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Metric connection on tangent bundle with Berger-type deformed Sasaki metric

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Abstract: Let TM be the tangent bundle over an almost antipara-Hermitian manifold endowed with Berger-type deformed Sasaki metric ${}^{BS}g$. In this paper, we introduce the deformed Sasaki metric which Berger-type and study the metric connection of this metric on the tangent bundle. We give some curvature properties of this metric and characterization of projective vector field which preserving the fiber of $\left(TM,{}^{BS}g\right)$. Next, we present some geometric results concerning them.

Key words: Berger-type deformed Sasaki metric, projective vector field, almost antipara-Hermitian manifold

1. Introduction

Let M be an $n-$dimensional Riemannian manifold with a metric g and TM be its tangent bundle. From the manifold (M,g) to its tangent bundle TM, different Riemannian and pseudo-Riemannian metrics may be distinguished using the natural lifts of Riemannian metric g. When used in this manner, metrics are referred to as natural metrics. The Sasaki metric, the most well-known of these metrics, was developed by Sasaki. Although the Sasaki metric is described as natural, it is rigid. For example, Kowalski [11] has shown that the tangent bundle with the Sasaki metric can never be locally symmetric if the base manifold is local flat. Next, Musso and Tricerri [13] showed that the necessary and sufficient condition for the tangent bundle with the Sasaki metric to have constant scalar curvature is that the base manifold is locally flat. In addition, Gudmundson and Kappos [8] also gave various results about the curvature of the tangent bundle with the Sasaki metric. Next, Gezer [3] characterized infinitesimal holomorphically projective transformations on the tangent bundle with the Sasaki metric and adapted an almost complex structure. Later on, different deformed forms of the Sasaki metric have been identified and studied by various authors (see [2], [4], [5], [10], [12], [17]). In [17], a method for determining accurate deformation and geodesic distances on certain manifolds was created by Yampolsky by use of the Berger deformation of a metric on a unit sphere and the Sasaki fiber-wise deformation of the Sasaki metric on slashed and unit tangent bundles on a Kähler manifold. In addition, recently, new studies have been carried out on this subject. Zagane [19] studied this metric on cotangent bundle and studied vertical rescaled Berger deformation metric on tangent bundle (see [20]).

Let M be a $2k$ dimension Riemannian manifold with a Riemannian metric g. A paracomplex manifold is an almost product manifold (M_{2k},φ), $\varphi^2 = id$, such that the two eigenbundles $T^{+}M$ and $T^{-}M$ linked to the two eigenvalues $+1$ and -1 of φ are of the same rank. The fact that the Nijenhuis tensor defined by

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$N_\varphi(X,Y)=[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+[X,Y]$ is zero means that an almost paracomplex structure is integrable. An almost paracomplex structure is a paracomplex structure that is integrable. An antipara-Hermitian metric is a Riemannian metric such that

$$g(\varphi X,\varphi Y)=g(X,Y)$$

or equivalent to this equation,

$$g(\varphi X,Y)=g(X,\varphi Y)\qquad \text{(purity condition)}$$

for any vector fields X,Y on M_{2k}. The triple (M_{2k},φ,g) is called an almost antipara-Hermitian manifold. (M_{2k},φ,g) is called antipara-Kähler if the almost paracomplex structure φ is paralel with regard to Levi–Civita ∇^g of g. Paraholomorphicity of g $(\Phi_\varphi g=0)$ is equivalent to the antipara-Kähler condition $(\nabla^g\varphi=0)$. In here Φ_φ is the Tachibana operator [14].

In [6], Gezer and Bilen gave a characterization of projective vector field which preserving the fiber on the tangent bundle with a class of Riemannian metric. Afterward in [1], Altunbaş, Şimşek and Gezer introduced a new metric which is named the Berger-type deformed Sasaki metric on the tangent bundle over an antipara-Kähler manifold. In addition to obtaining geodesics and curvature properties of the tangent bundle with the Berger-type deformed Sasaki metric, they also obtained and gave some result related to them. We focus a metric connection on the tangent bundle with the Berger-type deformed Sasaki metric in the current work. All forms of curvature tensors concerning the metric connection are obtained. Finally we give a characterization of projective vector field which preserving the fiber on $\left(TM,{}^{BS}g\right)$ concerning for to the metric connection ${}^{M}\nabla$.

2. The Berger-type deformed Sasaki metric on tangent bundle

Let M be an $n-$dimensional Riemannian manifold with a Riemannian metric g, TM be its tangent bundle and $\pi:TM\to M$ be a natural projection on M. The local coordinate system $\left(U,x^i\right)$ in M is induced to the local coordinate system $\left(\pi^{-1}(U),x^i,x^{\bar{i}}=u^i\right)$, $\bar{i}=n+i=n+1,...,2n$, on TM. Here $\left(u^i\right)$ are the cartesian coordinates in each tangent space T_PM of $\forall p\in U$. Also, p is an arbitrary point on U.

Let $X=X^i\frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift ${}^VX$ and the horizontal lift ${}^HX$ of X are given with respect to the induced coordinates, by

$$\begin{aligned}{}^VX &= X^i\partial_{\bar{i}},\\ {}^HX &= X^i\partial_i-y^s\Gamma^i_{sk}X^k\partial_{\bar{i}},\end{aligned}\tag{2.1}$$

where $\partial_i=\frac{\partial}{\partial x^i},\partial_{\bar{i}}=\frac{\partial}{\partial y^i}$ and Γ^i_{jk} are the coefficients of the Levi–Civita connection ∇ of g. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{aligned}\left[{}^HX,{}^HY\right] &= {}^H[X,Y]-{}^V(R(X,Y)u),\\ \left[{}^HX,{}^VY\right] &= {}^V(\nabla_XY),\\ \left[{}^VX,{}^VY\right] &= 0,\end{aligned}\tag{2.2}$$

for all vector fields M and Y on M , where R is the Riemannian curvature of g defined by $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ (for details, see [18]). We insert the adapted frame which allows the tensor calculus to be efficiently done in TM. With the connection ∇ of g on M, we can introduce adapted frames on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM. In each local chart $U \subset M$, we write $X_{(j)} = \frac{\partial}{\partial x^j}$, $j = 1, ..., n$. We see that these vector fields have, respectively, local expressions

$$\begin{aligned} {}^{H}X_{(j)} &= \delta_j^h \partial_h + \left(-y^s \Gamma_{sj}^h\right) \partial_{\overline{h}}, \\ {}^{V}X_{(j)} &= \delta_j^h \partial_{\overline{h}}, \end{aligned} \tag{2.3}$$

with respect to the natural frame $\{\partial_h, \partial_{\overline{h}}\}$, where δ_h^j denotes the Kronecker delta. These $2n$ vector fields are linearly independent and they generate the horizontal distribution of ∇g and the vertical distribution of TM, respectively. We call the set $\{{}^{H}X_{(j)}, {}^{V}X_{(j)}\}$ the frame adapted to the connection ∇ of g in $\pi^{-1}(U) \subset TM$. By denoting

$$\begin{aligned} E_j &= {}^{H}X_{(j)}, \\ E_{\overline{j}} &= {}^{V}X_{(j)}, \end{aligned} \tag{2.4}$$

we can write the adapted frame as $\{E_\beta\} = \{E_j, E_{\overline{j}}\}$. Using (2.1) and (2.4), we have

$$\begin{aligned} {}^{V}X &= \begin{pmatrix} 0 \\ X^h \end{pmatrix} = \begin{pmatrix} 0 \\ X^j \delta_j^h \end{pmatrix} = X^j \begin{pmatrix} 0 \\ \delta_j^h \end{pmatrix} = X^j E_{\overline{j}}, \\ {}^{H}X &= \begin{pmatrix} X^j \delta_j^h \\ -X^j \Gamma_{sj}^h y^s \end{pmatrix} = X^j \begin{pmatrix} \delta_j^h \\ -\Gamma_{sj}^h y^s \end{pmatrix} = X^j E_j, \end{aligned} \tag{2.5}$$

with respect to the adapted frame $\{E_\beta\}$, (for details see [18]).

Various Riemannian or pseudo-Riemannian metrics have been defined using the natural lifts of the Riemannian metric g. The most well-known example of these metrics is the Sasaki metric built in [15] by Sasaki on the tangent bundle TM. Various deformations of the Sasaki metric have been studied by some authors (see [2], [4],[17],[19],[20]). One of these deformations is the Berger-type deformed Sasaki metric.

Definition 2.1 *[1] Consider the almost antipara-Hermitian manifold (M_{2k}, φ, g) with arbitrary vector fields $X, Y \in \chi(M_{2k})$ and δ constant, and its tangent bundle TM. In the tangent bundle, the deformed Sasaki metric which Berger-type is defined as follows:*

$$\begin{aligned} {}^{BS}g\left({}^{H}X, {}^{H}Y\right) &= g(X, Y), \\ {}^{BS}g\left({}^{V}X, {}^{H}Y\right) &= {}^{BS}g\left({}^{H}X, {}^{V}Y\right), \\ {}^{BS}g\left({}^{V}X, {}^{V}Y\right) &= g(X, Y) + \delta^2 g(X, \varphi u)\, g(Y, \varphi u). \end{aligned} \tag{2.6}$$

In the adapted frame $\{E_\beta\}$, the Berger-type deformed Sasaki metric and its inverse respectively:

$$ {}^{BS}g = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} + \delta^2 g_{mo} g_{no} \varphi_i^m \varphi_j^n \end{pmatrix} \tag{2.7}$$

and

$$ {}^{BS}g^{-1} = \begin{pmatrix} g^{ij} & 0 \\ 0 & g^{ij} \frac{\delta^2}{1+\delta^2 g_{oo}} \varphi_o^i \varphi_o^j \end{pmatrix} \tag{2.8}$$

form is obtained. Here $g_{mo} = g_{mk}u^k, g_{oo} = g_{mk}u^m u^k, \varphi_o^i = \varphi_m^j u^m$.

For the Levi–Civita connection ${}^{BS}\nabla$ of the Berger-type deformed Sasaki metric ${}^{BS}g$, we give the following proposition.

Proposition 2.1: The ${}^{BS}\nabla$ Levi–Civita connection of the Berger-type deformed Sasaki metric ${}^{BS}g$ on tangent bundle is given as follows:

$$\begin{cases} {}^{BS}\nabla_{E_i} E_j = \Gamma_{ij}^h E_h - \frac{1}{2} R_{ijo}^h E_{\overline{h}}, \\ {}^{BS}\nabla_{E_{\overline{i}}} E_j = \frac{1}{2} R_{oij}^h E_h, \\ {}^{BS}\nabla_{E_i} E_{\overline{j}} = \frac{1}{2} R_{oji}^h E_h + \Gamma_{ij}^h E_{\overline{h}}, \\ {}^{BS}\nabla_{E_{\overline{i}}} E_{\overline{j}} = \frac{\delta^2}{1+\delta^2\alpha} \varphi_j^k \varphi_o^h g_{ik} E_{\overline{h}}, \end{cases} \tag{2.9}$$

where (M_{2k}, φ, g) is the antipara-Kähler manifold and TM its tangent bundle. Also R_{ijk}^h are components of the Riemannian curvature tensor field R of ∇ (see [1]).

3. The metric connection with nonvanishing torsion on the tangent bundle in accordance with Berger-type deformed Sasaki metric

Definition 3.1 *Let M be a Riemannian manifold and $\nabla : \Im_0^1(M) \times \Im_0^1(M) \longrightarrow \Im_0^1(M)$ be a linear connection on M. ∇ is called a metric connection on M if the Riemannian metric g that satisfies the condition $\nabla_X g = 0$ for every $X \in \Im_0^1(M)$. Otherwise, it is called a nonmetric connection. Here the condition $\nabla_X g = 0$ is equivalent to the condition $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for every $X, Y, Z \in \Im_0^1(M)$.*

As it is known, the only symmetrical connection that meets the condition $\nabla g = 0$ is the Levi–Civita connection. Thus, the Levi–Civita connection of the Berger-type deformed Sasaki metric is a torsion-free connection that meets the condition ${}^{BS}\nabla_\alpha \left({}^{BS}g\right)_{\beta\gamma} = 0$.

We will now deal with the metric connection ${}^{BS}\overline{\nabla}$ of the Berger-type deformed Sasaki metric, which is skew-symmetric in accordance with the torsion tensor $\left(\overline{T}_{\alpha\beta}^\gamma\right)$ indices α and β. Torsional metric connection in Riemannian manifolds was presented by Hayden [9].

We will express the coefficient of the metric connection ${}^{BS}\overline{\nabla}$ with $\overline{\Gamma}_{\alpha\beta}^\gamma$. Thus the metric connection ${}^{BS}\overline{\nabla}$ satisfies the followings:

$${}^{BS}\overline{\nabla}_\alpha \left({}^{BS}g\right)_{\beta\gamma} = 0 \qquad \text{and} \qquad \overline{\Gamma}_{\alpha\beta}^\gamma - \overline{\Gamma}_{\beta\alpha}^\gamma = \overline{T}_{\alpha\beta}^\gamma.$$

From the solution of the above equations, the following results are obtained [9]:

$$\overline{\Gamma}_{\alpha\beta}^\gamma = \widetilde{\Gamma}_{\alpha\beta}^\gamma + \overline{U}_{\alpha\beta}^\gamma, \tag{3.1}$$

where $\widetilde{\Gamma}_{\alpha\beta}^\gamma$ represents the coefficients of the Levi–Civita connection ${}^{BS}\nabla$.

$$\overline{U}_{\alpha\beta\gamma} = \frac{1}{2}\left(\overline{T}_{\alpha\beta\gamma} + \overline{T}_{\gamma\alpha\beta} + \overline{T}_{\gamma\beta\alpha}\right) \tag{3.2}$$

and

$$\overline{U}_{\alpha\beta\gamma} = \overline{U}_{\alpha\beta}^\varepsilon \left({}^{BS}g\right)_{\varepsilon\gamma}, \overline{T}_{\alpha\beta\gamma} = \overline{T}_{\alpha\beta}^\varepsilon \left({}^{BS}g\right)_{\varepsilon\gamma}.$$

If we put

$$\overline{T}_{ij}^{\overline{h}} = y^s R_{jis}^h, \tag{3.3}$$

all other $\overline{T}_{\alpha\beta}^{\gamma}$ not related to $\overline{T}_{ij}^{\overline{r}}$ is considered zero, by means of (3.2), from (3.3), we get

$$\begin{gathered}
\overline{U}_{ji}^{\overline{h}} = \tfrac{1}{2} y^s R_{jis}^h, \\
\overline{U}_{j\overline{i}}^{h} = -\tfrac{1}{2} y^s R_{sij}^h - \tfrac{\delta^2}{2} y^s R_{spj}^h g_{qi} \varphi_r^p \varphi_t^q u^r u^t, \\
\overline{U}_{\overline{j}i}^{h} = -\tfrac{1}{2} y^s R_{sji}^h - \tfrac{\delta^2}{2} y^s R_{spi}^h g_{qj} \varphi_r^p \varphi_t^q u^r u^t,
\end{gathered}$$

all other $\overline{U}_{\alpha\beta}^{\gamma}$ being zero. We can write the next proposition from the solutions of Equations (2.9) and (3.1).

Proposition 3.1: Let us take ∇ torsion-free affine connections on a Riemannian manifold M. Let TM be the tangent bundle of M with the Berger-type deformed Sasaki metric ${}^{BS}g$ on (M, ∇). According to the adapted frame $\{E_\beta\}$, the metric connection ${}^{BS}\overline{\nabla}$ provides the followings:

$$\begin{aligned}
{}^{BS}\overline{\nabla}_{E_i} E_j &= \Gamma_{ij}^h E_h, \\
{}^{BS}\overline{\nabla}_{E_{\overline{i}}} E_j &= 0, \\
{}^{BS}\overline{\nabla}_{E_i} E_{\overline{j}} &= \Gamma_{ij}^h E_{\overline{h}}, \\
{}^{BS}\overline{\nabla}_{E_{\overline{i}}} E_{\overline{j}} &= \frac{\delta^2}{1+\delta^2\alpha} g_{ik} \varphi_j^k \varphi_s^h u^s E_{\overline{h}},
\end{aligned} \tag{3.4}$$

where Γ_{ij}^h represent the component of ∇ on M and $\alpha = \|u\|^2 = g(u,u) = g_{ks} u^k u^s$.

Proof If the tensors $\overline{U}_{\alpha\beta}^{\gamma}$ obtained from the torsion tensor and Levi–Civita connection of Berger-type deformed Sasaki metric given by Equation (2.9) are used in Equation (3.1) the following results are obtained:

$$\begin{aligned}
\overline{\Gamma}_{ij}^h &= \Gamma_{ij}^h, \\
\overline{\Gamma}_{i\overline{j}}^{\overline{h}} &= \Gamma_{ij}^h, \\
\overline{\Gamma}_{\overline{i}\overline{j}}^{\overline{h}} &= \frac{\delta^2}{1+\delta^2\alpha} g_{ik} \varphi_j^k \varphi_s^h u^s, \\
\overline{\Gamma}_{ij}^{\overline{h}} &= \overline{\Gamma}_{\overline{i}j}^{\overline{h}} = \overline{\Gamma}_{\overline{i}j}^{h} = \overline{\Gamma}_{i\overline{j}}^{h} = \overline{\Gamma}_{\overline{i}\overline{j}}^{h} = 0.
\end{aligned} \tag{3.5}$$

If these results are used in the expression ${}^{BS}\overline{\nabla}_{E_\alpha} E_\beta = \overline{\Gamma}_{\alpha\beta}^{\gamma} E_\gamma$, we get the equations (3.5). Thus the proof is completed. □

Proposition 3.2: Let us take ∇ torsion-free affine connections on a Riemannian manifold M. Let TM be the tangent bundle of M with the Berger-type deformed Sasaki metric ${}^{BS}g$ on (M, ∇). According to the adapted frame $\{E_\beta\}$, the ${}^{M}R$ curvature tensor of the metric connection ${}^{BS}\overline{\nabla}$ in the tangent bundle has the followings:

$$\begin{aligned}
{}^{M}R(E_i,E_j)E_k &= R_{ijk}^{h}E_h,\\
{}^{M}R(E_i,E_j)E_{\overline{k}} &= \left(R_{ijk}^{h}+\frac{\delta^2}{1+\delta^2\alpha}R_{ijp}^{m}g_{mt}\varphi_k^t\varphi_s^h y^p y^s\right)E_{\overline{h}},\\
{}^{M}R(E_{\overline{i}},E_{\overline{j}})E_{\overline{k}} &= \left(\frac{\delta^2}{1+\delta^2\alpha}g_{ak}\left(\varphi_j^a\varphi_i^h-\varphi_i^a\varphi_j^h\right)+\left(\frac{\delta^2}{1+\delta^2\alpha}\right)^2 g_{ak}\varphi_s^h\left(g_{ip}\varphi_j^a-g_{jp}\varphi_i^a\right)y^p y^s\right)E_{\overline{h}},\\
{}^{M}R(E_{\overline{i}},E_j)E_k &= {}^{M}R(E_i,E_{\overline{j}})E_k={}^{M}R(E_{\overline{i}},E_{\overline{j}})E_k={}^{M}R(E_{\overline{i}},E_j)E_{\overline{k}}={}^{M}R(E_i,E_{\overline{j}})E_{\overline{k}}=0,
\end{aligned}$$

where Γ_{ij}^h and R_{ijk}^h respectively represent the component of ∇ and its curvature tensor field on M and $\alpha=\|u\|^2=g(u,u)=g_{ks}u^k u^s$.

Proof The Riemannian curvature tensor R of the connection ∇ is obtained from the well known formula

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_x Z-\nabla_{[X,Y]}Z,$$

for all $X,Y\in\Im_0^1(M)$ with respect to the adapted frame $\{E_\beta\}$, we write ${}^{BS}\nabla_{E_\alpha}E_\beta=\overline{\Gamma}_{\alpha\beta}^{\gamma}E_\gamma$, where $\overline{\Gamma}_{\alpha\beta}^{\gamma}$ is denote the metric connection constructed by ${}^{BS}g$. Then the Riemannian curvature tensor ${}^{M}R$ has the components

$${}^{M}R_{\alpha\beta\gamma}^{\ \ \ \sigma}=E_\alpha\overline{\Gamma}_{\beta\gamma}^{\sigma}-E_\beta\overline{\Gamma}_{\alpha\gamma}^{\sigma}+\overline{\Gamma}_{\alpha\varepsilon}^{\sigma}\overline{\Gamma}_{\beta\gamma}^{\varepsilon}-\overline{\Gamma}_{\beta\varepsilon}^{\sigma}\overline{\Gamma}_{\alpha\gamma}^{\varepsilon}-\Omega_{\alpha\beta}^{\ \ \varepsilon}\overline{\Gamma}_{\varepsilon\gamma}^{\sigma}, \tag{3.6}$$

where

$$\begin{aligned}
\Omega_{\overline{i}\overline{j}}^{\overline{k}} &= -\Omega_{\overline{j}i}^{\overline{k}}=\Gamma_{ji}^{k}, \qquad (3.7)\\
\Omega_{ij}^{\overline{k}} &= -\Omega_{ji}^{\overline{k}}=-y_s R_{ijs}^{\ \ \ k}
\end{aligned}$$

and $\Omega^{\alpha}{}_{\gamma\beta}={}^{BS}g^{\alpha\varepsilon}\ {}^{BS}g_{\delta\beta}\Omega_{\varepsilon\gamma}^{\ \ \delta}$, ${}^{BS}g_{\delta\beta}$ are the contravariant components of the metric ${}^{BS}g$ with respect to adapted frame (for details see [7]). $\left(\alpha=\left(i,\overline{i}\right),\ \beta=\left(j,\overline{j}\right),\ \gamma=\left(k,\overline{k}\right),\ \sigma=\left(h,\overline{h}\right)\right)$ From (3.6) and (3.5), we obtain the component of the Riemannian curvature tensor ${}^{M}R$ of the metric connection of Berger-type deformed Sasaki metric as follows:

$$\begin{aligned}
{}^{M}R_{ijk}^{\ \ \ h} &= R_{ijk}^{\ \ \ h},\\
{}^{M}R_{ij\overline{k}}^{\ \ \ \overline{h}} &= R_{ijk}^{\ \ \ h}+\frac{\delta^2}{1+\delta^2\alpha}R_{ijp}^{\ \ \ m}g_{mt}\varphi_k^t\varphi_s^h y^p y^s, \qquad (3.8)\\
{}^{M}R_{\overline{i}\overline{j}\overline{k}}^{\ \ \ \overline{h}} &= \frac{\delta^2}{1+\delta^2\alpha}g_{ak}\left(\varphi_j^a\varphi_i^h-\varphi_i^a\varphi_j^h\right)+\left(\frac{\delta^2}{1+\delta^2\alpha}\right)^2 g_{ak}\varphi_s^h\left(g_{ip}\varphi_j^a-g_{jp}\varphi_i^a\right)y^p y^s,
\end{aligned}$$

and all other terms are zero. Thus the proof is completed. □

Proposition 3.3: Let us take ∇ torsion-free affine connections on a Riemannian manifold M. Let TM be the tangent bundle of M with the Berger-type deformed Sasaki metric ${}^{BS}g$ on (M,∇). According to the adapted frame $\{E_\beta\}$, the Ricci curvature tensor of the metric connection ${}^{BS}\overline{\nabla}$ in the tangent bundle has the followings:

$$\begin{aligned}
{}^{M}R_{ij} &= R_{ij}, \\
{}^{M}R_{\bar{i}j} &= 0, \\
{}^{M}R_{i\bar{j}} &= 0, \\
{}^{M}R_{\overline{ij}} &= \left(\frac{\delta^2}{1+\delta^2\alpha}\right)^2 \left(g_{ks}g_{aj}\varphi_p^k\varphi_i^a - g_{ip}g_{js}\right)u^p u^s - \frac{\delta^2}{1+\delta^2\alpha}g_{ij},
\end{aligned}$$

where $R_{ij} = R_{kij}^k$ and $\alpha = \|u\|^2 = g(u,u) = g_{ks}u^k u^s$.

Proof As it is known, the Ricci curvature tensor is characterized by $R_{\alpha\beta} = R_{\gamma\alpha\beta}^{\ \ \ \ \gamma}$. Let ${}^{M}R_{\alpha\beta} = {}^{M}R_{\gamma\alpha\beta}^{\ \ \ \ \gamma}$ be the Ricci curvature tensor of the metric connection of Berger-type deformed Sasaki metric. $\left(\alpha = \left(i,\bar{i}\right),\ \beta = \left(j,\bar{j}\right),\ \gamma = \left(k,\bar{k}\right)\right)$ The equivalents of the Riemannian curvature tensor, given in (3.8) are used in this equation the following results are obtained:

$$\begin{aligned}
{}^{M}R_{ij} &= {}^{M}R_{kij}^{\ \ \ \ k} + {}^{M}R_{\bar{k}ij}^{\ \ \ \ \bar{k}} = R_{ij}, \\
{}^{M}R_{\overline{ij}} &= {}^{M}R_{k\overline{ij}}^{\ \ \ \ k} + {}^{M}R_{\overline{kij}}^{\ \ \ \ \bar{k}} = \left(\frac{\delta^2}{1+\delta^2\alpha}\right)^2 \left(g_{ks}g_{aj}\varphi_p^k\varphi_i^a - g_{ip}g_{js}\right)u^p u^s - \frac{\delta^2}{1+\delta^2\alpha}g_{ij}, \\
{}^{M}R_{\bar{i}j} &= {}^{M}R_{k\bar{i}j}^{\ \ \ \ k} + {}^{M}R_{\overline{ki}j}^{\ \ \ \ \bar{k}} = 0, \\
{}^{M}R_{i\bar{j}} &= {}^{M}R_{ki\bar{j}}^{\ \ \ \ k} + {}^{M}R_{\bar{k}i\bar{j}}^{\ \ \ \ \bar{k}} = 0,
\end{aligned} \tag{3.9}$$

the proof is completed. □

Proposition 3.4: Let us take ∇ torsion-free affine connections on a Riemannian manifold M. Let TM be the tangent bundle of M with the Berger-type deformed Sasaki metric ${}^{BS}g$ on (M,∇). According to the adapted frame $\{E_\beta\}$, the ${}^{M}r$ scalar curvature of the metric connection ${}^{BS}\overline{\nabla}$ in the tangent bundle satisfies the following:

$${}^{M}r = r + 2\alpha\left(\frac{\delta^2}{1+\delta^2\alpha}\right)^2 - 2n\left(\frac{\delta^2}{1+\delta^2\alpha}\right),$$

where r denotes the scalar curvature of ∇ on M and $\alpha = \|u\|^2 = g(u,u) = g_{ks}u^k u^s$.

Proof ${}^{M}r = {}^{BS}g^{\alpha\beta}\ {}^{M}R_{\alpha\beta}$ denote the scalar curvature of the metric connection of Berger-type deformed Sasaki metric. $\left(\alpha = \left(i,\bar{i}\right),\ \beta = \left(j,\bar{j}\right)\right)$ From (2.8) and (3.9) we have

$$\begin{aligned}
{}^{M}r &= {}^{BS}g^{\alpha\beta}\ {}^{M}R_{\alpha\beta} \\
&= {}^{BS}g^{ij}\ {}^{M}R_{ij} + {}^{BS}g^{\bar{i}j}\ {}^{M}R_{\bar{i}j} + {}^{BS}g^{i\bar{j}}\ {}^{M}R_{i\bar{j}} + {}^{BS}g^{\overline{ij}}\ {}^{M}R_{\overline{ij}} \\
&= r + 2\alpha\left(\frac{\delta^2}{1+\delta^2\alpha}\right)^2 - 2n\left(\frac{\delta^2}{1+\delta^2\alpha}\right).
\end{aligned}$$

Thus the proof of the proposition 3.4. is completed. □

4. Projective vector fields on the tangent bundle in accordance with Berger-type deformed Sasaki metric

Let ∇ be a linear connection on a manifold M. For every pair of vector fields X and Y on M, if the $1-$form θ satisfies the condition,

$$(L_V\nabla)(X,Y)=\theta(X)Y+\theta(Y)X, \tag{4.1}$$

then the vector field V is called projective vector field. Here L_V is the Lie derivative in accordance with the vector field V. In this situation, θ is named the associated $1-$form of the vector field V. In local coordinates, according to the natural frame $\{E_\beta\}$, Equation (4.1) is written on M as follows:

$$L_V\Gamma_{ij}^h=\theta_i\delta_j^h+\theta_j\delta_i^h.$$

Consider the vector field V with components $\left(v^h,v^{\overline{h}}\right)$ on the tangent bundle, in accordance with the adapted frame. v^h is the horizontal component of the vector field V and $v^{\overline{h}}$ is the vertical component of the vector field V. If the horizontal component of the vector field V, depends only on the variable $\left(x^h\right)$, then the vector field V is named the fibre-preserving vector field. We can clearly say that every fibre-preserving vector field V on the tangent bundle TM is reduced to a vector field with components $\left(v^h\right)$ on the base manifold M.

In this section, we have studied the projective vector field which preserving the fiber, according to ${}^{BS}\overline{\nabla}$, which is the metric connection of the Berger-type deformed Sasaki metric ${}^{BS}g$ on the tangent bundle. First, let us write the following Lemma, which we will have to use later.

Lemma 4.1 *[16] Let $X=v^hE_h+v^{\overline{h}}E_{\overline{h}}$ be a fibre-preserving vector field, the Lie derivatives of the adapted frame vectors in the direction of the vector field X are written as follows:*

$$\begin{aligned} L_XE_h &= -(\partial_hv^a)E_a+\left\{y^bv^cR_{hcb}{}^a-v^{\overline{b}}\Gamma_{bh}^a-\left(E_hv^{\overline{a}}\right)\right\}E_{\overline{a}}, \\ L_XE_{\overline{h}} &= \left\{v^b\Gamma_{bh}^a-\left(E_{\overline{h}}v^{\overline{a}}\right)\right\}E_{\overline{a}}. \end{aligned}$$

Theorem 4.2 *Let $\left(TM,{}^{BS}g\right)$ be the tangent bundle of Riemannian manifold (M,g). A necessary and sufficient condition for defining a X projective vector field which preserving the fiber with its associated 1-form θ on $\left(TM,{}^{BS}g\right)$ is the X vector field*

$$X={}^HV+{}^VB+\gamma A, \tag{4.2}$$

in addition, the associated 1-form θ provides the followings:

(i) $\theta=\theta_idx^i,\theta_{\overline{i}}=0$,

(ii) $L_XC_{ij}^a=C_{ij}^h\left(v^b\Gamma_{bh}^a+A_h^a\right)$,

(iii) $\theta_j=\frac{1}{n}\left(\nabla_jA_i^i+\frac{\delta^2}{1+\delta^2\alpha}\left(\nabla_jB^h\right)C_{ih}^i\right)$,

(iv) $\frac{\delta^2}{1+\delta^2\alpha}\left(v^cR_{jcs}^h-\nabla_jA_s^h\right)C_{ih}^i=0$,

(v) $L_X\Gamma_{ij}^a+\nabla_i\nabla_jv^a=\theta_i\delta_j^a+\theta_j\delta_i^a$,

(vi) $\nabla_i\left(v^cR_{cjs}^a+\nabla_jA_s^a\right)=0$,

(vii) $\nabla_i\ \nabla_jB^a=0$,

where the vector fields $V=(v^h), B=(B^h),$ *the* $(1,1)$*-tensor field* $A=(A_i^h)$, $\alpha=\|u\|^2=g(u,u)=g_{ks}u^k u^s$ *and* $C_{ih}^a=g_{it}\varphi_h^t\varphi_s^a u^s$.

Proof A vector field $X=v^h E_h+v^{\overline{h}}E_{\overline{h}}$ on TM is a projective vector field which preserving the fiber if and only if there exists a 1-form θ with components $(\theta_i,\theta_{\bar{i}})$ on TM such that

$$\begin{aligned}(L_X\nabla)(Y,Z) &= L_X(\nabla_Y Z)-\nabla_Y(L_X Z)-\nabla_{(L_X Y)}Z \\ &= \theta(Y)Z+\theta(Z)Y,\end{aligned} \tag{4.3}$$

for any vector fields Y and Z on TM. We compute the following three relations:

$$\begin{aligned}(L_X\nabla)(E_{\bar{i}},E_{\bar{j}}) &= L_X(\nabla_{E_{\bar{i}}}E_{\bar{j}})-\nabla_{E_{\bar{i}}}(L_X E_{\bar{j}})-\nabla_{(L_X E_{\bar{i}})}E_{\bar{j}} \\ &= \theta(E_{\bar{i}})E_{\bar{j}}+\theta(E_{\bar{j}})E_{\bar{i}},\end{aligned} \tag{4.4}$$

$$\begin{aligned}(L_X\nabla)(E_{\bar{i}},E_j) &= L_X(\nabla_{E_{\bar{i}}}E_j)-\nabla_{E_{\bar{i}}}(L_X E_j)-\nabla_{(L_X E_{\bar{i}})}E_j \\ &= \theta(E_{\bar{i}})E_j+\theta(E_j)E_{\bar{i}},\end{aligned} \tag{4.5}$$

$$\begin{aligned}(L_X\nabla)(E_i,E_j) &= L_X(\nabla_{E_i}E_j)-\nabla_{E_i}(L_X E_j)-\nabla_{(L_X E_i)}E_j \\ &= \theta(E_i)E_j+\theta(E_j)E_i.\end{aligned} \tag{4.6}$$

From Equation (4.4), by virtue of Equation (3.5) and Lemma 4.1, we obtain,

$$\frac{\delta^2}{1+\delta^2\alpha}\left\{L_X C_{ij}^h-C_{ij}^h\left(v^b\Gamma_{bh}^a+E_{\overline{h}}v^{\overline{a}}\right)\right\}+E_{\bar{i}}\left(E_{\bar{j}}v^{\overline{a}}\right)=\theta_{\bar{j}}\delta_i^a+\theta_{\bar{i}}\delta_j^a. \tag{4.7}$$

Similarly, from Equation (4.5) we get

$$\left[E_{\bar{i}}\left(E_j v^{\overline{a}}\right)+\left(E_{\bar{i}}v^{\overline{b}}\right)\Gamma_{bj}^a-v^c R_{jci}^a-\frac{\delta^2}{1+\delta^2\alpha}\left\{y^b v^c R_{jcb}^h-v^{\overline{b}}\Gamma_{bj}^h-E_j v^{\overline{h}}\right\}C_{ih}^a\right]E_{\overline{a}}=\theta_{\bar{i}}E_j+\theta_j E_{\bar{i}},$$

from which

$$\theta_{\bar{i}}=0 \tag{4.8}$$

and

$$E_{\bar{i}}\left(E_j v^{\overline{a}}\right)+\left(E_{\bar{i}}v^{\overline{b}}\right)\Gamma_{bj}^a-v^c R_{jci}^a-\frac{\delta^2}{1+\delta^2\alpha}\left\{y^b v^c R_{jcb}^h-v^{\overline{b}}\Gamma_{bj}^h-E_j v^{\overline{h}}\right\}C_{ih}^a=\theta_j\delta_i^a. \tag{4.9}$$

Due to Equations (4.8) and (4.7) reduces to

$$L_X C_{ij}^h-C_{ij}^h\left(v^b\Gamma_{bh}^a+E_{\overline{h}}v^{\overline{a}}\right)=0 \tag{4.10}$$

and

$$E_{\bar{i}}\left(E_{\bar{j}}v^{\overline{a}}\right)=0, \tag{4.11}$$

thus we can put

$$v^{\bar{a}}=y^s A_s^a+B^a, \tag{4.12}$$

where A is a tensor field of type $(1,1)$ which has components (A_a^s) and depends only on the variable $\left(x^h\right)$, B is a vector field with components (B^h) and depends only on the variable $\left(x^h\right)$. Thus, we can write a projective vector field which preserving the fiber X on the tangent bundle as follows:

$$\begin{aligned} X &= v^h E_h + v^{\overline{h}} E_{\overline{h}} \\ &= v^h E_h + \left(y^s A_s^h + B^h\right) E_{\overline{h}} \\ &= v^h E_h + B^h E_{\overline{h}} + y^s A_s^h E_{\overline{h}} \\ &= {}^H V + {}^V B + \gamma A. \end{aligned}$$

Also, substituting Equation (4.12) into Equations (4.10) and (4.9), we get

$$L_X C_{ij}^a = C_{ij}^h \left(v^b \Gamma_{bh}^a + A_h^a\right)$$

and

$$E_j A_i^a + A_i^b \Gamma_{bj}^a - v^c R_{jci}^a - \frac{\delta^2}{1+\delta^2\alpha}\left\{y^b v^c R_{jcb}^h - \left(y^s A_s^b + B^b\right)\Gamma_{bj}^h - E_j\left(y^s A_s^h + B^h\right)\right\} C_{ih}^a = \theta_j \delta_i^a.$$

Contracting i and a in last equation, we have

$$\theta_j = \frac{1}{n}\left(E_j A_i^a + \Gamma_{bj}^a A_i^b + \frac{\delta^2}{1+\delta^2\alpha}\left(\nabla_j B^h\right) C_{ih}^i\right),$$

$$\theta_j = \frac{1}{n}\left(\nabla_j A_i^i + \frac{\delta^2}{1+\delta^2\alpha}\left(\nabla_j B^h\right) C_{ih}^i\right)$$

and

$$\frac{\delta^2}{1+\delta^2\alpha}\left(v^c R_{jcs}^h - \nabla_j A_h^s\right) C_{ih}^i = 0.$$

From Equation (4.6), we get

$$L_X \Gamma_{ij}^a + \left(E_i v^h\right)\Gamma_{hj}^a + \nabla_i\left(E_j v^a\right) = \theta_i \delta_j^a + \theta_j \delta_i^a,$$

$$L_X \Gamma_{ij}^a + \nabla_i \nabla_j v^a = \theta_i \delta_j^a + \theta_j \delta_i^a$$

and

$$\nabla_i\left[y^b v^c R_{jcb}^a + v^{\bar{b}} \Gamma_{bj}^a + E_j v^{\bar{a}}\right] = 0.$$

Substituting Equation (4.12) into last equation, we have

$$\nabla_i\left(v^c R_{jcs}^a + \nabla_j A_s^a\right) = 0$$

and

$$\nabla_i \nabla_j B^a = 0.$$

Conversely, if B^h, v^h, θ_h and A_h^i providing $(i)-(vii)$ are given, we will see that $X = {}^{H}V + {}^{V}B + \gamma A$ is a projective vector field which preserving the fiber on $\left(TM, {}^{BS}g\right)$ when the above operations are reversed. This completes the proof. □

Corollary 4.1: Let $\left(TM, {}^{BS}g\right)$ be the tangent bundle of a Riemannian manifold (M, g). Every projective vector field which preserving the fiber written as in Equation (4.2) is naturally induced to a projective vector field on the base manifold M.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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