t)) = 0, \tag{3.9}$$

under the conditions $R(t) \in C([t_0, \infty), [0, \infty))$, $\tau(t) \in C([t_0, \infty), \mathbb{R})$, $\tau(t) < t$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$. From [19], if

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t R(s) ds > \frac{1}{e},$$

then (3.9) is oscillatory. Hence, combine Theorem 3.1 and this result to derive the following result.

Corollary 3.2 *Assume that conditions (D1)-(D4) hold. Furthermore, assume that there exists a function $g(t) \in C([t_0, \infty), \mathbb{R})$ such that $\sigma(t, \xi) \leq g(t) < \tau(t)$ for $t \geq t_0$. If for large $t \geq t_2 > t_1 \geq t_0$,*

$$\liminf_{t \rightarrow \infty} \int_{\tau^{-1}(\bar{\sigma}(t))}^t Q_2(s)\delta_2^\alpha(\tau^{-1}(\bar{\sigma}(s)), t_2) ds > \frac{1}{e},$$

and

$$\liminf_{t \rightarrow \infty} \int_{\tau^{-1}(g(t))}^t Q_1(s)G^\alpha(s) ds > \frac{1}{e}, \tag{3.10}$$

where $G(t)$ is defined as in Theorem 3.1, then (1.2) is oscillatory.

Directly from Theorem 3.1, we get an oscillatory criterion as below.

Theorem 3.3 Assume that conditions (D1)-(D4) hold, and $\sigma(t, \xi) < \tau(t)$ for $t \geq t_0$. If the two first-order delay differential equations

$$z'(t) + Q_2(t)\delta_2^\alpha(\tilde{\sigma}(t), t_2)z(\tilde{\sigma}(t)) = 0 \tag{3.11}$$

and

$$\omega'(t) + Q_1(t)\bar{G}^\alpha(t)\omega(\tau^{-1}(\bar{\sigma}(t))) = 0 \tag{3.12}$$

oscillate for large $t \geq t_2 > t_1 \geq t_0$, where

$$\bar{G}(t) = \int_{\tau^{-1}(\bar{\sigma}(t))}^{\infty} \delta_1(\tau^{-1}(\bar{\sigma}(t)), s)ds,$$

then (1.2) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of (1.2). We may assume that $x(t)$ is eventually positive. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t, \xi)) > 0$, $\xi \in [c, d]$ for $t \geq t_1$. Lemma 2.1 yields that $y(t)$ satisfies either case (I) or case (II).

Firstly, we assume that case (I) satisfies. Proceeding as in the proof of Theorem 3.1, we get (3.3). Since $\tau(t) \leq t$, we can see that $t \leq \tau^{-1}(t)$ and $\tilde{\sigma}(t) \leq \tau^{-1}(\tilde{\sigma}(t))$. Then we obtain $y(\tilde{\sigma}(t)) \leq y(\tau^{-1}(\tilde{\sigma}(t)))$, due to $y'(t) > 0$. Based on (2.3) and this inequality, we have

$$(b(t)(y''(t))^\alpha)' + Q_2(t)y^\alpha(\tilde{\sigma}(t)) \leq 0. \tag{3.13}$$

Substituting (3.3) into (3.13) yields that

$$(b(t)(y''(t))^\alpha)' + Q_2(t)\delta_2^\alpha(\tilde{\sigma}(t), t_2)b(\tilde{\sigma}(t))(y''(\tilde{\sigma}(t)))^\alpha \leq 0$$

for $\tilde{\sigma}(t) \geq t_2 > t_1$. Letting $z(t) = b(t)(y''(t))^\alpha$, we conclude that $z(t)$ is a positive solution of

$$z'(t) + Q_2(t)\delta_2^\alpha(\tilde{\sigma}(t), t_2)z(\tilde{\sigma}(t)) \leq 0,$$

which is a contradiction to that (4.1) oscillates.

Secondly, we assume that case (II) holds. Proceeding as in the proof of Theorem 3.1, we have (3.6). Since $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$ and $(b(t)(y''(t))^\alpha)' \leq 0$, we get for $t_2 \leq s \leq t$,

$$y'(t) - y'(s) = \int_s^t \frac{(b(u)(y''(u))^\alpha)^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}}(u)} du,$$

which means that

$$-y'(s) \geq b^{\frac{1}{\alpha}}(t)y''(t)\delta_1(t, s). \tag{3.14}$$

Integrating (3.14) from t to v ($v > t$) and letting $v \rightarrow \infty$, we get

$$y(t) \geq b^{\frac{1}{\alpha}}(t)y''(t) \int_t^\infty \delta_1(t, s)ds.$$

and then

$$y(\tau^{-1}(\bar{\sigma}(t))) \geq \bar{G}(t)b^{\frac{1}{\alpha}}(\tau^{-1}(\bar{\sigma}(t)))y''(\tau^{-1}(\bar{\sigma}(t))). \tag{3.15}$$

Substitute (3.15) into (3.6) to obtain

$$(b(t)(y''(t))^\alpha)' + Q_1(t)\bar{G}^\alpha(t)b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^\alpha \leq 0.$$

Similarly, letting $\omega(t) = b(t)(y''(t))^\alpha$, we get a contradiction to the fact that (4.2) oscillates. This completes the proof. \square

Remark 3.4 *The result of case (I) in Theorem 3.3 is directly obtained from Theorem 3.1. Hence, the corresponding results are similar. The main difference between Theorems 3.1 and 3.3 is the result about case (II). Especially, (4.2) does not need the reference function $g(t)$.*

Analogously, based on Theorem 3.3 and the well-known result in [19], we can get the following result.

Corollary 3.5 *Assume that conditions (D1)-(D4) hold, and $\sigma(t, \xi) < \tau(t)$ for $t \geq t_0$. If for large $t \geq t_2 > t_1 \geq t_0$,*

$$\liminf_{t \rightarrow \infty} \int_{\bar{\sigma}(t)}^t Q_2(s) \delta_2^\alpha(\bar{\sigma}(s), t_2) ds > \frac{1}{e},$$

and

$$\liminf_{t \rightarrow \infty} \int_{\tau^{-1}(\bar{\sigma}(t))}^t Q_1(s) \bar{G}^\alpha(s) ds > \frac{1}{e},$$

where $\bar{G}(t)$ is defined as in Theorem 3.3, then (1.2) is oscillatory.

In the following interesting result, we need to assume that the function $\sigma(t, \xi)$ is nondecreasing for t .

Theorem 3.6 *Assume that conditions (D1)-(D4) hold. Furthermore, assume that $\sigma(t, \xi)$ is nondecreasing for t with $\sigma(t, \xi) < \tau(t)$, $t \geq t_0$. If*

$$\limsup_{t \rightarrow \infty} \delta_1^\alpha(\tau^{-1}(\bar{\sigma}(t)), t_1) \int_t^\infty Q_2(s) \frac{\delta_2^\alpha(\tau^{-1}(\bar{\sigma}(s)), t_2)}{\delta_1^\alpha(\tau^{-1}(\bar{\sigma}(s)), t_1)} ds > 1, \tag{3.16}$$

and

$$\limsup_{t \rightarrow \infty} \int_{\tau^{-1}(\bar{\sigma}(t))}^t Q_1(s) H^\alpha(t, s) ds > 1 \tag{3.17}$$

for large $t \geq t_2 > t_1 \geq t_0$, where

$$H(t, s) = \int_{\tau^{-1}(\bar{\sigma}(s))}^{\tau^{-1}(\bar{\sigma}(t))} \delta_1(\tau^{-1}(\bar{\sigma}(t)), l) dl,$$

then (1.2) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of (1.2). We may assume that $x(t)$ is eventually positive. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t, \xi)) > 0$, $\xi \in [c, d]$ for $t \geq t_1$. Lemma 2.1 yields that $y(t)$ satisfies either case (I) or case (II).

Firstly, we assume that case (I) satisfies. Proceeding as in the proof of Lemma 2.2, we have (2.7), which means that

$$y(\tau^{-1}(\tilde{\sigma}(t))) \geq \frac{\delta_2(\tau^{-1}(\tilde{\sigma}(t)), t_2)}{\delta_1(\tau^{-1}(\tilde{\sigma}(t)), t_1)} y'(\tau^{-1}(\tilde{\sigma}(t))) \tag{3.18}$$

for $t \geq t_2 > t_1$. Substitute (3.18) into (2.3) to get

$$(b(t)(y''(t))^\alpha)' + Q_2(t) \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_1)} (y'(\tau^{-1}(\tilde{\sigma}(t))))^\alpha \leq 0. \tag{3.19}$$

Letting $z(t) = y'(t)$, we have

$$z(t) > 0, \quad z'(t) > 0, \quad (b(t)(z'(t))^\alpha)' \leq 0,$$

and (3.19) can be written as

$$(b(t)(z'(t))^\alpha)' + Q_2(t) \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_1)} z^\alpha(\tau^{-1}(\tilde{\sigma}(t))) \leq 0. \tag{3.20}$$

Integrating (3.20) from t to u ($u > t$) and letting $u \rightarrow \infty$, we obtain

$$\begin{aligned} b(t)(z'(t))^\alpha &\geq \int_t^\infty Q_2(s) \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_1)} z^\alpha(\tau^{-1}(\tilde{\sigma}(s))) ds \\ &\geq z^\alpha(\tau^{-1}(\tilde{\sigma}(t))) \int_t^\infty Q_2(s) \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_1)} ds. \end{aligned} \tag{3.21}$$

(2.6) can be rewritten as

$$z(t) \geq b^{\frac{1}{\alpha}}(t) z'(t) \delta_1(t, t_1). \tag{3.22}$$

Substitute (3.22) into (3.21) to get

$$\begin{aligned} b(t)(z'(t))^\alpha &\geq b(\tau^{-1}(\tilde{\sigma}(t))) (z'(\tau^{-1}(\tilde{\sigma}(t))))^\alpha \delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_1) \\ &\quad \times \int_t^\infty Q_2(s) \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_1)} ds. \end{aligned} \tag{3.23}$$

Since $(b(t)(z'(t))^\alpha)' \leq 0$ and $\tilde{\sigma}(t) < \tau(t)$, one can see that

$$b(t)(z'(t))^\alpha \leq b(\tau^{-1}(\tilde{\sigma}(t))) (z'(\tau^{-1}(\tilde{\sigma}(t))))^\alpha. \tag{3.24}$$

Hence, (3.23) and (3.24) yield that

$$\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_1) \int_t^\infty Q_2(s) \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_1)} ds \leq 1,$$

which contradicts (3.16).

Secondly, we assume that case (II) holds. Proceeding as in the proof of Theorem 3.1, we have (3.6) and (3.7). For $t_2 \leq s \leq t$, one can see that $\tau^{-1}(\tilde{\sigma}(t)) \geq \tau^{-1}(\tilde{\sigma}(s))$ due to the monotonicity of $\tilde{\sigma}(t)$ and $\tau^{-1}(t)$. Substitute $u = \tau^{-1}(\tilde{\sigma}(s))$ and $v = \tau^{-1}(\tilde{\sigma}(t))$ into (3.7) to obtain

$$y(\tau^{-1}(\tilde{\sigma}(s))) \geq H(t, s) b^{\frac{1}{\alpha}}(\tau^{-1}(\tilde{\sigma}(t))) y''(\tau^{-1}(\tilde{\sigma}(t))). \tag{3.25}$$

Integrating (3.6) from $\tau^{-1}(\bar{\sigma}(t))$ to t , we get

$$b(t)(y''(t))^\alpha - b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^\alpha + \int_{\tau^{-1}(\bar{\sigma}(t))}^t Q_1(s)y^\alpha(\tau^{-1}(\bar{\sigma}(s)))ds \leq 0.$$

Combining (3.25) and the above inequality, we conclude that

$$b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^\alpha \geq b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^\alpha \times \int_{\tau^{-1}(\bar{\sigma}(t))}^t Q_1(s)H^\alpha(t, s)ds,$$

and then

$$\int_{\tau^{-1}(\bar{\sigma}(t))}^t Q_1(s)H^\alpha(t, s)ds \leq 1.$$

Take limsup as $t \rightarrow \infty$ to yield a contradiction to (3.17). This completes the proof. □

4. Examples

Example 4.1. Consider the following equation

$$((y''(t))^{\frac{1}{3}})' + \int_{\frac{1}{2}}^1 8q_0\xi t^{\frac{1}{3}}x^{\frac{1}{3}}\left(\frac{t}{2} - \frac{\xi}{2}\right)d\xi = 0, \quad t > 8, \tag{4.1}$$

where

$$y(t) = x(t) + \frac{9t + 8}{t + 1}x\left(\frac{t}{2}\right), \quad m(t) = \frac{9t + 8}{t + 1}, \quad q(t, \xi) = 8q_0\xi t^{\frac{1}{3}},$$

$b(t) = 1$, $\alpha = 1/3$, $\tau(t) = t/2$, $f(x) = x^{1/3}$, $\sigma(t, \xi) = t/2 - \xi/2$, $c = 1/2$, $d = 1$ and $t_0 = 1$. We note that conditions (D1)-(D4) are satisfied. Choose $t_1 = 2$ and $t_2 = 4$. Then we get $8 \leq m(t) < 9$,

$$\bar{\sigma}(t) = \sigma(t, 1/2) = \frac{t}{2} - \frac{1}{4}, \quad \bar{\sigma}(t) = \sigma(t, 1) = \frac{t}{2} - \frac{1}{2},$$

$$\delta_1(t, t_1) = \int_2^t ds = t - 2,$$

$$\delta_2(t, t_2) = \int_4^t (s - 2)ds = \frac{1}{2}t^2 - 2t.$$

Furthermore, we deduce that

$$m_1(t) > \frac{1}{9}\left(1 - \frac{1}{8}\right) = \frac{7}{72} > 0,$$

$$m_2(t) > \frac{1}{9}\left(1 - \frac{1}{8} \cdot \frac{\frac{1}{2}(4t)^2 - 2 \cdot 4t}{\frac{1}{2}(2t)^2 - 2 \cdot 2t}\right) > \frac{1}{24} > 0,$$

$$Q_1(t) > \int_{\frac{1}{2}}^1 \left(\frac{7}{72}\right)^{\frac{1}{3}} 8q_0\xi t^{\frac{1}{3}}d\xi = 3q_0\left(\frac{7t}{72}\right)^{\frac{1}{3}},$$

$$Q_2(t) > \int_{\frac{1}{2}}^1 \left(\frac{1}{24}\right)^{\frac{1}{3}} 8q_0\xi t^{\frac{1}{3}}d\xi = 3q_0\left(\frac{t}{24}\right)^{\frac{1}{3}}.$$

Letting $g(t) = t/2 - 1/6$, we get

$$\begin{aligned} & \int_{\tau^{-1}(\tilde{\sigma}(t))}^t Q_2(s) \delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_2) ds \\ & > \int_{t-1}^t 3q_0 \left(\frac{s}{24}\right)^{\frac{1}{3}} \left(\frac{1}{2}(s-1)^2 - 2(s-1)\right)^{\frac{1}{3}} ds > 3q_0 \left(\frac{3}{2}\right)^{\frac{1}{3}} > \frac{1}{e}, \end{aligned}$$

if $q_0 > 2^{1/3}/(3^{4/3}e)$, and

$$\begin{aligned} & \int_{\tau^{-1}(g(t))}^t Q_1(s) G^\alpha(s) ds \\ & > \int_{t-\frac{1}{3}}^t 3q_0 \left(\frac{7s}{72}\right)^{\frac{1}{3}} \left(\int_{s-\frac{1}{2}}^{s-\frac{1}{3}} (s-\frac{1}{3}-l) dl\right)^{\frac{1}{3}} ds > q_0 \left(\frac{1}{12}\right)^{\frac{2}{3}} > \frac{1}{e}, \end{aligned}$$

if $q_0 > 12^{2/3}/e$. Hence, by Corollary 3.2, (4.1) is oscillatory with $q_0 > 12^{2/3}/e$.

Example 4.2. Consider the equation

$$((2t)^{-\frac{5}{3}}(y''(t))^{\frac{5}{3}})' + \int_1^3 \frac{q_0 \xi}{4t^{\frac{4}{3}}} x^{\frac{5}{3}} \left(\frac{t}{3} - \frac{4}{3} + \frac{1}{\xi}\right) d\xi = 0, \quad t > 9, \tag{4.2}$$

where $y(t) = x(t) + 30x(t/3)$, $m(t) = 30$, $b(t) = (2t)^{-5/3}$, $\alpha = 5/3$, $\tau(t) = t/3$, $q(t, \xi) = q_0 \xi / (4t^{4/3})$, $f(x) = x^{5/3}$, $\sigma(t, \xi) = t/3 - 4/3 + 1/\xi$, $c = 1$, $d = 3$ and $t_0 = 1$. Then conditions (D1)-(D4) also hold. Choosing $t_1 = 2$ and $t_2 = 3$, we obtain

$$\begin{aligned} \bar{\sigma}(t) &= \frac{t}{3} - \frac{1}{3}, \quad \tilde{\sigma}(t) = \frac{t}{3} - 1, \\ \delta_1(t, t_1) &= t^2 - 4, \quad \delta_2(t, t_2) = \frac{1}{3}t^3 - 4t + 3, \\ m_1(t) &= \frac{1}{30} \left(1 - \frac{1}{30}\right) = \frac{29}{900} > 0, \\ m_2(t) &= \frac{1}{30} \left(1 - \frac{1}{30} \cdot \frac{\frac{1}{3}(9t)^3 - 4 \cdot 9t + 3}{\frac{1}{3}(3t)^3 - 4 \cdot 3t + 3}\right) > \frac{1}{450} > 0, \\ Q_1(t) &> \int_1^3 \left(\frac{29}{900}\right)^{\frac{5}{3}} \frac{q_0 \xi}{4t^{\frac{4}{3}}} d\xi = \left(\frac{29}{900}\right)^{\frac{5}{3}} \frac{q_0}{t^{\frac{4}{3}}}, \\ Q_2(t) &> \int_1^3 \left(\frac{1}{450}\right)^{\frac{5}{3}} \frac{q_0 \xi}{4t^{\frac{4}{3}}} d\xi = \left(\frac{1}{450}\right)^{\frac{5}{3}} \frac{q_0}{t^{\frac{4}{3}}}. \end{aligned}$$

Letting

$$\rho(t) = \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_1)} = \left(\frac{\frac{1}{3}(t-3)^3 - 4(t-3) + 3}{(t-3)^2 - 4}\right)^{\frac{5}{3}},$$

it is easy to verify that $\rho(t)$ is increasing. Then $\rho(t) > (51/32)^{5/3}$ for $t > 9$. Furthermore, we see that

$$\begin{aligned} & \delta_1^\alpha(\tau^{-1}(\bar{\sigma}(t)), t_1) \int_t^\infty Q_2(s) \frac{\delta_2^\alpha(\tau^{-1}(\bar{\sigma}(s)), t_2)}{\delta_1^\alpha(\tau^{-1}(\bar{\sigma}(s)), t_1)} ds \\ & > ((t-3)^2 - 4)^{\frac{5}{3}} \int_t^\infty \left(\frac{1}{450}\right)^{\frac{5}{3}} \frac{q_0}{s^{\frac{4}{3}}} \cdot \left(\frac{51}{32}\right)^{\frac{5}{3}} ds \\ & > 3q_0 \left(\frac{1}{9 \cdot 32}\right)^{\frac{5}{3}} \frac{((t-3)^2 - 4)^{\frac{5}{3}}}{t^{\frac{1}{3}}} > 1, \end{aligned}$$

if $q_0 > 3^{7/3} \cdot 32^{5/3}$, and

$$\begin{aligned} & \int_{\tau^{-1}(\bar{\sigma}(t))}^t Q_1(s) H^\alpha(t, s) ds \\ & > \int_{t-1}^t \left(\frac{29}{900}\right)^{\frac{5}{3}} \frac{q_0}{s^{\frac{4}{3}}} \left(\int_{s-1}^{t-1} (t^2 - l^2) dl\right)^{\frac{5}{3}} ds \\ & > \left(\frac{29}{900}\right)^{\frac{5}{3}} q_0 \int_{t-1}^t s^{\frac{1}{3}} ds > \left(\frac{3}{100}\right)^{\frac{5}{3}} \frac{3q_0}{4} > 1, \end{aligned}$$

if $q_0 > 4 \cdot 100^{\frac{5}{3}} / 3^{\frac{8}{3}}$. Hence, by Theorem 3.6, (4.2) is oscillatory with $q_0 > 3^{7/3} \cdot 32^{5/3}$.

Acknowledgment

This research is supported by the National Natural Science Foundation of China (Grant No. 61803176, 61703180).

Conflict of interest

The authors declare that they have no competing interests.

References

- [1] Agarwal RP, Bohner M, Li TX, Zhang CH. Oscillation of third-order nonlinear delay differential equations. Taiwanese Journal of Mathematics 2013; 17 (2): 545-558.
- [2] Agarwal RP, Bohner M, Li TX, Zhang CH. A Philos-type theorem for third-order nonlinear retarded dynamic equations. Applied Mathematics and Computation 2014; 249: 527-531.
- [3] Baculíková B. Oscillation of second order half-linear differential equations with deviating arguments of mixed type. Applied Mathematics Letters 2021; 119: 107228.
- [4] Baculíková B, Džurina J. Oscillation and Proper B for third-order differential equations with advanced arguments. Electronic Journal of Differential Equations 2016; 244: 1-10.
- [5] Baculíková B, Džurina J. Oscillation of third-order neutral differential equations. Mathematical and Computer Modelling 2010; 52 (1-2): 215-226.
- [6] Bartušek M, Došlá Z. Oscillation of third order differential equation with damping term. Czechoslovak Mathematical Journal 2015; 65 (2): 301-316.
- [7] Bohner M, Grace SR, Sağır I, Tunç E. Oscillation of third-order nonlinear damped delay differential equations. Applied Mathematics and Computation 2016; 278: 21-32.

- [8] Chatzarakis GE, Grace SR, Jadlovská I, Li TX, Tunç E. Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients. *Complexity* 2019; 5691758 (2019): 1-7.
- [9] Dzurina J, Jadlovská I. Oscillation of n th order strongly noncanonical delay differential equations. *Applied Mathematics Letters* 2021; 115: 106940.
- [10] Elabbasy EM, Moaaz O. New oscillation results for class of third order neutral delay differential equations with distributed deviating arguments. *Global Journal of Science Frontier Research: F Mathematics and Decision Sciences* 2015; 15: 1-9.
- [11] Fu YL, Tian YZ, Jiang CM, Li TX. On the asymptotic properties of nonlinear third-order neutral delay differential equations with distributed deviating arguments. *Journal of Function Spaces* 2016; 2016: 1-5.
- [12] Grace SR. Oscillation criteria for third-order nonlinear delay differential equations with damping. *Opuscula Mathematica* 2015; 35 (4): 485-497.
- [13] Grace SR, Graef JR, Tunç E. Oscillatory behavior of a third-order neutral dynamic equation with distributed delays. *Electronic Journal of Qualitative Theory of Differential Equations* 2016; 14: 1-14.
- [14] Grace SR, Graef JR, Tunç E. On the oscillation of certain third order nonlinear dynamic equations with a nonlinear damping term. *Mathematica Slovaca* 2017; 67 (2): 501-508.
- [15] Grace SR, Jadlovská I, Tunç E. Oscillatory and asymptotic behavior of third-order nonlinear differential equations with a superlinear neutral term. *Turkish Journal of Mathematics* 2020; 44: 1317-1329.
- [16] Hassan TS, Agarwal RP, Mohammed WW. Oscillation criteria for third-order functional half-linear dynamic equations, *Advances in Difference Equations* 2017; 2017 (111): 1-28.
- [17] Jiang CM, Jiang Y, Li TX. Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments. *Advances in Difference Equations* 2016; 2016: 1-14.
- [18] Jiang CM, Li TX. Oscillation criteria for third-order nonlinear neutral differential equations with distributed deviating arguments. *Journal of Nonlinear Science and Applications* 2016; 9: 6170-6182.
- [19] Ladde GS, Lakshmikantham V, Zhang BG. *Oscillation Theory of Differential Equations with Deviating Arguments*. New York, NY, USA: Marcel Dekker, 1987.
- [20] Li TX, Rogovchenko YV. On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. *Applied Mathematics Letters* 2020; 105: 106293.
- [21] Philos ChG. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Archiv der Mathematik (Basel)* 1981; 36: 168-178.
- [22] Şenel MT, Utku N. Oscillation behavior of third-order nonlinear neutral dynamic equations on time scales with distributed deviating arguments. *Filomat* 2014; 28 (6): 1211-1223.
- [23] Şenel MT, Utku N. Oscillation of third-order quasilinear neutral dynamic equations on time scales with distributed deviating arguments. *Journal of Mathematics and Computer Science* 2017; 17: 41-52.
- [24] Sui Y, Han ZL. Oscillation of third-order nonlinear delay dynamic equation with damping term on time scales. *Journal of Applied Mathematics and Computing* 2018; 58: 577-599.
- [25] Sui Y, Yu HM. Oscillation of damped second order quasilinear wave equations with mixed arguments. *Applied Mathematics Letters* 2021; 117: 107060.
- [26] Sun YB, Han ZL, Sun Y, Pan YY. Oscillation theorems for certain third order nonlinear delay dynamic equations on time scales. *Electronic Journal of Qualitative Theory of Differential Equations* 2011; 75: 1-14.
- [27] Sun YB, Han ZL, Zhang YX. On the oscillation for third-order nonlinear neutral delay dynamic equations on time scales. *Journal of Applied Mathematics and Computing* 2017; 54 (1): 243-261.
- [28] Sun YB, Zhao YG. Oscillation and asymptotic behavior of third-order nonlinear neutral delay differential equations with distributed deviating arguments. *Journal of Applied Analysis and Computation* 2018; 8 (6): 1796-1810.

- [29] Sun YB, Zhao YG. Oscillatory behavior of third-order neutral delay differential equations with distributed deviating arguments. *Journal of Inequalities and Applications* 2019; 2019 (207): 1-16.
- [30] Sun YB, Zhao YG. Oscillatory and asymptotic behavior of third-order neutral delay differential equations with distributed deviating arguments. *AIMS Mathematics* 2020; 5: 5076-5093.
- [31] Sun YB, Zhao YG. Oscillation criteria for third-order nonlinear neutral differential equations with distributed deviating arguments. *Applied Mathematics Letters* 2021; 111: 106600.
- [32] Sun YB, Zhao YG. Oscillation theorems and asymptotic behaviour of certain third-order neutral differential equations with distributed deviating arguments. *International Journal of Dynamical Systems and Differential Equations* 2021; 11 (2): 174-189.
- [33] Tian YZ, Cai YL, Fu YL, Li TX. Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. *Advances in Difference Equations* 2015; 2015 (267): 1-14.
- [34] Tunç E. Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. *Electronic Journal of Differential Equations* 2017; 2017 (16): 1-12.
- [35] Tunç E, Grace SR. Oscillatory behavior of solutions to third-order nonlinear differential equations with a superlinear neutral term. *Electronic Journal of Differential Equations* 2020; 2020: 1-11.
- [36] Utku N, Şenel MT. Oscillation behavior of third-order quasilinear neutral delay dynamic equations on time scales. *Filomat* 2014; 28 (7): 1425-1436.
- [37] Wang PG. Oscillation criteria for second-order neutral equations with distributed deviating arguments. *Computers & Mathematics with Applications* 2004; 47 (12): 1935-1946.
- [38] Xiang HF. Oscillation of third-order nonlinear neutral differential equations with distributed time delay. *Italian Journal of Pure and Applied Mathematics* 2016; 36: 769-782.
- [39] Zhang QX, Gao L, Yu YH. Oscillation criteria for third-order neutral differential equations with continuously distributed delay. *Applied Mathematics Letters* 2012; 25 (10): 1514-1519.